Perturbational finite volume scheme for the one-dimensional Navier-Stokes equations

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Abstract. Starting from the second-order finite volume scheme,though numerical value perturbation of the cell facial fluxes, the perturbational finite volume (PFV) scheme of the Navier-Stokes (NS) equations for compressible flow is developed in this paper. The central PFV scheme is used to compute the one-dimensional NS equations with shock wave.Numerical results show that the PFV scheme can obtain essentially non-oscillatory solution.

1 Introduction

The finite volume (FV) method is widely used in commercial codes. In FV method, second order accurate central FV(2CFV) scheme is the simplest one and offers a good compromise among accuracy, simplicity and efficiency. Its disadvantage is that there is oscillation solution when the grid Reynolds number is larger than about 2[1]. The perturbation finite volume (PFV) method presented by Zhi Gao retains the advantages of 2CFV method, however, its interpolation approximations are of high order accurate. Numerical tests of using PFV schemes to compute the scalar transport equation and the Navier-Stokes equations for incompressible flow show that PFV schemes have better performances than 2CFV scheme[2,3]. Gao et al[4] discussed the significance of higher-order accuracy reconstruction approximation, offered numerically the practical effect and benefit in the upwind and central PFV schemes. Recently, Gao and Yang [5] developed a perturbational finite volume method for the convective-diffusion integral equation, the PFV scheme is an upwind and mixed scheme with any higher-order interpolation and second-order integration approximations. The PFV scheme uses the least nodes like the standard three-point schemes. The PFV scheme is applied on a number of 1-D linear and nonlinear problems, 2-D and 3-D flow model equations, its numerical accuracies are higher than second-order central scheme, the power law scheme (PLS) and QUICK scheme.

Starting from the second-order central finite volume scheme,though numerical value perturbation of the cell facial fluxes, the perturbational finite volume (PFV) scheme of the Navier-Stokes (NS) equations for compressible flow is developed in this paper. Numerical results of the one-dimensional NS equations with shock wave show that the PFV scheme can obtain essentially non-oscillatory solution.

2 Control equations and numerical methods

2.1 The PFV scheme of the transport equation

The integral form of the scalar transport equation is

$$\int_S \rho \vec{u} \cdot \vec{n} ds = \int_S \Gamma \Delta\phi \cdot \vec{n} ds + \int_\Omega q dv \tag{1}$$

The finite volume scheme of eq(1) with a parameter α is as following

$$\Sigma_j^{J_p}\{[\frac{\Gamma \vec{d}_j}{|\vec{d}_j|^2} \cdot \vec{S}_j)_{jf} - \frac{1-\alpha}{2} m_{jf}]\phi_{jp} - [\frac{\Gamma \vec{d}_j}{|\vec{d}_j|^2} \cdot \vec{S}_j)_{jf} - \frac{1+\alpha}{2} m_{jf}]\phi_p\} + q_p\Omega = 0 \tag{2}$$

where ϕ expresses any transported variable, such as temperature, energy and components of flow velocity, ρ is the density, $\vec{u}$ is the velocity vector, Γ is the diffusion coefficient, q is the source term. S and Ω are the surface and the volume or area of control volume, respectively. A typical CV's face labeled "jf" is considered. Suppose the line connecting the node P (CV center) and the node jp (adjacent CV center) is nearly orthogonal to the interface of the control volumes j and jp. m_{jf} is the mass flux through the j-face

In PFV method [2-4], in order to improve FV accuracy, high order accurate of the interpolation approximation is obtained by a numerical value perturbation technique, i.e. the mass fluxes of the cell faces m_{jf} and the source term q_p are expanded into power series of the grid spacing and the coefficients of the power series are determined with the aid of the conservation equation itself. Supposing there are perturbations exerted on the values of the mass fluxes m_{jf} and the source term q_p, i.e.

$$m_{jfp} = m_{jf} + \Sigma_{n=1}^{N} A_{jn} |\vec{d}_j|^n, \quad q_{pp} = q_p + \Sigma_{n=1}^{N} B_{jn} |\vec{d}_j|^n \tag{3}$$

By using the relations

$$\frac{\Gamma \vec{d}_j \cdot \vec{S}_j}{|\vec{d}_j|^2} \phi'_{jf} - m_{jf}\phi_{jf} + q_p = 0 \tag{4}$$

we can obtain the coefficients

$$A_{j1} = \frac{\alpha}{2|\vec{d}_j|} m_{jf} R_{jf}, A_{j2} = \frac{1}{6|\vec{d}_j|^2} m_{jf} R_{jf}, A_{j3} = \frac{\alpha}{24|\vec{d}_j|^3} m_{jf} R_{jf}^3, \ldots$$

$$B_{j1} = \frac{\alpha}{2|\vec{d}_j|} m_{jf} R_{jf}, B_{j2} = \frac{1}{6|\vec{d}_j|^2} m_{jf} R_{jf}, B_{j3} = \frac{\alpha}{24|\vec{d}_j|^3} m_{jf} R_{jf}^3, \ldots$$

where $R_{jf} = \frac{m_{jf}}{\Gamma S_j} |\vec{d}_j| = \frac{\rho U_d |\vec{d}_j|}{\Gamma}$. R_{jf} can be regarded as cell Reynolds number. So we obtain the perturbational FV scheme

$$\Sigma_j^{J_p} \frac{1}{G_j} \{ [(\frac{\Gamma \vec{d}_j}{|\vec{d}_j|^2} \cdot \vec{S}_j)_{jf} - \frac{1-\alpha}{2} m_{jfp}] \phi_{jp} - [(\frac{\Gamma \vec{d}_j}{|\vec{d}_j|^2} \cdot \vec{S}_j)_{jf} - \frac{1+\alpha}{2} m_{jfp}] \phi_p \} + q_p \Omega = 0 \tag{5}$$

where

$$G_j = 1 + \frac{\alpha}{2} R_{jf} + \frac{1}{6} R_{jf}^2 + \frac{\alpha}{24} R_{jf}^3 + \ldots$$

Let $\alpha = 0$, we obtain the central perturbational finite volume scheme.

2.2 The PFV scheme of the Navier-Stokes equation

The integral form of the NS equations for compressible flow is as follows

$$\frac{\partial}{\partial t} \int_V \rho dv + \int_S \rho \vec{u} \cdot \vec{n} ds = 0 \,, \tag{6}$$

$$\frac{\partial}{\partial t} \int_V \rho \vec{u} dv + \int_S \rho \vec{u} \cdot \vec{u} \cdot \vec{n} ds = - \int_S \vec{n} ds + \int_S \Psi \cdot \vec{n} ds \,, \tag{7}$$

$$\frac{\partial}{\partial t} \int_V \rho E dv + \int_S \rho \vec{u} H \vec{n} ds = \int_S k \frac{\partial T}{\partial t} \vec{n} ds + \int_S [(uT_{xx} + vT_{xy}) \vec{e}_x + (uT_{xy} + vT_{yy}) \vec{e}_y] \vec{n} ds \,, \tag{8}$$

For a perfect gas with constant specific heats, the equation of state is

$$p = \rho RT \,, \tag{9}$$

Similar to the process of the PFV scheme for the scalar transport equation, the (2N+2)th-order accurate central PFV scheme for the momentum equation is as follows

$$
\begin{aligned}
V_p \tfrac{\partial}{\partial t}(\rho\phi)_p = \textstyle\sum_j \tfrac{1}{G_{jp}}\{[\tfrac{\mu \vec{d}_j\cdot\vec{S}_j}{|\vec{d}_j|^2} - \tfrac{1}{2}\dot{m}_{jf}G^{+}_{jp}]\phi_{jp} \\
-[\tfrac{\mu \vec{d}_j\cdot\vec{S}_j}{|\vec{d}_j|^2} + \tfrac{1}{2}\dot{m}_{jf}G^{-}_{jp}]\phi_p\} \\
-\textstyle\sum_j(\mu\tfrac{\partial\phi}{\partial\xi})_{jf}S_j - \sum p_{jf}S_j + \Phi\,,
\end{aligned}
\tag{10}
$$

where

$$G^{+}_{jp} = \sum_{n=0}^{2N+1}\frac{(-1)^n}{(n+1)!}R^n_{jf}$$

$$G^{-}_{jp} = \sum_{n=0}^{2N+1}\frac{1}{(n+1)!}R^n_{jf}$$

$$G_{jp} = \sum_{n=0}^{N}\frac{1}{(2n+1)!}R^{2n}_{jf}$$

The (2N+2)th order accurate central PFV scheme for the energy equation also can be written as following

$$
\begin{aligned}
v_p\ \tfrac{\partial}{\partial t}(\rho E)_p = \textstyle\sum_j \tfrac{1}{Q_{jp}}\{[\tfrac{k \vec{d}_j\cdot\vec{S}_j}{|\vec{d}_j|^2} - \tfrac{1}{2}\dot{m}_{jf}C_pQ^{+}_{jp}]T_{jp} \\
-[\tfrac{k \vec{d}_j\cdot\vec{S}_j}{|\vec{d}_j|^2} + \tfrac{1}{2}\dot{m}_{jf}C_pQ^{-}_{jp}]T_p\} \\
-\textstyle\sum(k\tfrac{\partial\phi}{\partial\xi})_{jf}S_j + \sum_j k(\tfrac{\partial T}{\partial x}\vec{e}_x + \tfrac{\partial T}{\partial y}\vec{e}_y)\cdot\vec{S}_j \\
-[\rho u(H - C_pT)\vec{e}_x + \rho v(H - C_pT)\vec{e}_y]_{jf}\cdot\vec{S}_j \\
+[(uT_{xx} + vT_{xy})\vec{e}_x + (uT_{xy} + vT_{yy})\vec{e}_y]_{jf}\vec{S}_j\,,
\end{aligned}
\tag{11}
$$

where

$$Q^{+}_{jp} = \sum_{n=0}^{2N+1}\frac{(-1)^n}{(n+1)!}P^n_{jf}$$

$$Q^{-}_{jp} = \sum_{n=0}^{2N+1}\frac{1}{(n+1)!}P^n_{jf}$$

$$Q_{jp} = \sum_{n=0}^{N}\frac{1}{(2n+1)!}P^{2n}_{jf}$$

ϕ expresses u and v, ξ_j is the directional variable along the line connecting juncture nodes p and jp. $\dot{m}_{jf}$ is the mass flux of the cell j-face, $R_{jf} = \dot{m}_{jf}\vec{d}_j/\mu S_j$,$P_{jf} = \dot{m}_{jf}C_p|\vec{d}_j|/kS_j$. R_{jf} and P_{jf} are respectively the cell Reynolds number and Peclet number in the $\vec{d}_j$-direction. When ϕ expresses u, $\Phi = \sum(\psi_{xx}\vec{e}_x + \psi_{xy}\vec{e}_y)_{jf}\cdot\vec{S}_j$; if ϕ expresses v, $\Phi = \sum(\psi_{xy}\vec{e}_x + \psi_{yy}\vec{e}_y)_{jf}\cdot\vec{S}_j$. $\vec{\Psi}$ is the stress tensor, $\psi_{ij} = \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) - \frac{2}{3}\mu\frac{\partial u_j}{\partial x_j}\delta_{ij}$, $\vec{e}_x$ and $\vec{e}_y$ are the unit vectors in the x- and y-direction respectively, μ is the dynamic viscousity, k is the thermal conductivity, H is the total enthalpy per unit mass, T is the absolute temperature(K), p is the pressure. The other terms in eq.(10) and (11) are discreted by central scheme.

The discrete continuity equation is expressed as the 2CFV scheme or fourth order accurate artificial-viscosity PFV scheme.

To solve the ordinary differential equation

$$\frac{dv}{dt} = L(v), \tag{12}$$

where L(v) is a discretization of the spatial opertor, the second-order Runge-Kutta is applied,

$$v^{n+1/2} = v^n + \delta t(L(v))^n,$$
$$v^{n+1} = \tfrac{1}{2}(v^n + v^{n+1/2}) + \tfrac{1}{2}\delta t(L(v))^{n+1/2}, \tag{13}$$

3 Numerical results

The PFV scheme (10) and (11) are applied into computing the one-dimensional Navier-Stokes equation.

The boundary conditions and initial conditions are as following[6]

$$\rho(0,t) = u(0,t) = T(0,t) = 1,$$
$$u(1,t) = \frac{2/(\gamma-1)+M^2}{(\gamma+1)/(\gamma-1)M^2},$$
$$T(1,t) = (\frac{2\gamma}{(\gamma+1)M^2} - \frac{\gamma-1}{\gamma+1})(\frac{\gamma-1}{\gamma+1} + \frac{2\gamma}{(\gamma+1)M^2} \tag{14}$$

When $t = 0, \rho(1,0) = 1/u(1,0)$,the other variables is obtained by linear interpolation of boundary conditions. In our computation, $\rho(1,t)$ is given by extrapolation of interior points.The grid number $N = 200$.

First, we using both the second-order central finite volume (2-CFV)scheme and the second-order perturbational finite volume (2-PFV) scheme to compute the case with $M = 2$, $Re = 800$. The variations of the density ρ, velocity u, pressure p and temperature T with the distance are given in Figs.1 and 2.

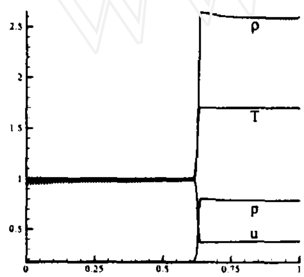

Fig. 1. 2-CFV scheme, Re=800

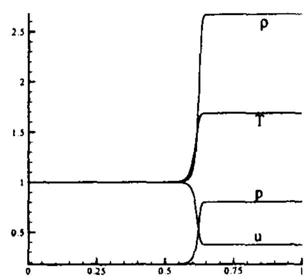

Fig. 2. 2-PFV scheme, Re=800

In Figs.1 and 2, the average grid Reynolds number is 4, there are small oscillations domain in the solutions of second order accurate CFV scheme, however, there are no oscillations in the solutions of 2-PFV scheme.

Then, the case with $M = 2, Re = 3000$ is computed by using 2CFV, fourth- and sixth-order perturbational finite (4-PFV and 6-PFV) volume schemes. Figs.3 and 4 given the comparisons of the pressure and velocity.

Figs.3 and 4 show that the 2-CFV scheme emergences the more oscillatory solutions than the Fig.1, when Re from 800 to 3000, especially on the downstream of shock-wave. The 4-PFV and 6-PFV can remain the property of essentially non-oscillatory. The 6-PFV scheme has higher resolution of shock wave than the 4-PFV does.

Figs.5 and 6 given the numerical results of 2-CFV and 6-PFV schemes with $Re = 5000$.

In Fig.5, there are obvious oscillations in whole computational domain in the contributions of density ρ, velocity u, pressure p and temperature T. Fig.6 shows that 6-PFV scheme is an essentially non-oscillatory scheme, even through the grid Reynolds number is about 30.

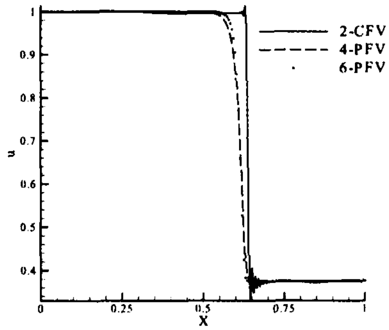

Fig. 3. Comparison of Velocity, Re=3000

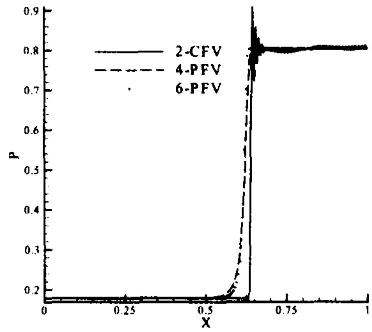

Fig. 4. Comparison of pressure, Re=3000

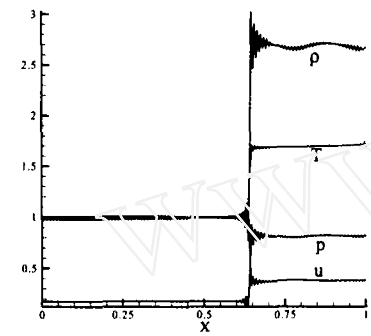

Fig. 5. 2-CFV, Re=5000

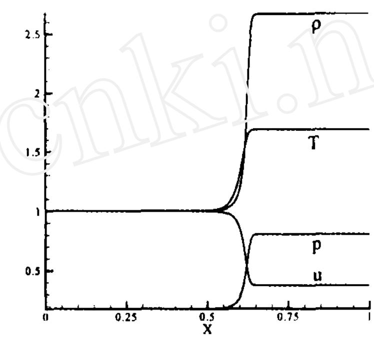

Fig. 6. 6-PFV, Re=5000

4 Conclusions

Starting from the second-order finite volume scheme, though numerical value perturbation of the cell facial fluxes, the perturbational finite volume (PFV) scheme of the Navier-Stokes (NS) equations for compressible flow is developed. The formulation of the PFV scheme is consistent with that of the second-order central finite volume (2-CFV). Numerical results of the one-dimensional Navier-Stokes equation show that the PFV scheme can obtain essentially non-oscillatory solution even much larger cell Reynolds number, thus it has a wider applicable range of Reynolds number than that of the second order finite volume scheme. Our next work is to use the PFV scheme computing 2-D and 3-D Navier-Stokes equations and to solve practical engineering flow problems.

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