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Analysis of Damage Moments in the Collective Evolution of Short Fatigue Cracks

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ABSTRACT

The concept of equilibrium in crack numerical density is applied to analyze the process of collective damage evolution for short fatigue cracks. The damage moments with respect to crack numerical density and crack length are adopted to evaluate the damage extent of dispersive short cracks. The analytical results show that the crack growth rate is in relation to the "zero-th" order of damage moment D_0, and the "first" order of damage moment D_1 increases exponentially with fatigue cycles whereas $\dot{D}_1$ is linearly proportional to D_1.

1. INTRODUCTION

It has been noticed that the formation and the development of short fatigue cracks in some metallic materials present collective evolution characteristics, namely that the damage cumulation due to surface short cracks in the primary stage of fatigue failure is dominated by the initiation and propagation of a large number of dispersed short cracks [1-6]. The most essential character of the collective damage is that the number of short cracks gradually increases with an increasing number of fatigue cycles [1-6].

To deal with such a problem, a few investigators [5-7] carried out statistical simulations and illustrated graphical displays of short-crack development so as to describe the collective evolution of short cracks. In addition, the concept based on the balance of crack-numerical-density was invoked to analyze the behaviour of the collective damage [8], which seemed to be an applicable approach to account for the collective behaviour.

At the primary stage of fatigue damage in some materials, short cracks may initiate sporadically and the interaction between cracks is negligible. Thus, the equilibrium equation based on the balance of crack-numerical-density within the concerned phase-space is [8] :

$$\frac{\partial}{\partial t} n(c,t) + \frac{\partial}{\partial c}[A(c) \cdot n(c,t)] = N_g \cdot n_N(c) \ , \tag{1}$$

where A is the crack growth rate, n_N is the crack nucleation rate, and n is the crack numerical density which is such defined that at time t, the number of cracks with sizes between c and $c + dc$ within a unit volume is $n dc$. The above equation is in the form of a non-dimensional expression, and the non-dimensional coefficient $N_g = (n_N^* \cdot d)/(n^* \cdot A^*)$, with n_N^* being the characteristic crack nucleation rate, A^* the characteristic crack growth rate, n^* the characteristic crack numerical density and d the characteristic microstructural dimension of the material concerned (e.g. grain diameter). Eq. 1 is established based on the consideration that the total number of short cracks is produced by both the crack nucleation and the crack growth.

If the initial condition for Eq. 1 is that $n(c,0)=0$ and that the initial crack length before growth is zero, then the theoretical solution of Eq. 1 is [9] :

$$n(c,t) = \frac{1}{A(c)} \int_{\eta(c,t)}^{c} N_g \cdot n_N(c')\, dc' \ . \tag{2}$$

The implication of $\eta(c,t)$ is that the crack with its length of η at $t=0$, will grow to length c at time t under the growth rate of $A(c)$.

In order to describe the damage extent of dispersive short cracks, the damage moments may serve as promising parameters. A series of damage moments are generally defined:

$$D_m = \gamma \int_0^\infty n(c,t) \cdot c^m \, dc \ , \tag{3}$$

where γ is a non-dimensional coefficient and $m = 0, 1, 2, \cdots$. It is evident that D_0, the "zero-th" order of damage moment, is in relation to the total number of cracks and D_1, the "first" order of damage moment, corresponds to the total crack lengths. In the following, the characteristics of damage moments in the collective evolution of short fatigue cracks are theoretically analyzed. The relation between crack growth rate A and D_0 is inferred, and the variation of D_0 and D_1 with respect to time t is evaluated.

2. CORRELATION BETWEEN CRACK GROWTH RATE AND DAMAGE MOMENTS

In Eqs. 1 and 2, the crack growth rate A is only regarded as a function of crack length c, i.e. A is independent of time t or the progression of fatigue cycling. However, at different stages of fatigue damage, A may vary not only with c but also with t.

Consider that A is a function of c and t with the two variables acting independently:

$$A(c,t) = G(c) \cdot T(t) \ , \tag{4}$$

where the value of $T(t)$ is always positive and $\int_0^\infty T(t) dt$ is not rational in the physical sense. Substituting Eq. 4 into Eq. 1, one may re-construct the equilibrium equation as,

$$\frac{\partial}{\partial t} n(c,t) + T(t) \frac{\partial}{\partial c} [G(c) \cdot n(c,t)] = N_g \cdot n_N(c) \ . \tag{5}$$

Assuming that the particular solution of Eq. 5 is $n_0(c,t)$, then one may write,

$$\tilde{n}(c,t) = n(c,t) - n_0(c,t) \ . \tag{6}$$

Introducing Eq. 6 into Eq. 5, one can show,

$$\frac{\partial}{\partial t} \tilde{n}(c,t) + T(t) \frac{\partial}{\partial c} [G(c) \cdot \tilde{n}(c,t)] = 0 \ . \tag{7}$$

Since the effects of c and t on the crack evolution are assumed to be acting separately, thus,

$$G(c) \cdot \tilde{n}(c,t) = P(c) \cdot Q(t) \ . \tag{8}$$

Introducing Eq. 8 into Eq. 7, one obtains,

$$Q(t) = \alpha_1 \cdot \exp\left\{-ik \int_0^t T(\tau)\, d\tau\right\} \ , \quad \text{and} \quad P(c) = \alpha_2 \cdot \exp\left\{ik \int_0^c \frac{dc'}{G(c')}\right\} \ , \tag{9}$$

where α_1 and α_2 are constants, and k is a coefficient related to the solution. Hence, the solution to the equilibrium equation is

$$n(c,t) = n_0(c,t) + \tilde{n}(c,t) = n_0(c,t) + \int_0^\infty \frac{F(k)}{G(c)} \exp[it \cdot \Psi(k)]\,\mathrm{d}k \ , \tag{10}$$

where $F(k)$, a function of α_1 and α_2, is determined by the boundary and the initial conditions [assuming that $F(k)$ is always integrable], $T(k)$, corresponding to k, is the portion of crack-growth-rate relevant to time t, and

$$\Psi(k) = \frac{k}{t}\left[\int_0^c \frac{\mathrm{d}c'}{G(c')} - \int_0^t T_k(\tau)\,\mathrm{d}\tau\right] \ .$$

Consider that $T_k(t)$ is independent of k, and t tends to infinity. Making use of the Riemann-Lobergue lemma [10], one sees that,

$$\int_0^\infty \frac{F(k)}{G(c)} \exp[it \cdot \Psi(k)]\,\mathrm{d}k \to 0 \ . \tag{11}$$

Thus, the solution (Eq. 10) tends to $n_0(c,t)$. If $T(t)|_{t\to\infty} = T_0$ (a bounded parameter), referring to Eq. 2, one finds,

$$n(c,t) \to n_0(c,t) \to \overline{n}_0(c) = \frac{1}{T_0 \cdot G(c)} \int_0^c N_\mathrm{g} \cdot n_\mathrm{N}(c')\,\mathrm{d}c' \ . \tag{12}$$

The physical implication of the irrelevance between $T_k(t)$ and k is that the crack growth rate A is independent of the **distribution pattern** of crack numerical density $n(c,t)$, although A depends on $n(c,t)$. Note that $D_0(t)$ is the total number of cracks:

$$D_0(t) = \int_0^\infty n(c,t)\,\mathrm{d}c = t \cdot \int_0^\infty n_\mathrm{N}(c)\,\mathrm{d}c \ . \tag{13}$$

Obviously, $T_k(t) = T_k(D_0)$ is in accordance with T_k being irrelevant to k. In this case,

$$A = A[c, D_0(t)] = G(c) \cdot T[D_0(t)] \ . \tag{14}$$

On the other hand, consider that A is relevant to the **distribution pattern** of $n(c,t)$, i.e. T_k depends on k. If there is a series of k_j which enable $\Psi(k_j) \neq 0$, $\Psi'(k_j) = 0$ and $\Psi''(k_j) \neq 0$, then let $\theta_j = (\pi/4)\mathrm{sgn}\Psi''(k_j)$. When t tends to infinity, according to the stationary-phase-theorem [10], one derives,

$$n(c,t) \to \overline{n}_0(c) + \frac{1}{G(c)} \sum_{j=1}^{N} \left\{ F(k_j)\cdot\tilde{T}_j(t)\cdot\exp\left[i\left(k_j \int_0^c \frac{\mathrm{d}c'}{G(c')} - k_j \int_0^t T_{k_j}(\tau)\,\mathrm{d}\tau + \theta_j\right)\right]\right\} \ , \tag{15}$$

where k_j $(j = 1, 2, \cdots N)$ are stationary-phase-points, and

$$\tilde{T}_j(t) = \sqrt{\pi}\left|4\int_0^t \left(\frac{\partial T_k(\tau)}{\partial k}\right)_{k_j} \mathrm{d}\tau - 2k_j \int_0^t \left(\frac{\partial^2 T_k(\tau)}{\partial k^2}\right)_{k_j} \mathrm{d}\tau\right|^{-\frac{1}{2}} \ .$$

Eq. 15 is the solution for the case when T_k is dependent on k, which is equivalent to the case when A is correlated with D_m $(m \geq 1)$. The analysis of Eq. 15 suggests that $n(c,t)$ will vary with t in an oscillating pattern of distribution.

Our previous investigation [8] indicated that, crack numerical density $n(c,t)$ increased with time t (fatigue cycles) monotonically and tended to a stable distribution. For such a situation, therefore, one may argue that the crack growth rate A is dominated by the crack length and the zero-th order of damage moment D_0:

$$A = A[c, D_0(t)] \; . \tag{16}$$

3. RESPONSES OF THE FIRST ORDER OF DAMAGE MOMENT

Similar to Eq. 1, the equilibrium equation with respect to $[c \cdot n(c,t)]$ can be expressed as

$$\frac{\partial}{\partial t}[c \cdot n(c,t)] + \frac{\partial}{\partial c}[A \cdot c \cdot n(c,t)] = N_g \cdot c \cdot n_N(c) + A \cdot n(c,t) \; . \tag{17}$$

By taking into account the propensity of short-crack behaviour, the following expressions for A and n_N are proposed, which are required in the derivation of the solution for D_1:

$$A(c) = \begin{cases} 1 - (1 - A_d) \cdot c & (c \le 1) \\ \bar{d} \cdot c & (c > 1) \end{cases} \tag{18}$$

and

$$n_N(c) = \begin{cases} 1 - \dfrac{c}{2} & (c \le 2) \\ 0 & (c > 2) \end{cases} \tag{19}$$

where A_d is the crack growth rate at $c = 1$, and $\bar{d}$ is the ratio of non-dimensional average grain diameter d to $A(0)$. Since c is normalized by d, thus A_d is the crack velocity at the time the crack tip reaches the first grain boundary.

Regarding the definition of D_1 and the balance of the crack system, one may have,

$$D_1 = \gamma \int_0^\infty n(c,t) \cdot c \, \mathrm{d}c = \gamma \left(\tilde{\alpha} \cdot t + D_1^*\right) \; , \tag{20}$$

where

$$\tilde{\alpha} = \int_0^\infty N_g \cdot c \cdot n_N(c) \, \mathrm{d}c \; , \tag{21}$$

and

$$D_1^* = \int_0^t \mathrm{d}t' \int_0^1 [A - \bar{d} \cdot c] \cdot n(c,t') \, \mathrm{d}c + \frac{\bar{d}}{\gamma} \int_0^t D_1(t') \, \mathrm{d}t' \; . \tag{22}$$

Eq. 20 can be rearranged as

$$D_1 - \bar{d} \int_0^t D_1(t') \, \mathrm{d}t' = \gamma \left[\tilde{\alpha} \cdot t + \Theta(t)\right] \; , \tag{23}$$

with

$$\Theta(t) = \int_0^t \mathrm{d}t' \int_0^1 [A - \bar{d} \cdot c] \, n(c,t') \, \mathrm{d}c \; . \tag{24}$$

Let $\bar{A}_d = 1 - A_d$, $\chi = \bar{A}_d + \bar{d}$, $\beta_1 = \int_0^1 N_g \cdot n_N(c) \mathrm{d}c$ and $\beta_2 = \int_0^1 N_g \cdot c \cdot n_N(c) \mathrm{d}c$. Then, introducing Eqs. 2 and 18 into Eq. 24, one obtains

$$\Theta(t) = \int_0^t \mathrm{d}t' \int_0^1 n(c,t') \, \mathrm{d}c - \chi \int_0^t \mathrm{d}t' \int_0^1 c \cdot n(c,t') \, \mathrm{d}c \; , \tag{25}$$

in which

$$\int_0^1 n(c,t') \, \mathrm{d}c = \beta_1 \cdot t' - \int_0^{t'} A_d \cdot n(1,\tau) \, \mathrm{d}\tau \; , \tag{26}$$

and

$$\int_0^1 c\cdot n(c,t')\mathrm{d}c = \beta_2\cdot t' + \int_0^1 A\cdot n(c,t')\,\mathrm{d}c - \int_0^{t'} A_\mathrm{d}\cdot n(1,\tau)\,\mathrm{d}\tau\ . \tag{27}$$

Eqs. 26 and 27 were developed such that the balance of $n(c,t)$ for cracks with lengths between 0 and 1 was considered. After the derivation of the integrals from Eq. 25 through Eq. 27, the following result is achieved,

$$\Theta(t) = \begin{cases} \xi_1^* + \xi_2^*\cdot t + \xi_3^*\cdot t^2 + \xi_4^*\cdot\exp(\bar{A}_\mathrm{d}\cdot t) + \xi_5^*\cdot\exp(2\bar{A}_\mathrm{d}\cdot t) & (t\le t_0) \\ \xi_6\cdot t^2 & (t>t_0)\end{cases} \tag{28}$$

where

$$\xi_1^* = -\frac{A_\mathrm{d}\cdot q\cdot N_\mathrm{g}\left[8\,(2\bar{A}_\mathrm{d}-1)+A_\mathrm{d}\right]}{16\bar{A}_\mathrm{d}^4}\ , \qquad \xi_2^* = \frac{A_\mathrm{d}\cdot q\cdot N_\mathrm{g}\left[4\,(2\bar{A}_\mathrm{d}-1)+A_\mathrm{d}\right]}{8\bar{A}_\mathrm{d}^3}\ ,$$

$$\xi_3^* = \frac{(1-\bar{d})\,\beta_1-\gamma\cdot\beta_2}{2(1+\bar{A}_\mathrm{d})} - \frac{q\cdot N_\mathrm{g}\,(3\bar{A}_\mathrm{d}^2-4\bar{A}_\mathrm{d}+1)}{8\bar{A}_\mathrm{d}^2}\ , \qquad \xi_4^* = -\frac{A_\mathrm{d}\cdot q\cdot N_\mathrm{g}\,(2\bar{A}_\mathrm{d}-1)}{2\bar{A}_\mathrm{d}^4}\ ,$$

$$\xi_5^* = -\frac{A_\mathrm{d}^2\cdot q\cdot N_\mathrm{g}}{16\bar{A}_\mathrm{d}^4}\ , \qquad \xi_6 = \frac{(1-\bar{d})\,\beta_1-\gamma\cdot\beta_2}{2\,(1+\bar{A}_\mathrm{d})} - \frac{3q\cdot N_\mathrm{g}}{8}\ , \quad \text{and} \quad q = \frac{1-\gamma-\bar{d}}{1+\bar{A}_\mathrm{d}}\ .$$

Using the initial condition of $D_1(0) = 0$ and substituting Eq. 28 into Eq. 23, one finally obtains the solution for $D_1(t)$:

$$D_1(t) = \begin{cases} \left[\,\xi_1+\xi_2\cdot t+\xi_3\cdot\exp(\bar{A}_\mathrm{d}\cdot t)+\xi_4\cdot\exp(2\bar{A}_\mathrm{d}\cdot t)+\xi_5\cdot\exp(\bar{d}\cdot t)\right]\cdot\gamma & (t\le t_0) \\ -\dfrac{2\gamma\cdot t\cdot\xi_6}{\bar{d}} + \dfrac{\gamma\cdot\left[\exp(\bar{d}\cdot t)-1\right]\left(2\xi_6+\tilde{\alpha}\cdot\bar{d}\right)}{\bar{d}^2} & (t>t_0)\end{cases} \tag{29}$$

where

$$\xi_1 = -\frac{\xi_2^*+\tilde{\alpha}}{\bar{d}} - \frac{2\xi_3^*}{\bar{d}^2}\ , \qquad \xi_2 = -\frac{2\xi_3^*}{\bar{d}}\ , \qquad \xi_3 = \xi_4^*\left(1+\frac{\bar{A}_\mathrm{d}}{\bar{A}_\mathrm{d}-\bar{d}}\right)\ ,$$

$$\xi_4 = \frac{2\xi_5^*\cdot\bar{A}_\mathrm{d}}{2\bar{A}_\mathrm{d}-\bar{d}}\ , \qquad \xi_5 = -\left(\frac{\xi_2^*+\tilde{\alpha}}{\bar{d}} + \frac{2\xi_3^*}{\bar{d}^2} + \frac{\xi_4^*\cdot\bar{A}_\mathrm{d}}{\bar{A}_\mathrm{d}-\bar{d}} + \frac{\xi_5^*\cdot\bar{A}_\mathrm{d}}{2\bar{A}_\mathrm{d}-\bar{d}}\right)\ .$$

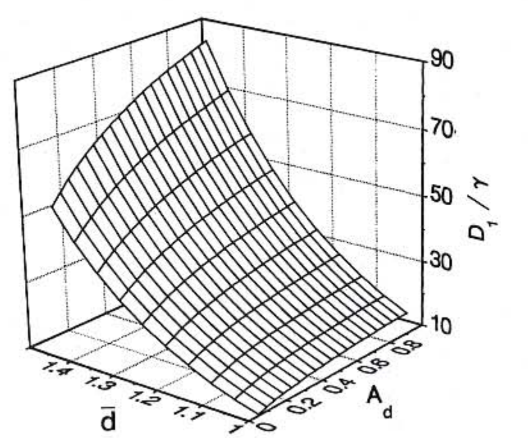

Fig. 1 Variation of D_1/γ with A_d and $\bar{d}$ at $t=3.0$

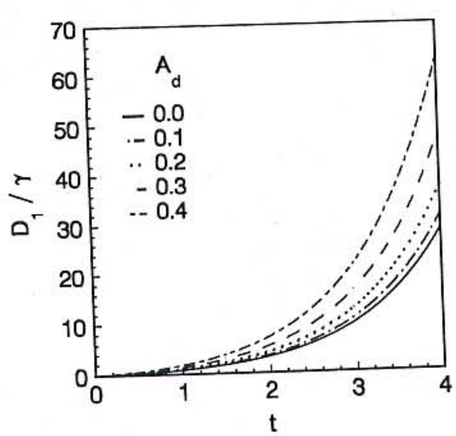

Fig. 2 D_1/γ versus t at $\bar{d}=1$

Fig. 1 indicates that D_1 increases with increasing value of A_d and $\bar{d}$ with the influence of A_d being less sensitive in comparison with that of $\bar{d}$. Fig. 2 shows that D_1 increases sharply with t, in which the curve is approximately of exponential tendency. Upon comparing ξ_6 with $(\tilde{\alpha} \cdot \bar{d})$, it is noted that the former is relatively small. Consequently, when $t > t_0$, Eq. 29 reduces to

$$D_1 \doteq \frac{\tilde{\alpha} \cdot \gamma}{\bar{d}} [\exp(\bar{d} \cdot t) - 1] \ . \tag{30}$$

The above equation is in accordance with the resultant expression for D_1 from a numerical simulation in Reference [8], in which

$$D_1 = \frac{\omega}{\kappa} [\exp(\kappa \cdot t) - 1] \ , \tag{31}$$

with ω and κ being material parameters. Eqs. 30 and 31 suggest that $\dot{D}_1$ and D_1 are linearly correlated, i.e.

$$\dot{D}_1 = \kappa \cdot D_1 + \omega \ . \tag{32}$$

4. CONCLUSIONS

The main conclusions of this study are drawn as follows:

(1) For the case that the crack numerical density converges to a stable distribution as the number of fatigue cycles becomes large enough, the crack growth rate is predominantly correlated with the zero-th order of damage moment and with the crack length.

(2) The first order of damage moment becomes larger with increasing values of grain size and with escalating obstacle effect of grain boundary against crack growth.

(3) The zero-th order of damage moment increases linearly with an increasing number of fatigue cycles. The first order of damage moment enlarges exponentially with an increasing number of fatigue cycles, whereas $\dot{D}_1$ and D_1 are linearly interrelated, when the number of fatigue cycles is beyond a critical value.

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