Thermocapillary Flow in a Jet of Liquid Film Painted on a Moving Boundary

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In the present paper, a liquid (or melt) film of relatively higher temperature ejected from a vessel and painted on the moving solid boundary is analyzed. The thermocapillary flow is driven by the temperature gradient on the free surface of a liquid film, because of the heat transfer from the liquid with higher temperature to the environmental gas with relatively lower temperature. The thermocapillary flow changes the height profile of the liquid film. The analysis is based on the approximations of lubrication theory and perturbation theory, and the equation of liquid height and the process of thermal hydrodynamics in the liquid film are solved for a given temperature distribution on the solid boundary.

1. Introduction

The processing of film and polymer requires the understanding of the hydrodynamic process, such as the temperature, pressure, flow field, and diameter or height distributions; see, for example, refs 1−3. The temperature of the liquid or the melt, ejected from the nozzle of a manufacture vessel, is relatively higher in comparison with the environmental gas temperature, and then, there is a strong heat exchange, especially in the region near the exit of vessel. The heat transfer from the liquid film to the environmental gas forms a temperature gradient and then the surface tension gradient on the free surface and induces thermocapillary flow, which will, of course, change the cross section of the jet liquid. The Barus effect or Die Swell effect on the change of cross section in a polymer jet is often observed in the processing and is explained usually by the rheology property of the polymer melt. However, there are many suggestions on the viscoelastic constitutive relationships and the rheology properties of the polymer, and suitable and acceptable descriptions on the relationships and properties are still open problems that need to be studied in the future.[4]

It is known that thermocapillary flow is induced in a thin liquid layer which lays on a solid wall with nonuniform temperature distribution and the heat transfer process will result the nonuniformity of the liquid thickness. By using the approximation of lubrication theory, an ordinary differential equation of liquid thickness and its solution for given boundary temperature were obtained for unsteady cases in a thin liquid layer.[5] A similar method was applied to discuss the steady cases where a smooth condition at the symmetric cross section was improved.[6]
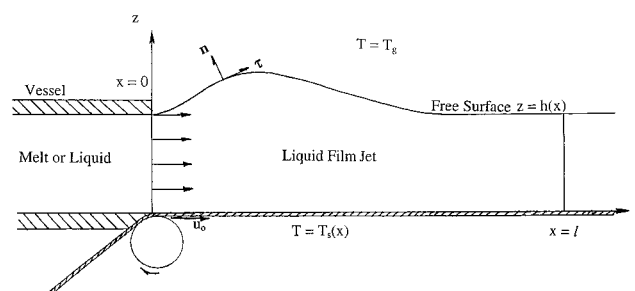

Figure 1. Schematic diagram of the physical model of a jet liquid film.

The problem of motionless and infinitely extending solid boundary was studied in refs 5 and 6. Furthermore, the solutal capillary flow induced by the surfactant of the liquid may increase the thickness of the liquid layer.[7]

In the present paper, the change of cross section due to the thermocapillary convection in a two-dimensional and steady model is discussed for the case of jet liquid film painting on a moving solid boundary, and the similar approximation of lubrication theory is assumed. In the case of a small geometrical aspect ratio and small capillary number, the perturbation method is applied. The first-order solutions of the problem can be obtained analytically, and a fourth-order ordinary differential equation of the liquid layer height can be demonstrated. Four boundary conditions are given to solve the equation of liquid height, and the solution clearly shows the effect of thermocapillary convection on the thickness variation of the liquid layer. The conclusion of this paper suggests that a large enlargement of the cross section, which is usually explained by the rheology property of the liquid, may be due to the thermocapillary effect in the liquid jet of a Newtonian fluid.

2. Model of Jet Liquid Film

A simplified model is suggested as shown in Figure 1, where the liquid layer with the height h_0 is ejected from a nozzle of a melt or liquid vessel and then attached to a

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moving solid film. This model avoids the complexity in detailed proceedings and emphasizes the influence of the thermocapillary convection on the height variation. The Cartesian coordinate system (x, y, z) is adopted, the problem is assumed two-dimensional, and $\partial/\partial y = 0$. The melt or liquid flows in a distance l far from the exit of the vessel, and there is a small geometrical aspect ratio $\epsilon = h_0/l \ll 1$. The liquid temperature T_e at the vessel exit is higher than the environmental gas temperature T_g and the temperature T_l at $x = l$. A moving solid film is touched smoothly with the lower boundary of the vessel and moves with the same velocity u_0 of the liquid at the lower boundary of the vessel exit. This assumption idealizes the model to avoid mathematical complexity at the contact and is beneficial to analyze the thermocapillary effects. The heat transfer between moving solid film, liquid jet, and environmental gas results in a large temperature gradient on the free surface, especially near the exit of the vessel, and then, the thermocapillary flow will be driven in addition to the jet flow.

The liquid is assumed to be incompressible and Newtonian with constant kinematics viscosity ν and thermal diffusivity κ. The relationships of the mass conservation, the momentum conservation, and the energy conservation will be given as

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{2.1}$$

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right) \tag{2.2}$$

$$u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2}\right) \tag{2.3}$$

$$u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) \tag{2.4}$$

where $(u, 0, w)$ is the velocity vector and ρ, p, and T are respectively the density, pressure, and temperature of the liquid.

The boundary condition of the liquid layer may be summarized as follows:

$z = 0$:

$$u = u_0 \quad w = 0 \quad T = T_s(x) \tag{2.5}$$

$z = h(x)$:

$$u\frac{\mathrm{d}h}{\mathrm{d}x} = w \tag{2.6}$$

$$\frac{\rho\nu}{(1 + h'^2)}\left[(1 - h'^2)\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + 2h'\left(\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}\right)\right] =$$
$$\sigma_T'\left(\frac{\partial T}{\partial x} + h'\frac{\partial T}{\partial z}\right) \tag{2.7}$$

$$p = \frac{2\rho\nu}{1 + h'^2}\left[\left(\frac{\partial w}{\partial z} - h'\frac{\partial u}{\partial x}\right) - h'\left(\frac{\partial w}{\partial x} + h'\frac{\partial u}{\partial x}\right)\right] -$$
$$[\sigma_0 + \sigma_T'(T - T_*)]\frac{h''}{(1 + h'^2)^{3/2}} \tag{2.8}$$

$$k\frac{\partial T}{\partial n} = -H(T - T_g) \tag{2.9}$$

$x = 0$:

$$u = u_e(z) \quad w = w_e(z) \quad T = T_e(z) \tag{2.10}$$

Here the superscript prime denotes the differential such as $h' = \mathrm{d}h/\mathrm{d}x$, $\mathbf{n}$ is the unit vector in the normal direction, T_* is a constant reference temperature, and k and H are respectively the thermal conductivity of liquid and the heat transfer coefficient in the gas. Conditions (2.5)−(2.8) describe respectively the free surface $z = h(x)$ as a stream surface, the viscous stress in the tangent direction balancing with the thermocapillary force driven by the surface tension gradient, the momentum conservation in the normal direction, and the heat transfer across the free surface where the radiation effect is omitted. In addition to boundary conditions (2.4)−(2.9), the boundary conditions for the height of liquid layer should be given. And the problem of jet liquid film may, then, be solved.

Nondimensional quantities and parameters may be introduced on the basis of lubrication theory as follow:[5]

$$\xi = \frac{x}{l} \quad \zeta = \frac{z}{h_0} \quad \eta = \frac{h}{h_0} \quad \epsilon = \frac{h_0}{l}$$

$$U = \frac{u}{v_*} \quad \epsilon W = \frac{w}{v_*} \quad \epsilon^2 P = \frac{pl}{\mu v_*} \quad \Theta = \frac{T}{T_*} \tag{2.11}$$

Here the typical temperature T_* and the typical velocity v_* are adopted respectively as

$$T_* = T_s(0) \quad v_* = -\epsilon\sigma_T'T_*/\rho\nu \tag{2.12}$$

The basic feature of the lubrication problem is that there are two typical scales of different orders of magnitude; that is, one of the typical scales l is much larger than the another h_0, and then, other quantities have different orders of magnitude. The nondimensional parameters in this case are the Reynolds number Re and Peclet number Pe:

$$Re = \frac{v_*l}{\nu} \quad Pe = \frac{v_*l}{\kappa} \tag{2.13}$$

The Prandtl number and Marangoni number may be given as

$$Re = PrMa \quad Ma = -\sigma_T'T_*l/\kappa\nu = \epsilon Pe$$

Nondimensional equations and boundary conditions may then be written as

$$\frac{\partial U}{\partial \xi} + \frac{\partial W}{\partial \zeta} = 0 \tag{2.14}$$

$$Re\epsilon^2\left(U\frac{\partial U}{\partial \xi} + W\frac{\partial U}{\partial \zeta}\right) = -\frac{\partial P}{\partial \xi} + \frac{\partial^2 U}{\partial \zeta^2} + \epsilon^2\frac{\partial^2 U}{\partial \xi^2} \tag{2.15}$$

$$Re\epsilon^4\left(U\frac{\partial W}{\partial \xi} + W\frac{\partial W}{\partial \zeta}\right) = -\frac{\partial P}{\partial \zeta} + \epsilon^2\frac{\partial^2 W}{\partial \zeta^2} + \epsilon^4\frac{\partial^2 W}{\partial \xi^2} \tag{2.16}$$

$$Pe\epsilon^2\left(U\frac{\partial \Theta}{\partial \xi} + W\frac{\partial \Theta}{\partial \zeta}\right) = \frac{\partial^2 \Theta}{\partial \zeta^2} + \epsilon^2\frac{\partial^2 \Theta}{\partial \xi^2} \tag{2.17}$$

and the boundary conditions are listed as follows:

$\zeta = 0$:

$$U = U_o(\text{const}) \quad W = 0 \quad \Theta = \Theta_s(\xi) \quad (2.18)$$

$\zeta = \eta(\xi)$:

$$\eta(0) = 1$$

$$W(\xi, \eta) = \eta' U(\xi, \eta)$$

$$\frac{\partial U}{\partial \zeta} + \epsilon^2 \left[\left(\frac{\partial W}{\partial \xi} - \eta'^2 \frac{\partial U}{\partial \zeta} \right) + 2\eta' \left(\frac{\partial W}{\partial \zeta} - \frac{\partial U}{\partial \xi} \right) - \epsilon^2 \eta' \frac{\partial W}{\partial \xi} \right] = \sqrt{1 + \epsilon^2 \eta'^2} \left(\frac{\partial \Theta}{\partial \xi} + \eta' \frac{\partial \Theta}{\partial \zeta} \right)$$

$$P = -\frac{\epsilon^2}{C} \frac{\eta''}{(1 + \epsilon^2 \eta'^2)^{3/2}} + \frac{2\epsilon^2}{1 + \epsilon^2 \eta'^2} \left[\left(\frac{\partial W}{\partial \zeta} - \eta' \frac{\partial U}{\partial \zeta} \right) + \epsilon^2 \eta' \left(-\frac{\partial W}{\partial \xi} + \eta' \frac{\partial U}{\partial \xi} \right) \right]$$

$$\frac{\partial \Theta}{\partial \zeta} - \epsilon^2 \eta' \frac{\partial \Theta}{\partial \xi} = -Bi(\Theta - \Theta_g)(1 + \epsilon^2 \eta'^2)^{1/2} \quad (2.19)$$

$\xi = 0$:

$$U = U_e(\zeta) \quad W = W_e(\zeta) \quad \Theta = \Theta_e(\zeta) \quad (2.20)$$

where the Capillary number $C = -\sigma'_T T_*/\sigma_0$ is usually much smaller than unity and the nondimensional Boit number is defined by $Bi = Hh_0/k$.

3. Perturbation Method

The nondimensional equations and boundary conditions show the relationships of the orders of magnitude, and the perturbation method can be applied by the expansion based on the small parameter ϵ. Due to the order of magnitude analysis (OMA), it requires that

$$Re = O(1) \quad Pe = O(1) \quad Bi = O(1) \quad C = \epsilon^2/\alpha = O(\epsilon^2) \quad (3.1)$$

where α is a constant. It is noted that definitions of the Reynolds number and the Peclet number in (2.13) are ϵ times smaller than the usual definitions, because the typical velocity v_* is ϵ times smaller than the usual typical thermocapillary velocity. The quantities are expanded as follows:

$$U = \sum_{n=0}^{\infty} \epsilon^n U^{(n)} \quad W = \sum_{n=0}^{\infty} \epsilon^n W^{(n)} \quad \Theta = \sum_{n=0}^{\infty} \epsilon^n \Theta^{(n)}$$

$$\eta = \sum_{n=0}^{\infty} \epsilon^n \eta^{(n)} \quad (3.2)$$

If one substitutes relations (3.2) into the equations and boundary conditions, the problem can be solved order by order.

The zero's order relationships can be written as

$$\frac{\partial U^{(0)}}{\partial \xi} + \frac{\partial W^{(0)}}{\partial \zeta} = 0 \quad (3.3)$$

$$\frac{\partial P^{(0)}}{\partial \xi} = \frac{\partial^2 U^{(0)}}{\partial \zeta^2} \quad (3.4)$$

$$\frac{\partial P^{(0)}}{\partial \zeta} = 0 \quad (3.5)$$

$$\frac{\partial^2 \Theta^{(0)}}{\partial \zeta^2} = 0 \quad (3.6)$$

and the boundary conditions are

$\zeta = 0$:

$$U^{(0)} = U_o \quad W^{(0)} = 0 \quad \Theta^{(0)} = \Theta_s(\xi) \quad (3.7)$$

$\zeta = \eta^{(0)}$:

$$W^{(0)} = U^{(0)} \eta^{(0)\prime} \quad (3.8)$$

$$\frac{\partial U^{(0)}}{\partial \zeta} = -\frac{\partial \Theta^{(0)}}{\partial \xi} - \frac{d\eta^{(0)}}{d\xi} \frac{\partial \Theta^{(0)}}{\partial \zeta} \quad (3.9)$$

$$P^{(0)} = -\alpha \frac{d^2 \eta^{(0)}}{d\xi^2} \quad (3.10)$$

$$\frac{\partial \Theta^{(0)}}{\partial \zeta} = -Bi(\Theta^{(0)} - \Theta_g) \quad (3.11)$$

$\xi = 0$:

$$U^{(0)} = U_e(\zeta) \quad W^{(0)} = W_e(\zeta) \quad \Theta^{(0)} = \Theta_e(\zeta) \quad \eta^{(0)} = 1 \quad (3.12)$$

$\xi = 1$:

$$\eta^{(0)} = \eta_m \quad (3.13)$$

where η_m is the height at the cross section $z = l$ and is adopted typically as $\eta_m = 1$. Basic equations (3.3)−(3.6) of the zero's order of magnitude are similar to the ones given in refs 4 and 5.

Equation (3.6) means that the temperature is a linear function of ζ and may be obtained as follows by using boundary condition (3.11):

$$\Theta^{(0)}(\xi, \zeta) = \Theta_s - \frac{Bi(\Theta_s - \Theta_g)}{1 + Bi\eta^{(0)}} \zeta \quad (3.14)$$

Relationship (3.14) shows that the heat transfers from the solid of temperature Θ_s to the free surface of temperature $\Theta^{(0)}(\xi, \eta^{(0)})$ and then to the gas of temperature Θ_g. The temperature in the liquid film will be uniform if the solid boundary $\zeta = 0$ keeps a uniform temperature the same as the environmental gas temperature. In this case, there will be no thermocapillary flow in the liquid layer.

Equation (3.5) implies that the pressure is a function of ξ only, and then condition (3.10) at the free surface gives

$$P^{(0)}(\xi) = -\alpha \frac{d^2 \eta^{(0)}}{d\xi^2} \quad (3.15)$$

Substituting (3.15) into equations (3.3) and (3.4) and using free surface condition (3.8), the velocity field in the liquid layer may be written as

$$U^{(0)} = -\frac{\alpha}{2} \frac{d^3 \eta^{(0)}}{d\xi^3} \zeta^2 + A(\xi)\zeta + U_o \quad (3.16)$$

$$W^{(0)} = \frac{\alpha}{6} \frac{d^4 \eta^{(0)}}{d\xi^4} \zeta^3 - \frac{1}{2} \frac{dA}{d\xi} \zeta^2 \quad (3.17)$$

and the function A is

$$A(\xi) = \alpha \eta^{(0)} \frac{d^3 \eta^{(0)}}{d\xi^3} - \frac{d}{d\xi} \left[\Theta_s - \frac{Bi(\Theta_s - \Theta_g)}{1 + Bi\eta^{(0)}} \eta^{(0)} \right] \quad (3.18)$$

Relationship (3.16) shows that longitude velocity consists of a main jet velocity U_o in addition to the contribution

of the cross-section variation and the thermocapillary effect. By using solutions (3.16) and (3.17), a stream function $\Psi(\xi, \zeta)$ may be easily obtained as

$$\Psi(\xi,\zeta) = \frac{\alpha}{6}\frac{d^3\eta^{(0)}}{d\xi^3}(3\eta^{(0)} - \zeta)\zeta^2 -$$

$$\frac{1}{2}\frac{d}{d\xi}\left[\Theta_s - \frac{Bi(\Theta_s - \Theta_g)}{1 + Bi\eta^{(0)}}\eta^{(0)}\right]\zeta^2 + U_o\zeta + \Psi_o \quad (3.19)$$

where Ψ_o is an integral constant and may be selected as zero. Solutions (3.14)−(3.17) depend on the height of the liquid layer, which equation is determined by boundary condition (3.9).

The velocity and temperature boundary conditions in (3.12) and (3.13) cannot be given arbitrarily, and must satisfy the distributions of general solutions (3.14), (3.16), and (3.17).

4. Height Equation and Its Solution

By substitution of solution (3.14), (3.16), and (3.17) into boundary condition (3.9) at the free surface, the height equation of the liquid layer is demonstrated as

$$\frac{d}{d\xi}\left(\frac{\alpha}{6}\eta^{(0)3}\frac{d^3\eta^{(0)}}{d\xi^3} - \frac{A(\xi)}{2}\eta^{(0)2} - U_o\eta^{(0)}\right) = 0 \quad (4.1)$$

The height equation is a fourth-order ordinary differential equation, and its solution needs four boundary conditions on the height. The initial integration of eq (4.1) gives a third-order ordinary differential equation including an integral constant.

$$\frac{d^3\eta^{(0)}}{d\xi^3} = \frac{3}{2\alpha\eta^{(0)}(1 + Bi\eta^{(0)})}\left[\frac{d\Theta_s}{d\xi} - \frac{Bi(\Theta_s - \Theta_g)}{1 + Bi\eta^{(0)}}\frac{d\eta^{(0)}}{d\xi}\right] -$$
$$\frac{3U_o}{\alpha}\frac{1}{\eta^{(0)2}} + \frac{c_1}{\eta^{(0)3}} \quad (4.2)$$

where integral constant c_1 may be given by the boundary condition at $\xi = 0$:

$$c_1 = \frac{3U_o}{\alpha} + \frac{d^3\eta(0)}{d\xi^3} -$$
$$\frac{3}{2\alpha(1 + Bi)}\left[\frac{d\Theta_s(0)}{d\xi} - \frac{Bi(1 - \Theta_g)}{1 + Bi}\frac{d\eta^{(0)}(0)}{d\xi}\right], \quad (4.3)$$

Here the boundary conditions $\eta^{(0)}(0) = 1$ and $\Theta_s(0) = 1$ are used.

Four boundary conditions of the liquid layer height at the exit of the melt vessel $\xi = 0$ and at the longitudinal exit cross section $\xi = 1$ may be selected by different options and then give different distributions of the height. In the present paper, the boundary conditions are given as follows:

$$\eta^{(0)}(0) = 1 \quad \frac{d\eta^{(0)}(0)}{d\xi} = \gamma \quad \frac{d^2\eta^{(0)}(0)}{d\xi^2} = \beta \quad \eta^{(0)}(1) = 1$$
$$(4.5)$$

Here γ is the expansion angle of the jet and β is the inclination variation of the expanding angle at the exit of melt vessel. Different selections of (γ, β) on the variation of height are analyzed for understanding the main feature of the thermocapillary effect in the jet liquid film. The first and last conditions in (4.5) require that the cross sections are the same at the exit of melt vessel $\xi = 0$ and
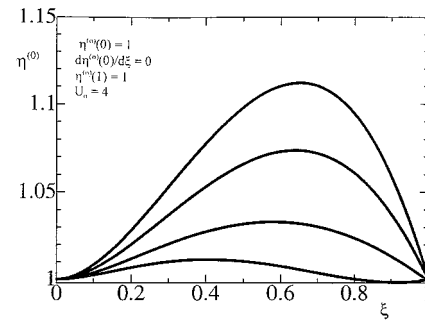

Figure 2. Influence of the expending angle gradients on the height profiles of the jet liquid layer for cases $\eta^{(0)}(0) = 1$, $d\eta^{(0)}(0)/d\xi = 0$, $\eta^{(0)}(1) = 1$, and $U_o = 4$. The profiles relate to $\beta = 0.7$, 1.0, 1.5, and 2.0 respectively from bottom to top.

at the longitudinal exit position $\xi = 1$. The last condition in (4.5) is reasonable, and relates to a specified value of $d^3\eta(0)/d\xi^3$. This condition has no limitation in discussing the general features and may be easily changed to a given value, which may be designed by the engineering consideration.

The temperature $\Theta_s(\xi)$ on the solid boundary $\zeta = 0$ should be obtained by analyzing the heat transfer process between liquid layer, solid film, and environmental gas, and this process is coupled with the process in the liquid layer. For simplification, the temperature at the solid boundary will be given, for example, as follows:

$$\Theta_s = 1 - (1 - \Theta_l)\frac{1 - \exp(-n\xi)}{1 - \exp(-n)} \quad (4.6)$$

Here $1/n$ denotes a typical scale of heat transfer. The heat transfer and the temperature decrease are relatively large in the region near the vessel exit and then become slower afterward if the melt temperature T_o at the vessel exit is larger than the environmental gas temperature T_g and the temperature T_l at $x = l$. Distribution (4.6) shows that the temperature on the solid film changes from the exit value T_o at $x = 0$ to the temperature T_l at $x = l$, and the most important change appears in the region $0 \leq x \leq l/n$.

Substituting temperature distribution (4.6) into height eq (4.4), the determined solution of eq (4.4) under boundary conditions (4.5) may then be obtained. Typical parameters in eq (4.4) are adopted as

$$\alpha = 0.5 \quad \Theta_g = 0.1 \quad \Theta_l = 0.1 \quad Bi = 0.5 \quad n = 5 \quad (4.7)$$

and, then, the equation is reduced as

$$\frac{d^3\eta^{(0)}}{d\xi^3} = \frac{c_1}{\eta^{(0)3}} - \frac{6U_o}{\eta^{(0)2}} -$$
$$\left[\frac{ne^{-n\xi}}{1 - e^{-n}} + \frac{1}{2 + \eta^{(0)}}\frac{e^{-n\xi} - e^{-n}}{1 - e^{-n}}\frac{d\eta^{(0)}}{d\xi}\right] \quad (4.8)$$

where the integral constant c_1 is determined by the last condition of (4.5).

Ordinary differential eq (4.8) under the boundary condition and parameter conditions is solved by using the Runge−Kutta method, and the last condition of (4.5) is satisfied by an interaction process. The cases of zero expanding angle $\gamma = 0$ are interesting sometimes. In these cases, the influence of gradient of expanding angle on the height distribution is shown in Figure 2 for $U_o = 4$. The larger the gradient of expanding angle at the vessel exit, the larger the height in the jet film. In the case of the zero
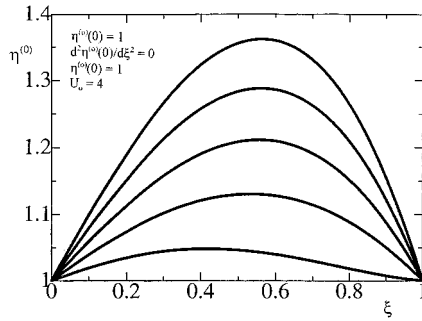
Figure 3. Influence of the expending angles on the height profiles of the jet liquid layer for cases $\eta^{(0)}(0) = 1$, $d^2\eta^{(0)}(0)/d\xi^2 = 0$, $\eta^{(0)}(1) = 1$, and $U_o = 4$. The profiles relate to $\gamma = 0.2$, 0.4, 0.6, 0.8, and 1.0 respectively from bottom to top.
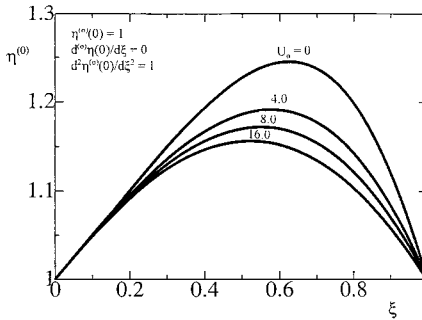

Figure 4. Influence of jet velocities on the height profiles of the jet liquid layer for case $\eta^{(0)}(0) = 1$, $d\eta^{(0)}(0)/d\xi = 0$, $d^2\eta^{(0)}(0)/d\xi^2 = 1.0$, and $\eta^{(0)}(1) = 1$. The profiles relate to $U_o = 0$, 4.0, 8.0, and 16.0 respectively from top to bottom.

gradient of expanding angle $\beta = 0$, the influence of expanding angle γ at the vessel exit is given in Figure 3. The larger the expanding angle γ, the larger the height. The results show that the influence of expanding angle is relatively more direct and important in comparison with the one of the gradient of expanding angle. The influence of jet velocity U_o is summarized in Figure 4 for the case $\gamma = 0$ and $\beta = 1.0$. There is only thermocapillary flow and no jet flow in the special case of zero jet velocity $U_o = 0$, and the liquid film with thermocapillary flow persists on a motionless solid film.

With substitution of boundary temperature distribution (4.6) into the temperature distribution (3.14), the temperature in the jet liquid layer may be demonstrated as

$$\Theta^{(0)}(\xi,\zeta) = 1 - (1 - \Theta_l)\frac{1 - e^{-n\xi}}{1 - e^{-n}} - $$
$$\frac{Bi[(1 - \Theta_g)(1 - e^{-n}) - (1 - \Theta_l)(1 - e^{-n\xi})]}{(1 + Bi\eta^{(0)})(1 - e^{-n})}\zeta \quad (4.9)$$

The first two terms on the right-hand side are independent of ζ, and the third term is a linear function of ζ. On the basis of the solutions of liquid layer height, if

$$\eta^{(0)}(\xi) = 1 + \delta(\xi) \quad (4.10)$$

where δ is a small quantity and satisfies the relation in order of magnitude $\delta = O(\epsilon)$. By using parameters adoptions (4.8) and taking $U = 4$ and $n = 5$, the equation of temperature distribution (4.9) is reduced as

$$\Theta^{(0)}(\xi,\zeta) = (0.093895 + 0.906105e^{-5\xi}) - $$
$$(0.302035e^{-5\xi} - 0.002035)(1 - \delta)\,\zeta + O(\delta^2) \quad (4.11)$$

Relationship (4.11) gives a linear temperature distribution at the vessel exit $\Theta^{(0)}(0,\zeta) = 1 - 0.3\zeta$ and a uniform temperature at the longitudinal exit cross section $\Theta^{(0)}(1,\zeta) = 0.1$. These are the reasonable boundary conditions required by (3.12) and (3.13). The temperature on the free surface is obtained as

$$\Theta^{(0)}(\xi,\eta^{(0)}(\xi)) = 0.095930 + 0.604070e^{-5\xi} + O(\delta^2) \quad (4.12)$$

The nondimensional temperature on the free surface is mainly dependent on the heat transfer process decreasing from a value 0.7 at the vessel exit $\xi = 0$ to a value 0.1 at the solidification cross section $\xi = 1$ and is decoupled from the jet flow in an accuracy of $O(\delta^2)$.

Similar discussions may be applied to the flow fields, which depend mainly on both thermocapillary effect and the jet flow. By using height eq (4.8), the equation of the stream function (3.19) is reduced to

$$\Psi^{(0)}(\xi,\zeta) = \frac{(\eta^{(0)} - \zeta)\zeta^2}{4\eta^{(0)}(1 + Bi\eta^{(0)})}\frac{d\Theta_s}{d\xi} - $$
$$\frac{Bi(\Theta_s - \Theta_g)(\eta^{(0)} - \zeta)\zeta^2}{4\eta^{(0)}(1 + Bi\eta^{(0)})^2}\frac{d\eta^{(0)}}{d\xi} + $$
$$\frac{U_o(2\eta^{(0)} - \zeta)(\eta^{(0)} - \zeta)\zeta}{2\eta^{(0)}} + \frac{c_1\alpha(3\eta^{(0)} - \zeta)\zeta^2}{6\eta^{(0)3}} \quad (4.13)$$

where the integral constant Ψ_o in relationship (3.19) is adopted as zero. The stream functions $\Psi(\xi,0) = 0$ and $\Psi(\xi,\eta^{(0)}(\xi)) = c_1\alpha/3$ (constant) correspond to that both the solid boundary $\zeta = 0$ and the free surface $\zeta = \eta^{(0)}(\xi)$ are stream surfaces. With substitution of parameters (4.5) and expression (4.6), the stream function is given as

$$\Psi^{(0)}(\xi,\zeta) = -4.530527e^{-5\xi}\frac{(\eta^{(0)} - \zeta)\zeta^2}{\eta^{(0)}(2 + \eta^{(0)})^2} + $$
$$(0.453053e^{-5\xi} - 0.003052)\frac{(\eta^{(0)} - \zeta)\zeta^2}{\eta^{(0)}(2 + \eta^{(0)})^2}\frac{d\eta^{(0)}}{d\xi} + $$
$$\frac{U_o(2\eta^{(0)} - \zeta)(\eta^{(0)} - \zeta)\zeta}{2\eta^{(0)}} + \frac{c_1(3\eta^{(0)} - \zeta)\zeta^2}{12\eta^{(0)3}} \quad (4.14)$$

In considerations of relationship (4.10) and the small variation of liquid layer height $|d\eta^{(0)}/d\xi| = O(\delta)$, the stream function may be expanded as

$$\Psi^{(0)}(\xi,\zeta) = -0.503363e^{-5\xi}(1 - \zeta)\zeta^2 + \frac{U_o}{2}(2 - \zeta)(1 - $$
$$\zeta)\zeta + \frac{c_1}{12}(3 - \zeta)\zeta^2 + \left\{(0.05034e^{-5\xi} - 0.00339)(1 - \right.$$
$$\zeta)\zeta^2\frac{d\eta^{(0)}}{d\xi} + \left[0.167788e^{-5\xi}(2 - 5\zeta)\zeta^2 + \frac{U_o}{2}(1 - \zeta)(2 + \right.$$
$$\left.\left.\zeta) + \frac{c_1}{4}(\zeta - 2)\right]\delta\right\} + O(\delta^2) \quad (4.15)$$

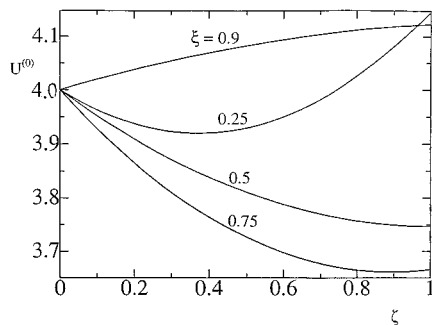The field distributions may be calculated directly from eq (4.14), and Figure 5 gives an example.

Figure 5. Velocity profiles in the jet liquid layer for case $\eta^{(0)}(0)$ = 1, $d\eta^{(0)}(0)/d\xi = 0.5$, $d^2\eta^{(0)}(0)/d\xi^2 = 0$, $\eta^{(0)}(1) = 1$, and $U_o = 4$.
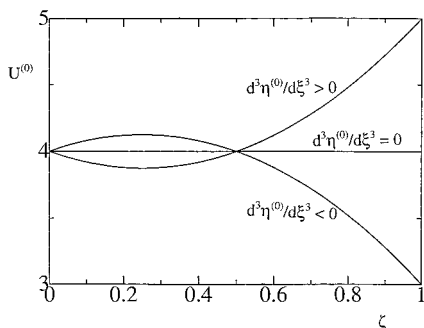

Figure 6. Typical velocity profiles in the downward region of jet liquid film.

By substitution of relationships (4.6) and (4.7), relationship (3.16) for the velocity component in the jet direction may be written as

$$U^{(o)} = U_o + a\zeta - b\zeta^2 \qquad (4.16)$$

$$\alpha = \left[\frac{\eta^{(0)}}{2}\frac{d^3\eta^{(0)}}{d\xi^2} + \frac{9.061052}{2+\eta^{(0)}}e^{-5\xi} + \right.$$

$$\left.(e^{-5\xi} - e^{-5})\frac{1.812211}{(2+\eta^{(0)})^2}\frac{d\eta^{(0)}}{d\xi}\right] \quad b = \frac{1}{4}\frac{d^3\eta^{(0)}}{d\xi^3}$$

where the values of $\eta^{(0)}$, $d\eta^{(0)}/d\xi$, and $d^3\eta^{(0)}/d\xi^3$ are given by the solution of height eq (4.8). Then, the velocity component as a second-order function of ζ may be obtained, as an example in Figure 5, which gives the velocity profiles at cross section $\xi = 2.5$, 0.5, 0.75, and 0.9 for the case $\gamma = 0.5$, $\beta = 0$, and $U_o = 4$. The results show that a return flow is induced by the thermocapillary effect and cross-section variation in addition to the jet flow U_o in the region near the exit of vessel. The values of a and b are listed as follows:

ξ	0	0.25	0.50	0.75	0.90
a	−0.006 323	−0.428 667	−0.759 072	−0.503 546	0.203 776
b	1.557 014	0.575 915	0.425 068	0.250 218	−0.080 106

It could be seen that the last two terms in the bracket are relatively smaller in the downward region of jet, and the velocity component may be approximately written as

$$U^{(0)}(\xi,\zeta) \cong U_o + \frac{2\eta^{(0)} - \zeta}{4}\frac{d^3\eta^{(0)}(\xi)}{d\xi^3}\zeta \quad \xi > 0.5 \quad (4.17)$$

The typical velocity profiles are show in Figure 6.

If the temperature difference between T_e at the vessel exit $x = 0$ and T_g of the environmental gas or the temperature T_l at $x = l$ is not large enough, the temper-
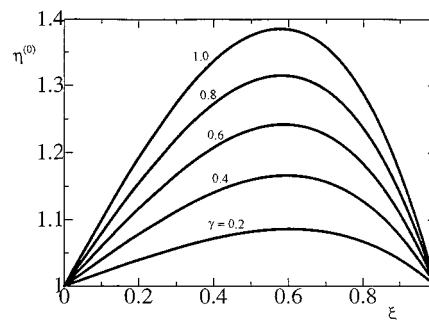
ature distribution T_s is better to be described approximately by a linear function of x such as

$$\Theta_s = 1 - (1 - \Theta_l)\xi \qquad (4.18)$$

Substituting relationship (4.18) into height eq (4.2) gives

$$\frac{d^3\eta^{(0)}}{d\xi^3} = \frac{c_1}{\eta^{(0)3}} - \frac{6U_o}{\eta^{(0)2}} - 5.4\left[\frac{1}{(2+\eta^{(0)})\eta^{(0)}} + \right.$$

$$\left. \frac{1-\xi}{(2+\eta^{(0)})^2\eta^{(0)}}\frac{d\eta^{(0)}}{d\xi}\right] \quad (4.19)$$

Ordinary differential eq (4.19) can be solved under boundary conditions (4.5) and parameter conditions (4.7), and similar conclusions are obtained. As an example, Figure 7 gives the influence of the expanding angle on the height distributions in the case of $U_o = 4$ for temperature distribution (4.18). The heights of the jet liquid film are relatively larger, and their maximums shift relatively into the downward direction of the jet in the case of linear temperature distributions (4.18) in comparison with the results of Figure 3 for temperature distributions (4.6).

5. Discussion

The lubrication approximation and the perturbation method are applied to discuss the problem of jet liquid film, and the analytical solutions of temperature, pressure, and velocity depending on the height of liquid film were obtained. The determined problem of the liquid height is described by a third-order ordinary differential equation with an integral constant, and 4 boundary conditions of liquid height are required to determine the 4 integral constants. The problem of liquid height was solved by using the Runge–Kutta method, and then, the temperature, pressure, and velocity fields in the jet liquid film were obtained. The results show that the influence of thermocapillary effect may enlarge the cross section of the jet liquid. The influence of typical parameters on the height distribution is discussed specifically.

It should be noted that the approximations of lubrication theory and perturbation theory have some limitations. The Barus effect has a larger variation of cross section, which may be two or three times, and the region of the enlarging cross section is closer to the vessel exit. However, the assumption of last relationship in (3.1) permits only a small variation of cross section. Therefore, the results of this paper show mainly the mechanism of the cross section variation due to the thermocapillary effect and jet flow. The larger variation of cross section needs to be studied by omitting the assumption and should be studied in the future.

Newtonian fluid is assumed in this paper. Thermocapillary flow may increase the cross section of the liquid layer on the basis of this assumption. This conclusion does not imply that the rheology of the liquid layer is not important. In fact, more attention should be paid to study the rheology, the manufacture model, and the detailed parameter region if the fiber processing is considered. However, the main purpose of this paper is concentrated in the mechanism study, which shows that the heat transfer may induce thermocapillary flow, which will increase the cross section even for a Newtonian fluid. An experiment to check the results of this paper is important. Usually, the thermocapillary effect will be coupled with the effect of rheological properties during the die swell, and both effects depend on the temperature gradient. However, the thermocapillary effect may be nearly distinguished in an experiment from the effect of the rheological effect if either the experimental condition is limited in the initial Newtonian region, and second Newtonian region[8] or a modeling medium of Newtonian fluid is adopted as the experimental medium. More studies should be initiated, especially in connecting with the experimental studies and the manufacture processing.

(8) Lenk, R. S. *Polymer Rheology*; Applied Science Publishers Ltd.: London, 1978; Chapter 1.