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Wedge and twist disclinations in second strain gradient elasticity

Shouchun Deng *, Jinxing Liu, Naigang Liang

State Key Laboratory of Nonlinear Mechanics (LNM), Institute of Mechanics, Chinese Academy of Sciences (CAS), Graduate School of CAS, Beijing 100080, China

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Abstract

The aim of this paper is to study disclinations in the framework of a second strain gradient elasticity theory. This second strain gradient elasticity has been proposed based on the first and second gradients of the strain tensor by Lazar et al. [Lazar, M., Maugin, G.A., Aifantis, E.C., 2006. Dislocations in second strain gradient elasticity. Int. J. Solids Struct. 43, 1787–1817]. Such a theory is an extension of the first strain gradient elasticity [Lazar, M., Maugin, G.A., 2005. Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. Int. J. Eng. Sci. 43, 1157–1184] with triple stress. By means of the stress function method, the exact analytical solutions for stress and strain fields of straight disclinations in an infinitely extended linear isotropic medium have been found. An important result is that the force stress, double stress and triple stress produced by wedge and twist disclinations are nonsingular. Meanwhile, the corresponding elastic strain and its gradients are also nonsingular. Analytical results indicate that the second strain gradient theory has the capacity of eliminating all unphysical singularities of physical fields.

Keywords: Wedge disclination; Twist disclination; Double stress; Triple stress; Strain gradient; Stress function

1. Introduction

Classical elasticity breaks down at small distances from crystal defects like dislocations and disclinations and leads to unphysical singularities. As an extension of the classical elasticity, strain gradient elasticity (Kröner, 1963, 1967; Kröner and Datta, 1966; Green and Rivlin, 1964a,b; Mindlin, 1964, 1965; Mindlin and Eshel, 1968) can be used to eliminate singularities.

Gradient elasticity and other theories were used to calculate the stress and the strain fields produced by dislocations and disclinations (Aifantis, 2003; Gutkin and Aifantis, 1996, 1997, 1999; Gutkin, 2000; Lazar and Maugin, 2004a,b, 2005; Lazar, 2003a,b,c,d). The gradient elasticity solutions have no singularity in both the stress and the strain fields. On the other hand, in first gradient elasticity the double stresses of twist disclinations, e.g. τ_{xxx}, τ_{zzx}, etc., still have singularities at the defect line (Lazar and Maugin, 2005). Recently, Lazar

* Corresponding author. Fax: +86 10 6257 9511.
E-mail address: dengsc@lnm.imech.ac.cn (S. Deng).

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et al. (2006) proposed a special second strain gradient theory and calculated the stress and strain fields produced by edge dislocations and screw dislocations. In which, the force stress, double stress and triple stress are all nonsingular. In this paper, we want to extend second strain gradient theory to investigate disclinations.

The plan of the paper is as follows. In Section 2, the framework of the second strain gradient elasticity is introduced. In Section 3, the classical solutions of wedge disclination are presented first and subsequently its elastic stress, double stress and triple stress in second strain gradient elasticity theory are calculated. The Section 4 treats twist disclination, and its structure is same as in Section 3. These fields in the framework of second strain gradient elasticity have no singularities within the disclination core region. Therefore, it regularizes all elastic fields within the framework of this theory. In Section 5, a concise conclusion has been provided.

2. Framework of the second strain gradient elasticity

In this section, the framework of second strain gradient elasticity will be introduced, and the details can be found in references [e.g., Lazar et al. (2006)]. E_{ij} denotes the symmetric elastic strain, and it is incompatible in continuum theory of defects (deWit, 1973; Kröner, 1981; Mura, 1982). The double and triple strains are defined by

$$\begin{cases} \eta_{ijk} = \partial_k E_{ij}, \\ \eta_{ijkl} = \partial_l \partial_k E_{ij}. \end{cases} \tag{2.1}$$

They fulfill the following compatibility conditions:

$$e_{mlk}\partial_l \eta_{ijk} = 0, \tag{2.2a}$$

$$e_{mnl}e_{pqk}\partial_n \partial_q \eta_{ijkl} = 0. \tag{2.2b}$$

Herein e_{ijk} denotes the permutation tensor.

For a linear elastic solid, the potential energy function, W, is assumed to be a quadratic function in terms of elastic strain, double strain and triple strain (Lazar and Maugin, 2005; Polizzotto, 2003)

$$W = W(E_{ij}, \eta_{ijk}, \eta_{ijkl}). \tag{2.3}$$

Since the strain E_{ij} is incompatible, we deal with an incompatible strain gradient elasticity which is valid for defects in linear elasticity. Then

$$\sigma_{ij} = \frac{\partial W}{\partial E_{ij}}, \quad \sigma_{ij} = \sigma_{ji}, \tag{2.4}$$

$$\tau_{ijk} = \frac{\partial W}{\partial(\partial_k E_{ij})} = \frac{\partial W}{\partial \eta_{ijk}}, \quad \tau_{ijk} = \tau_{jik}, \tag{2.5}$$

$$\tau_{ijkl} = \frac{\partial W}{\partial(\partial_l \partial_k E_{ij})} = \frac{\partial W}{\partial \eta_{ijkl}}, \quad \tau_{ijkl} = \tau_{jikl} = \tau_{ijlk} = \tau_{jilk} \tag{2.6}$$

are the response quantities with respect to the elastic, double and triple strains. τ_{ijk} and τ_{ijkl} are called the double and triple stresses, respectively.

In order to connect the higher gradient elasticity with the nonlocal isotropic elasticity proposed by Eringen (1992, 2002), the double and triple stresses are just simple gradients of the Cauchy-like stress tensor multiplied by two gradient coefficients:

$$\sigma_{ij} = C_{ijkl}E_{kl}, \tag{2.7}$$

$$\tau_{ijk} = \varepsilon^2 C_{ijpq}\partial_k E_{pq} = \varepsilon^2 \partial_k \sigma_{ij}, \tag{2.8}$$

$$\tau_{ijkl} = \gamma^4 C_{ijpq}\partial_l \partial_k E_{pq} = \gamma^4 \partial_l \partial_k \sigma_{ij}. \tag{2.9}$$

Both ε and γ are gradient coefficients with the dimension of a length, and $\varepsilon \geqslant \sqrt{2}\gamma$. For simplicity and generality, $\varepsilon > \sqrt{2}\gamma$ is considered in the paper. The corresponding stress functions under $\varepsilon = \sqrt{2}\gamma$ have also been given in Appendices A and B for wedge and twist disclinations, respectively. In the isotropic case, C_{ijkl}, the elasticity tensor reads

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.10}$$

where λ and μ are the Lamé's constants.

The force equilibrium condition (the body forces excluded) follows from the variation of W with respect to the displacement vector u_i:

$$\partial_j(\sigma_{ij} - \partial_k\tau_{ijk} + \partial_l\partial_k\tau_{ijkl}) = 0. \tag{2.11}$$

Eq. (2.11) takes the form

$$\partial_j\sigma^0_{ij} = 0 \tag{2.12}$$

if we define the total stress tensor

$$\sigma^0_{ij} = \sigma_{ij} - \partial_k\tau_{ijk} + \partial_l\partial_k\tau_{ijkl}, \tag{2.13}$$

σ^0_{ij} may be identified with the "classical" stress tensor. Substitution of Eqs. (2.8) and (2.9) into (2.11), yields the expression

$$(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)\partial_j\sigma_{ij} = 0, \tag{2.14}$$

herein Δ denotes the Laplacian. Considering Eq. (2.13), Eq. (2.14) is rewritten as

$$(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)\sigma_{ij} = \sigma^0_{ij}. \tag{2.15}$$

Moreover, for the elastic strain, the stress function f, etc., they have the same form

$$(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)E_{ij} = E^0_{ij}, \tag{2.16}$$

$$(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)f = f^0. \tag{2.17}$$

Herein, E^0_{ij} and f^0 denote the "classical" elastic strain and the "classical" stress function, respectively. The basic equations in second strain gradient elasticity like Eqs. (2.15)–(2.17) are typical inhomogeneous bi-Helmholtz equations.

3. Wedge disclination

3.1. Classical solutions

Here, we consider a straight wedge disclination inside an infinitely long cylinder with outer radius R. The z-axis is along the disclination line and coincides with the axis of the cylinder. For a wedge disclination, the Frank vector is parallel to the disclination line, $\mathbf{\Omega} = (0, 0, \Omega)$.

In absence of body forces, the force equilibrium condition can be identically satisfied by using the stress function (Hirth and Lothe, 1982). To the plane problem, we use the classical stress field of a straight wedge disclination in terms of the Airy stress function f^0 (Lazar, 2003c)

$$\sigma^0_{ij} = \begin{pmatrix} \partial^2_{yy}f^0 & -\partial^2_{xy}f^0 & 0 \\ -\partial^2_{xy}f^0 & \partial^2_{xx}f^0 & 0 \\ 0 & 0 & v\Delta f^0 \end{pmatrix}. \tag{3.1}$$

Here $\Delta = \partial^2_{xx} + \partial^2_{yy}$ and v the Poisson's ratio. In addition, the strain is given in terms of the stress function as

$$E^0_{ij} = \frac{1}{2\mu}\begin{pmatrix} \partial^2_{yy}f^0 - v\Delta f^0 & -\partial^2_{xy}f^0 & 0 \\ -\partial^2_{xy}f^0 & \partial^2_{xx}f^0 - v\Delta f^0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.2}$$

The stress function of a "classical" wedge disclination is given by

$$f^0 = Ar^2\left\{\ln r - \frac{1-4v}{2(1-2v)} - C\right\}, \quad A = \frac{\mu\Omega}{4\pi(1-v)}, \tag{3.3}$$

herein $r^2 = x^2 + y^2$. In the case of $C = 0$, the stress function reproduces the stress of a wedge disclination given by deWit (1973). Notes

$$B = -\frac{1-4v}{2(1-2v)}A, \quad B' = A\left\{-\frac{1-4v}{2(1-2v)} - C\right\} = B - AC. \tag{3.4}$$

Substitution of Eq. (3.3) into (3.1), yields the classical stress (Lazar, 2003c)

$$\sigma^0_{rr} = A\left\{2\ln r + \frac{A+2B'}{A}\right\}, \tag{3.5}$$

$$\sigma^0_{\theta\theta} = A\left\{2\ln r + \frac{3A+2B'}{A}\right\}, \tag{3.6}$$

$$\sigma^0_{zz} = 4vA\left\{\ln r + \frac{A+B'}{A}\right\}. \tag{3.7}$$

Similarly, the classical strain can be gotten by using Eqs. (3.2) and (3.3)

$$E^0_{r\theta} = E^0_{rz} = E^0_{\theta z} = E^0_{zz} = 0, \tag{3.8}$$

$$E^0_{rr} = \frac{A}{2\mu}\left\{[2-4v]\ln r + \frac{(1-4v)A + (2-4v)B'}{A}\right\}, \tag{3.9}$$

$$E^0_{\theta\theta} = \frac{A}{2\mu}\left\{(2-4v)\ln r + \frac{(3-4v)A + (2-4v)B'}{A}\right\}. \tag{3.10}$$

For satisfying the boundary condition, $\sigma^0_{rr}(R) = 0$, there is

$$B' = -\frac{A}{2}[1 + 2\ln R] \tag{3.11}$$

and

$$C = \ln R + \frac{v}{1-2v}. \tag{3.12}$$

3.2. Nonsingular solutions in second strain gradient elasticity

In this subsection we want to consider the wedge disclination in second strain gradient elasticity to find modified solutions without the classical singularities.

We make a hypothesis in terms of unknown stress function f which has the same form as the classical stress field

$$\sigma_{ij} = \begin{pmatrix} \partial^2_{yy}f & -\partial^2_{xy}f & 0 \\ -\partial^2_{xy}f & \partial^2_{xx}f & 0 \\ 0 & 0 & v\Delta f \end{pmatrix}. \tag{3.13}$$

In addition, the strain is given as

$$E_{ij} = \frac{1}{2\mu}\begin{pmatrix} \partial^2_{yy}f - v\Delta f & -\partial^2_{xy}f & 0 \\ -\partial^2_{xy}f & \partial^2_{xx}f - v\Delta f & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.14}$$

Substitution of Eqs. (3.1), (3.3), and (3.13) into (2.15) yields the results

$$(1 - \varepsilon^2\Delta + \gamma^4\Delta\Delta)f = Ar^2\ln r + B'r^2. \tag{3.15}$$

Eq. (3.15) is rewritten as

$$(1-c_1^2\varDelta)(1-c_2^2\varDelta)f = Ar^2\ln r + B'r^2, \tag{3.15$'$}$$

with

$$c_1^2 = \frac{\varepsilon^2+\sqrt{\varepsilon^4-4\gamma^4}}{2}, \tag{3.16}$$

$$c_2^2 = \frac{\varepsilon^2-\sqrt{\varepsilon^4-4\gamma^4}}{2}. \tag{3.17}$$

Eq. (3.15′) is an inhomogeneous bi-Helmholtz equation, and its solution can be expressed in the form (the solution procedure has been given in Appendix C)

$$f = A\left\{r^2\ln r + 4(c_1^2+c_2^2)\ln r + \frac{4}{c_1^2-c_2^2}\left[c_1^4K_0\left(\frac{r}{c_1}\right)-c_2^4K_0\left(\frac{r}{c_2}\right)\right]\right\} + B'r^2 + 4(A+B')(c_1^2+c_2^2), \tag{3.18}$$

herein K_n denotes the modified Bessel function of the second kind and n is the order of this function. Eq. (3.18) is the stress function under $c_1 \neq c_2$, and the corresponding one under $c_1 = c_2$ has been given in Appendix A.

Substitution of Eq. (3.18) into Eq. (3.13), yields the modified elastic stress

$$\sigma_{rr} = \frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r} = A\left\{2\ln r + \left(1+\frac{2B'}{A}\right) + \frac{4(c_1^2+c_2^2)}{r^2} - \frac{4}{r(c_1^2-c_2^2)}\left[c_1^3K_1\left(\frac{r}{c_1}\right)-c_2^3K_1\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.19}$$

$$\begin{aligned}\sigma_{\theta\theta} = \frac{\mathrm{d}^2f}{\mathrm{d}r^2} = A\Bigg\{&2\ln r + \left(3+\frac{2B'}{A}\right) - \frac{4(c_1^2+c_2^2)}{r^2} - \frac{4}{r(c_1^2-c_2^2)}\left[c_1^3K_1\left(\frac{r}{c_1}\right)-c_2^3K_1\left(\frac{r}{c_2}\right)\right]\\ &+\frac{4}{c_1^2-c_2^2}\left[c_1^2K_2\left(\frac{r}{c_1}\right)-c_2^2K_2\left(\frac{r}{c_2}\right)\right]\Bigg\},\end{aligned} \tag{3.20}$$

$$\tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial f}{\partial\theta}\right) = 0, \tag{3.21}$$

$$\begin{aligned}\sigma_{zz} = v\varDelta f = 4vA\Bigg\{&\ln r + \frac{A+B'}{A} - \frac{2}{r(c_1^2-c_2^2)}\left[c_1^3K_1\left(\frac{r}{c_1}\right)-c_2^3K_1\left(\frac{r}{c_2}\right)\right]\\ &+\frac{1}{c_1^2-c_2^2}\left[c_1^2K_2\left(\frac{r}{c_1}\right)-c_2^2K_2\left(\frac{r}{c_2}\right)\right]\Bigg\}.\end{aligned} \tag{3.22}$$

Similarly, it has to satisfy the boundary condition, $\sigma_{rr}(R) = 0$, and there is

$$B' = A\left\{-\ln R - \frac{1}{2} - \frac{2(c_1^2+c_2^2)}{R^2} + \frac{2}{R(c_1^2-c_2^2)}\left[c_1^3K_1\left(\frac{R}{c_1}\right)-c_2^3K_1\left(\frac{R}{c_2}\right)\right]\right\}, \tag{3.23}$$

The stresses in different theories are illustrated by Figs. 1–3, respectively. In these figures, $\frac{c_1}{c_2} = 2$, $\frac{R}{c_2} = 20$, $v = \frac{1}{3}$, and these particular values will be used to drawing in this paper. In addition, all figures in the paper are plotted according to the independent variable(s) as $\frac{r}{c_2}$, $\frac{x}{c_2}$ and/or $\frac{y}{c_2}$. The "classical" stresses are singular at the disclination line, but the "modified" stresses are nonsingular and they have finite minimum values in the disclination core region. In addition, the "modified" stresses change slowly near the disclination line comparing to the classical stresses, and they have little difference away from the defect line.

Similarly, the modified strains are expressed as follows

$$\begin{aligned}E_{rr} = \frac{A}{2\mu}\Bigg\{&[2-4v]\ln r + \frac{(1-4v)A+(2-4v)B'}{A} + \frac{4(c_1^2+c_2^2)}{r^2} - \frac{4v}{c_1^2-c_2^2}\left[c_1^2K_2\left(\frac{r}{c_1}\right)-c_2^2K_2\left(\frac{r}{c_2}\right)\right]\\ &-\frac{4-8v}{r(c_1^2-c_2^2)}\left[c_1^3K_1\left(\frac{r}{c_1}\right)-c_2^3K_1\left(\frac{r}{c_2}\right)\right]\Bigg\},\end{aligned} \tag{3.24}$$

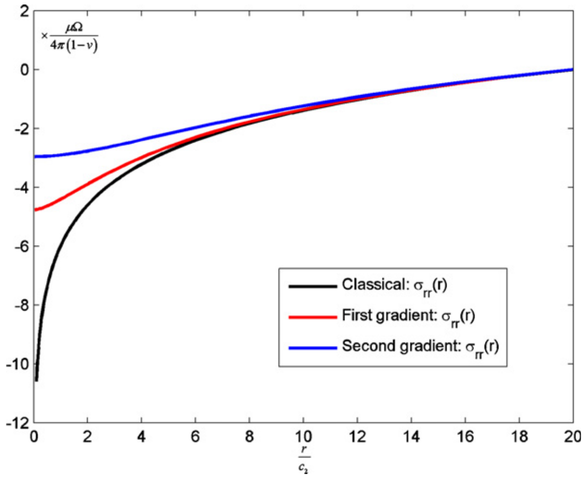

Fig. 1. The classical and "modified" $\sigma_{rr}(r)$.

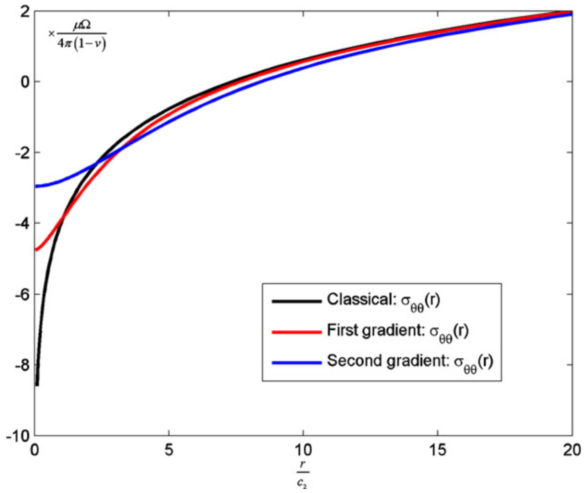

Fig. 2. The classical and "modified" $\sigma_{\theta\theta}\,(r)$.

$$E_{\theta\theta} = \frac{A}{2\mu}\left\{(2-4v)\ln r + \frac{(3-4v)A+(2-4v)B'}{A} - \frac{4(c_1^2+c_2^2)}{r^2} - \frac{4-8v}{r(c_1^2-c_2^2)}\left[c_1^3K_1\left(\frac{r}{c_1}\right) - c_2^3K_1\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. + \frac{4(1-v)}{c_1^2-c_2^2}\left[c_1^2K_2\left(\frac{r}{c_1}\right) - c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.25}$$

$$E_{r\theta} = E_{rz} = E_{\theta z} = E_{zz} = 0. \tag{3.26}$$

The strains in different theories are plotted in Fig. 4. In which, the "classical" strains change quickly but the "modified" ones change steadily at the disclination line.

For linear isotropic media, the relations of the double strain and the double stress and the ones of the triple strain and the triple stress are linear. Therefore, only the hyperstresses (double and triple stresses) are investigated.

Substitution of Eqs. (3.19), (3.20) and (3.22) into (2.8), yields the components of double stress

$$\tau_{rrr} = \varepsilon^2\frac{\mathrm{d}\sigma_{rr}}{\mathrm{d}r} = \varepsilon^2 A\left\{\frac{2}{r} - \frac{8(c_1^2+c_2^2)}{r^3} + \frac{4}{r(c_1^2-c_2^2)}\left[c_1^2K_2\left(\frac{r}{c_1}\right) - c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.27}$$

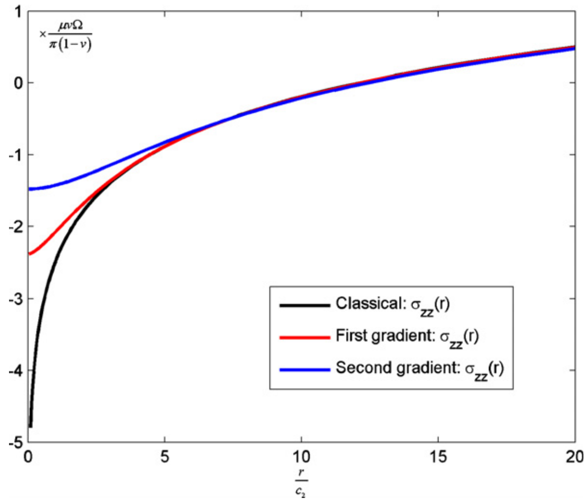

Fig. 3. The classical and "modified" $\sigma_{zz}(r)$.

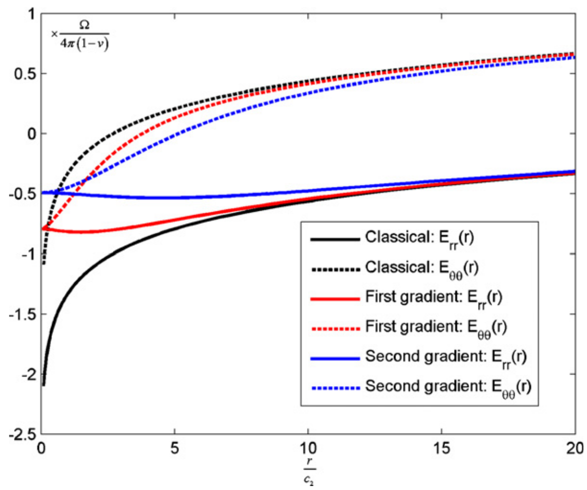

Fig. 4. The classical and "modified" $E_{ij}(r)$.

$$\tau_{\theta\theta r} = \varepsilon^2 \frac{\mathrm{d}\sigma_{\theta\theta}}{\mathrm{d}r} = \varepsilon^2 A\left\{\frac{2}{r} + \frac{8(c_1^2+c_2^2)}{r^3} + \frac{12}{r(c_1^2-c_2^2)}\left[c_1^2 K_2\left(\frac{r}{c_1}\right) - c_2^2 K_2\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. - \frac{4}{c_1^2-c_2^2}\left[c_1 K_3\left(\frac{r}{c_1}\right) - c_2 K_3\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.28}$$

$$\tau_{zzr} = \varepsilon^2 \frac{\mathrm{d}\sigma_{zz}}{\mathrm{d}r} = 4v\varepsilon^2 A\left\{\frac{1}{r} + \frac{4}{r(c_1^2-c_2^2)}\left[c_1^2 K_2\left(\frac{r}{c_1}\right) - c_2^2 K_2\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. - \frac{1}{c_1^2-c_2^2}\left[c_1 K_3\left(\frac{r}{c_1}\right) - c_2 K_3\left(\frac{r}{c_2}\right)\right]\right\}. \tag{3.29}$$

The double stresses are illustrated by Fig. 5. The figure shows that the double stresses have zero value at disclination line, and they change quickly in the defect core region.

Similarly, the triple stresses are given

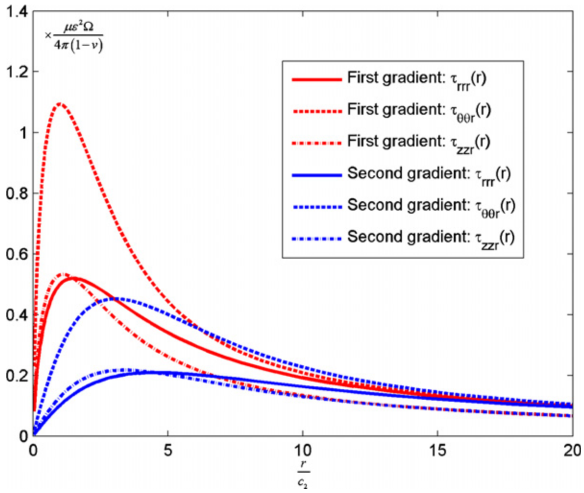

Fig. 5. The double stress in first and second gradient elasticities.

$$\tau_{rrrr} = \gamma^4 A\left\{-\frac{2}{r^2} + \frac{24(c_1^2+c_2^2)}{r^4} + \frac{4}{r^2(c_1^2-c_2^2)}\left[c_1^2 K_2\left(\frac{r}{c_1}\right) - c_2^2 K_2\left(\frac{r}{c_2}\right)\right] - \frac{4}{r(c_1^2-c_2^2)}\left[c_1 K_3\left(\frac{r}{c_1}\right) - c_2 K_3\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.30}$$

$$\tau_{\theta\theta rr} = \gamma^4 A\left\{-\frac{2}{r^2} - \frac{24(c_1^2+c_2^2)}{r^4} + \frac{12}{r^2(c_1^2-c_2^2)}\left[c_1^2 K_2\left(\frac{r}{c_1}\right) - c_2^2 K_2\left(\frac{r}{c_2}\right)\right] - \frac{24}{r(c_1^2-c_2^2)}\left[c_1 K_3\left(\frac{r}{c_1}\right) - c_2 K_3\left(\frac{r}{c_2}\right)\right] + \frac{4}{c_1^2-c_2^2}\left[K_4\left(\frac{r}{c_1}\right) - K_4\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.31}$$

$$\tau_{zzrr} = 4\upsilon\gamma^4 A\left\{-\frac{1}{r^2} + \frac{4}{r^2(c_1^2-c_2^2)}\left[c_1^2 K_2\left(\frac{r}{c_1}\right) - c_2^2 K_2\left(\frac{r}{c_2}\right)\right] - \frac{7}{r(c_1^2-c_2^2)}\left[c_1 K_3\left(\frac{r}{c_1}\right) - c_2 K_3\left(\frac{r}{c_2}\right)\right] + \frac{1}{c_1^2-c_2^2}\left[K_4\left(\frac{r}{c_1}\right) - K_4\left(\frac{r}{c_2}\right)\right]\right\}. \tag{3.32}$$

The components of the triple stress are illustrated by Fig. 6, and they have finite values at the disclination line ($r = 0$). Moreover, the triple stress goes to zero away from the disclination line.

The elastic bend–twist may be determined from the condition that the dislocation density (disclination torsion) has to be zero for a straight wedge disclination

$$\alpha_{xz} = -\frac{1-\upsilon}{2\mu}\partial_y \Delta f - \kappa_{zx} \equiv 0, \quad \alpha_{yz} = \frac{1-\upsilon}{2\mu}\partial_x \Delta f - \kappa_{zy} \equiv 0. \tag{3.33}$$

The effective Frank vector of the wedge disclination is given by (see Fig. 7)

$$\Omega_r(r) = \oint_\Gamma (\kappa_{zx}\,\mathrm{d}x + \kappa_{zy}\,\mathrm{d}y) = \Omega\left\{1 - \frac{1}{c_1^2-c_2^2}\left[c_1 r K_1\left(\frac{r}{c_1}\right) - c_2 r K_1\left(\frac{r}{c_2}\right)\right]\right\}, \tag{3.34}$$

From Fig. 7, we find that the "modified" Frank vectors are no longer constants and they have zero value at the disclination line. On the other hand, when $r \to \infty$, they approach to the classical Frank vector Ω.

The contribution of double and triple stresses to strain energy density has been investigated. For simplicity, strain energy density function in part 2 can be rewritten as

$$W = W(E_{ij}, \eta_{ijk}, \eta_{ijkl}) = \frac{1}{2}\left\{\sigma_{ij}E_{ij} + \frac{1}{\varepsilon^2}\tau_{ijk}\eta_{ijk} + \frac{1}{\gamma^4}\tau_{ijkl}\eta_{ijkl}\right\}, \tag{3.35}$$

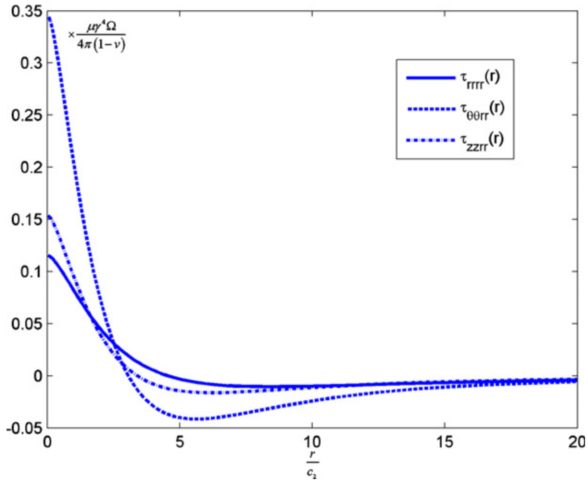

Fig. 6. The triple stress.

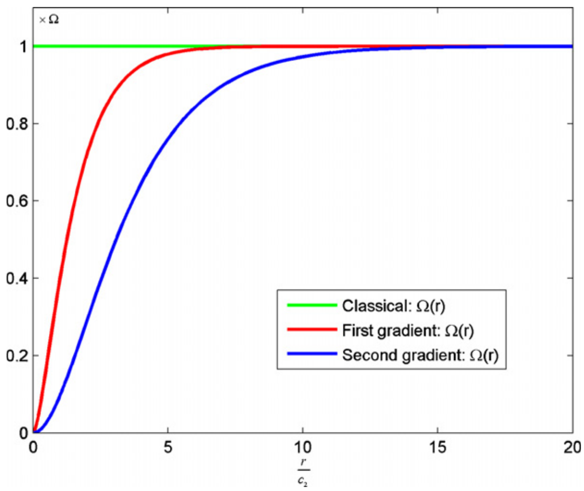

Fig. 7. The effective Frank vectors.

and it has been illustrated in Fig. 8. In which, we can find that the contribution of double and triple stresses changes quickly within the disclination core. Moreover, the effect of triple stress is neglectable out of disclination core. Certainly, the contribution of double and triple stresses will increase as the gradient coefficients ε and γ increase.

4. Twist disclination

4.1. Classical solutions

In this subsection we present the "classical" stress field for a straight twist disclination in an infinitely extended isotropic body by the help of the stress function method. We assume the disclination line is along the z-axis and the Frank vector has the form, $\Omega = (0, \Omega, 0)$.

The classical solution for the elastic stress field was given by deWit (1973) and Lazar (2003d)

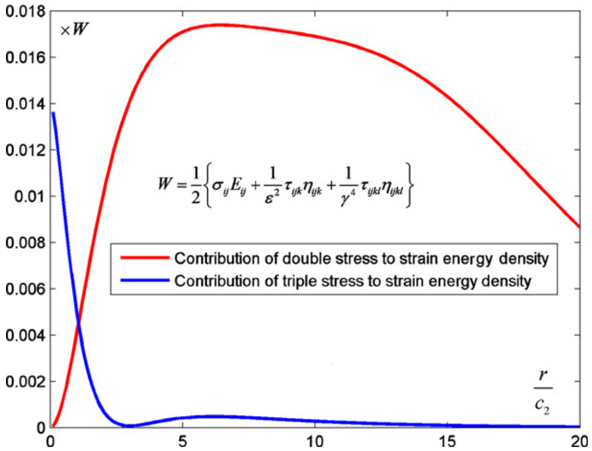

Fig. 8. The contribution of double and triple stresses to strain energy density.

$$\sigma^0_{xx} = -\frac{\mu\Omega}{2\pi(1-v)}\frac{zy(y^2+3x^2)}{r^4}, \tag{4.1}$$

$$\sigma^0_{yy} = -\frac{\mu\Omega}{2\pi(1-v)}\frac{zy(y^2-x^2)}{r^4}, \tag{4.2}$$

$$\sigma^0_{xy} = \frac{\mu\Omega}{2\pi(1-v)}\frac{zx(x^2-y^2)}{r^4}, \tag{4.3}$$

$$\sigma^0_{zz} = -\frac{\mu\Omega v}{\pi(1-v)}\frac{zy}{r^2}, \tag{4.4}$$

$$\sigma^0_{zx} = \frac{\mu\Omega}{2\pi(1-v)}\frac{xy}{r^2}, \tag{4.5}$$

$$\sigma^0_{zy} = -\frac{\mu\Omega}{2\pi(1-v)}\left\{(1-2v)\ln r+\frac{x^2}{r^2}\right\}. \tag{4.6}$$

The components of the classical elastic strain have been given in Appendix D. Obviously, the expressions (4.1)–(4.4) contain the classical singularity $\sim r^{-1}$ and a logarithmic singularity $\sim \ln r$ in (4.6). For the situation of the strain condition, $E^0_{zz}=0$, Eqs. (4.1)–(4.6) can be calculated by using the so-called stress function method in the following form

$$\sigma^0_{ij} = \begin{bmatrix} \partial^2_{yy}f^0 & -\partial^2_{xy}f^0 & -\partial_y F^0 \\ -\partial^2_{xy}f^0 & \partial^2_{xx}f^0 & \partial_x F^0+\partial_z g^0 \\ -\partial_y F^0 & \partial_x F^0+\partial_z g^0 & v\Delta f^0 \end{bmatrix}. \tag{4.7}$$

In order to satisfy the force equilibrium the stress σ^0_{zz} has to fulfill the condition

$$v\Delta f^0 = -\partial_y g^0. \tag{4.8}$$

The classical stress functions for the stress fields Eq. (4.7) are

$$f^0 = -\frac{\mu\Omega}{2\pi(1-v)}zy\ln r, \tag{4.9}$$

$$F^0 = -\frac{\mu\Omega}{2\pi(1-v)}x\ln r, \tag{4.10}$$

$$g^0 = \frac{\mu\Omega v}{\pi(1-v)}z\ln r. \tag{4.11}$$

4.2. Nonsingular solutions in second strain gradient elasticity

After giving the classical solutions the twist disclination in second strain gradient elasticity theory will be considered to find the modified solutions without the classical singularities.

In second strain gradient elasticity, the stress functions are related to the stress tensor (Lazar and Maugin, 2005)

$$\sigma_{ij} = \begin{bmatrix} \partial^2_{yy}f & -\partial^2_{xy}f & -\partial_y F \\ -\partial^2_{xy}f & \partial^2_{xx}f & \partial_x F + \partial_z g \\ -\partial_y F & \partial_x F + \partial_z g & v\Delta f \end{bmatrix} \tag{4.12}$$

with the relation

$$v\Delta f = -\partial_y g. \tag{4.13}$$

Consequently, three inhomogeneous bi-Helmholtz equations for the unknown stress functions have been obtained

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)f = (1-c_1^2\Delta)(1-c_2^2\Delta)f = f^0 = -\frac{\mu\Omega}{2\pi(1-v)}zy\ln r, \tag{4.14}$$

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)F = (1-c_1^2\Delta)(1-c_2^2\Delta)F = F^0 = -\frac{\mu\Omega}{2\pi(1-v)}x\ln r, \tag{4.15}$$

$$(1-\varepsilon^2\Delta+\gamma^4\Delta\Delta)g = (1-c_1^2\Delta)(1-c_2^2\Delta)g = g^0 = \frac{\mu\Omega v}{\pi(1-v)}z\ln r, \tag{4.16}$$

where c_1^2 and c_2^2 are given by Eqs. (3.14) and (3.15), respectively.

The solutions for the stress functions of a straight twist disclination are given by (its solution procedure is same as in wedge disclination but more complicated, see Appendix E)

$$f = -\frac{\mu\Omega}{2\pi(1-v)}zy\left\{\ln r + \frac{2(c_1^2+c_2^2)}{r^2} - \frac{1}{(c_1^2-c_2^2)r}\left[2c_1^3K_1\left(\frac{r}{c_1}\right) - 2c_2^3K_1\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.17}$$

$$F = -\frac{\mu\Omega}{2\pi(1-v)}x\left\{\ln r + \frac{2(c_1^2+c_2^2)}{r^2} - \frac{1}{(c_1^2-c_2^2)r}\left[2c_1^3K_1\left(\frac{r}{c_1}\right) - 2c_2^3K_1\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.18}$$

$$g = \frac{\mu\Omega v}{\pi(1-v)}z\left\{\ln r + \frac{1}{c_1^2-c_2^2}\left[c_1^2K_0\left(\frac{r}{c_1}\right) - c_2^2K_0\left(\frac{r}{c_2}\right)\right]\right\}. \tag{4.19}$$

Eqs. (4.17)–(4.19) indicate $c_1 \neq c_2$. The corresponding stress functions under $c_1 = c_2$ have been given in Appendix B. By means of Eq. (4.12) and the stress functions (4.17)–(4.19), yields the elastic stress in Cartesian coordinates

$$\begin{aligned}\sigma_{xx} = -\frac{\mu\Omega}{2\pi(1-v)}z\left\{\frac{3y}{r^2} - \frac{2y^3}{r^4} - \frac{12(c_1^2+c_2^2)}{r^4}y + \frac{16(c_1^2+c_2^2)}{r^6}y^3 + \frac{3y}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right.\\ \left. - \frac{y^3}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\},\end{aligned} \tag{4.20}$$

$$\begin{aligned}\sigma_{xy} = \frac{\mu\Omega}{2\pi(1-v)}z\left\{\frac{x}{r^2} - \frac{2xy^2}{r^4} - \frac{4(c_1^2+c_2^2)}{r^4}x + \frac{16(c_1^2+c_2^2)}{r^6}xy^2 + \frac{x}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right.\\ \left. - \frac{xy^2}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\},\end{aligned} \tag{4.21}$$

$$\begin{aligned}\sigma_{yy} = -\frac{\mu\Omega}{2\pi(1-v)}z\left\{\frac{y}{r^2} - \frac{2x^2y}{r^4} - \frac{4(c_1^2+c_2^2)}{r^4}y + \frac{16(c_1^2+c_2^2)}{r^6}x^2y + \frac{y}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right.\\ \left. - \frac{x^2y}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\},\end{aligned} \tag{4.22}$$

$$\sigma_{xz} = \frac{\mu\Omega}{2\pi(1-v)} xy \left\{ \frac{1}{r^2} - \frac{4(c_1^2+c_2^2)}{r^4} + \frac{1}{(c_1^2-c_2^2)r^2} \left[2c_1^2 K_2\left(\frac{r}{c_1}\right) - 2c_2^2 K_2\left(\frac{r}{c_2}\right) \right] \right\}, \tag{4.23}$$

$$\begin{aligned}\sigma_{yz} = -\frac{\mu\Omega}{2\pi(1-v)} \Bigg\{ & (1-2v)\ln r + \frac{2(c_1^2+c_2^2)}{r^2} + \frac{x^2}{r^2} - \frac{4(c_1^2+c_2^2)}{r^4}x^2 - \frac{2v}{c_1^2-c_2^2}\left[c_1^2 K_0\left(\frac{r}{c_1}\right) - c_2^2 K_0\left(\frac{r}{c_2}\right)\right] \\ & - \frac{1}{(c_1^2-c_2^2)r}\left[2c_1^3 K_1\left(\frac{r}{c_1}\right) - 2c_2^3 K_1\left(\frac{r}{c_2}\right)\right] + \frac{x^2}{(c_1^2-c_2^2)r^2}\left[2c_1^2 K_2\left(\frac{r}{c_1}\right) - 2c_2^2 K_2\left(\frac{r}{c_2}\right)\right] \Bigg\},\end{aligned} \tag{4.24}$$

$$\sigma_{zz} = -\frac{\mu\Omega v}{\pi(1-v)} zy \left\{ \frac{1}{r^2} - \frac{1}{(c_1^2-c_2^2)r}\left[c_1 K_1\left(\frac{r}{c_1}\right) - c_2 K_1\left(\frac{r}{c_2}\right)\right] \right\}. \tag{4.25}$$

The components of the force stress in classical, the first and second gradient theories are plotted in Figs. 9–13, respectively.

When $r \to 0$, there are $K_0(\frac{r}{c}) \to -[\xi + \ln\frac{r}{2c}]$, $K_1(\frac{r}{c}) \to \frac{c}{r}$ and $K_2(\frac{r}{c}) \to -\frac{1}{2} + \frac{2c^2}{r^2}$, etc. Here ξ denotes the Euler constant, and thus σ_{ij} goes to finite value. Moreover, Figs. 9–13 have also illustrated that all "modified" force stresses are nonsingular, and the amplitudes of the stress in second gradient elasticity are less than the corre-

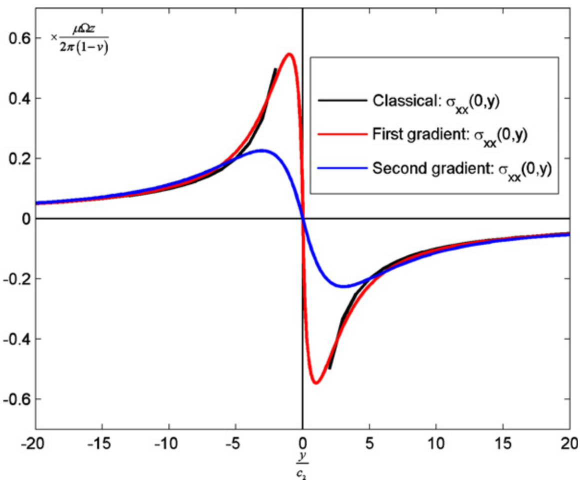

Fig. 9. The classical and "modified" σ_{xx} $(0, y)$.

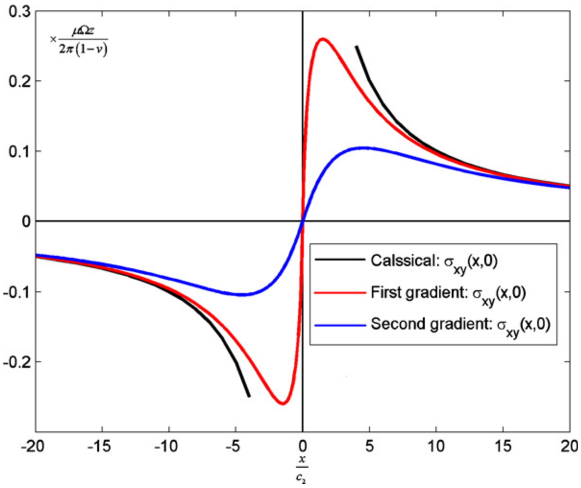

Fig. 10. The classical and "modified" $\sigma_{xy}(x, 0)$.

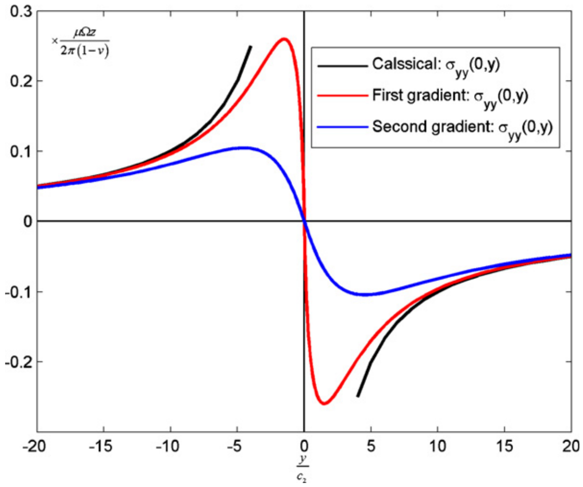

Fig. 11. The classical and "modified" $\sigma_{yy}(0, y)$.

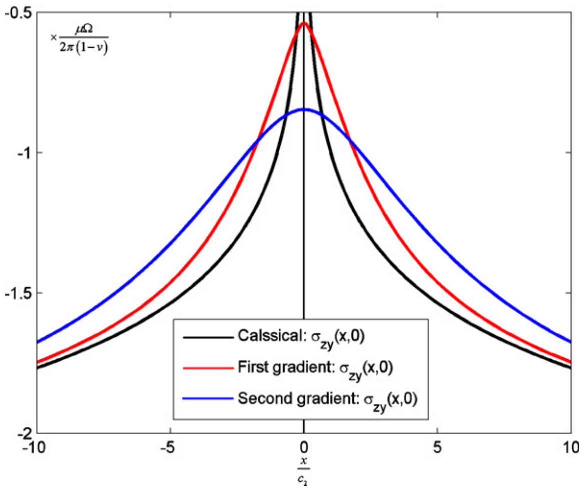

Fig. 12. The classical and "modified" $\sigma_{zy}(x, 0)$.

sponding ones in first gradient elasticity near the disclination line. Simultaneously, the elastic strain given in terms of stress functions

$$E_{ij} = \frac{1}{2\mu}\begin{bmatrix} \partial_{yy}^2 f - v\Delta f & -\partial_{xy}^2 f & -\partial_y F \\ -\partial_{xy}^2 f & \partial_{xx}^2 f - v\Delta f & \partial_x F + \partial_z g \\ -\partial_y F & \partial_x F + \partial_z g & 0 \end{bmatrix} \tag{4.26}$$

is nonsingular, and its components are written as follows:

$$E_{xz} = \frac{\Omega}{4\pi(1-v)}xy\left\{\frac{1}{r^2} - \frac{4(c_1^2+c_2^2)}{r^4} + \frac{1}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.27}$$

$$\begin{aligned} E_{xx} = &-\frac{\Omega}{4\pi(1-v)}zy\left\{\frac{3-2v}{r^2} - \frac{2y^2}{r^4} - \frac{12(c_1^2+c_2^2)}{r^4} + \frac{16(c_1^2+c_2^2)}{r^6}y^2 + \frac{2v}{(c_1^2-c_2^2)r}\left[c_1K_1\left(\frac{r}{c_1}\right) - c_2K_1\left(\frac{r}{c_2}\right)\right]\right. \\ &\left. + \frac{3}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right] - \frac{y^2}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\}, \end{aligned} \tag{4.28}$$

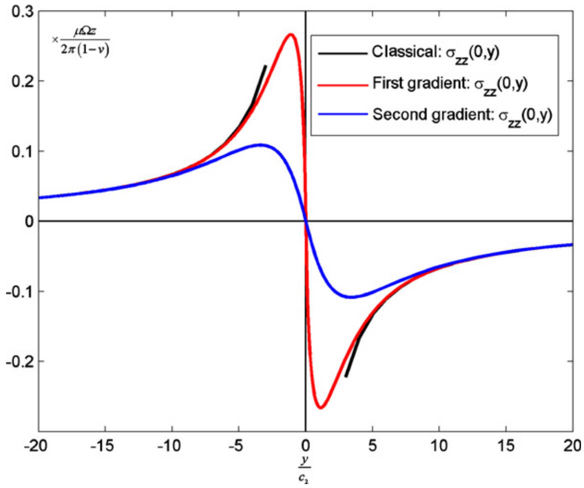

Fig. 13. The classical and "modified" $\sigma_{zz}\,(0, y)$.

$$E_{xy} = \frac{\Omega}{4\pi(1-v)} zx\left\{\frac{1}{r^2} - \frac{2y^2}{r^4} - \frac{4(c_1^2+c_2^2)}{r^4} + \frac{16(c_1^2+c_2^2)}{r^6}y^2 + \frac{1}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. - \frac{y^2}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.29}$$

$$E_{yy} = -\frac{\Omega}{4\pi(1-v)} zy\left\{\frac{1-2v}{r^2} - \frac{2x^2}{r^4} - \frac{4(c_1^2+c_2^2)}{r^4} + \frac{16(c_1^2+c_2^2)}{r^6}x^2 + \frac{2v}{(c_1^2-c_2^2)r}\left[c_1K_1\left(\frac{r}{c_1}\right) - c_2K_1\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. + \frac{1}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right] - \frac{x^2}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.30}$$

$$E_{yz} = -\frac{\Omega}{4\pi(1-v)}\left\{(1-2v)\ln r + \frac{2(c_1^2+c_2^2)}{r^2} + \frac{x^2}{r^2} - \frac{4(c_1^2+c_2^2)}{r^4}x^2 - \frac{2v}{c_1^2-c_2^2}\left[c_1^2K_0\left(\frac{r}{c_1}\right) - c_2^2K_0\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. - \frac{1}{(c_1^2-c_2^2)r}\left[2c_1^3K_1\left(\frac{r}{c_1}\right) - 2c_2^3K_1\left(\frac{r}{c_2}\right)\right] + \frac{x^2}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right]\right\}. \tag{4.31}$$

Here, we have to emphasize that in first strain gradient elasticity only the components of the double stress of twist disclinations τ_{xxz}, τ_{yyz}, τ_{xyz}, τ_{zzz}, τ_{zxx}, τ_{zxy}, τ_{zyx} and τ_{zyy} are nonsingular. The other components τ_{xxx}, τ_{yyx}, τ_{xyx}, τ_{zzx}, τ_{xxy}, τ_{yyy}, τ_{xyy} and τ_{zzy} are singular at $r = 0$. Moreover, these components are zero at $z = 0$. On the contrary, all components of the double and triple stresses in second strain gradient theory are nonsingular, and the double and triple strains have finite values. For verifying this important result, we illustrate it by many representative components

$$\tau_{zyy} = -\frac{\mu\Omega\varepsilon^2}{2\pi(1-v)} y\left\{\frac{1-2v}{r^2} - \frac{4(c_1^2+c_2^2)}{r^4} - \frac{2x^2}{r^4} + \frac{16(c_1^2+c_2^2)}{r^6}x^2 + \frac{2v}{(c_1^2-c_2^2)r}\left[c_1K_1\left(\frac{r}{c_1}\right) - c_2K_1\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. + \frac{1}{(c_1^2-c_2^2)r^2}\left[2c_1^2K_2\left(\frac{r}{c_1}\right) - 2c_2^2K_2\left(\frac{r}{c_2}\right)\right] - \frac{x^2}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.32}$$

$$\tau_{xxx} = -\frac{\mu\Omega\varepsilon^2}{2\pi(1-v)} zx\left\{-\frac{6y}{r^4} + \frac{8y^3}{r^6} + \frac{48(c_1^2+c_2^2)}{r^6}y - \frac{96(c_1^2+c_2^2)}{r^8}y^3 - \frac{3y}{(c_1^2-c_2^2)r^3}\left[2c_1K_3\left(\frac{r}{c_1}\right) - 2c_2K_3\left(\frac{r}{c_2}\right)\right]\right.$$
$$\left. + \frac{y^3}{(c_1^2-c_2^2)r^4}\left[2K_4\left(\frac{r}{c_1}\right) - 2K_4\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.33}$$

$$\tau_{zzx} = -\frac{\mu\Omega v\varepsilon^2}{\pi(1-v)} xyz\left\{-\frac{2}{r^4} + \frac{1}{(c_1^2-c_2^2)r^2}\left[K_2\left(\frac{r}{c_1}\right) - K_2\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.34}$$

$$\tau_{zzxx} = -\frac{\mu\Omega v\gamma^4}{\pi(1-v)} zy\left\{-\frac{2}{r^4}+\frac{8x^2}{r^6}+\frac{1}{(c_1^2-c_2^2)r^2}\left[K_2\left(\frac{r}{c_1}\right)-K_2\left(\frac{r}{c_2}\right)\right]-\frac{x^2}{(c_1^2-c_2^2)r^3}\left[\frac{1}{c_1}K_3\left(\frac{r}{c_1}\right)-\frac{1}{c_2}K_3\left(\frac{r}{c_2}\right)\right]\right\}, \tag{4.35}$$

$$\tau_{zzxy} = -\frac{\mu\Omega v\gamma^4}{\pi(1-v)} zx\left\{-\frac{2}{r^4}+\frac{8y^2}{r^6}+\frac{1}{(c_1^2-c_2^2)r^2}\left[K_2\left(\frac{r}{c_1}\right)-K_2\left(\frac{r}{c_2}\right)\right]-\frac{y^2}{(c_1^2-c_2^2)r^3}\left[\frac{1}{c_1}K_3\left(\frac{r}{c_1}\right)-\frac{1}{c_2}K_3\left(\frac{r}{c_2}\right)\right]\right\}. \tag{4.36}$$

These components are plotted in Figs. 14–18, and the figures have also illustrated that the components of the double and triple stresses are regularized.

The effective Frank vector of twist disclination is defined by

$$\Omega_y(r) = \oint_\Gamma (\kappa_{yx}\,\mathrm{d}x + \kappa_{yy}\,\mathrm{d}y) \tag{4.37}$$

with the components of the elastic bend–twist tensor

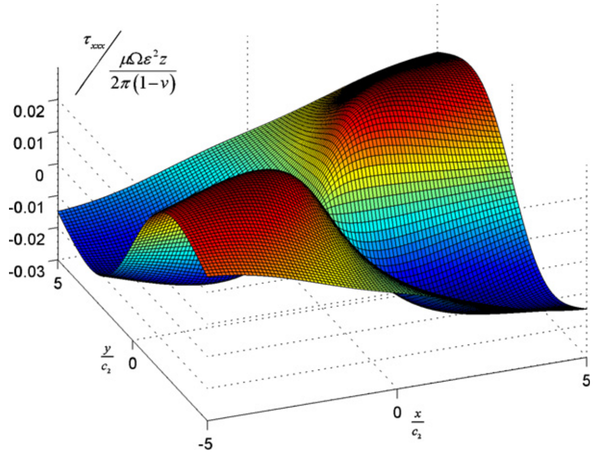

Fig. 14. τ_{xxx}.

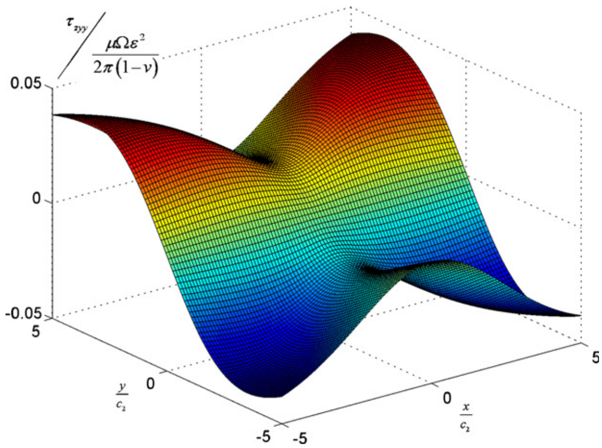

Fig. 15. τ_{zyy}.

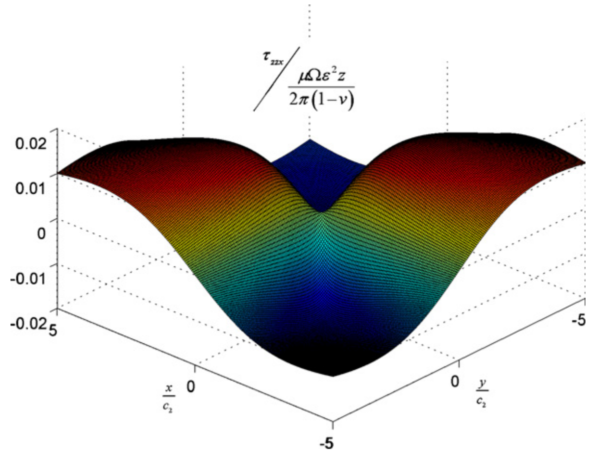

Fig. 16. τ_{zzx}.

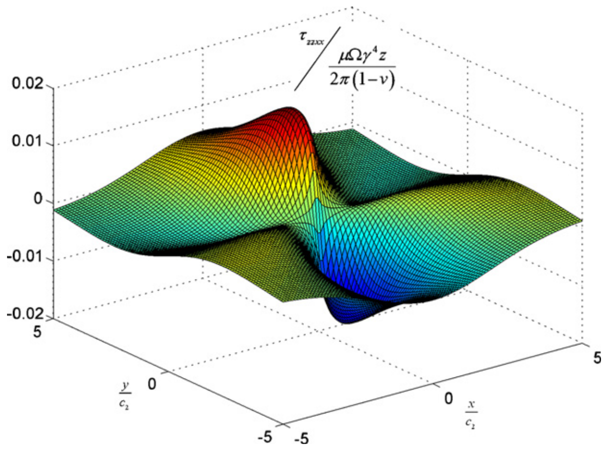

Fig. 17. τ_{zzxx}.

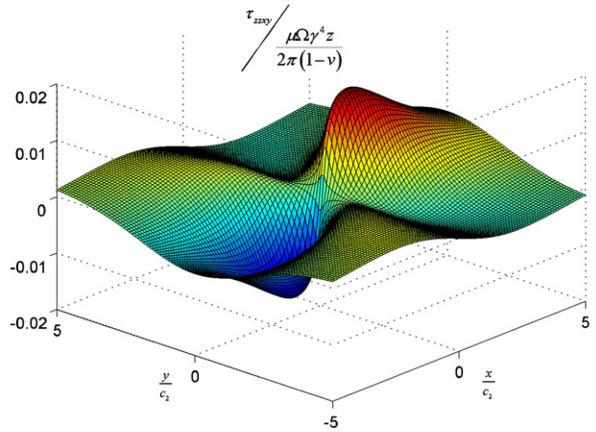

Fig. 18. τ_{zzxy}.

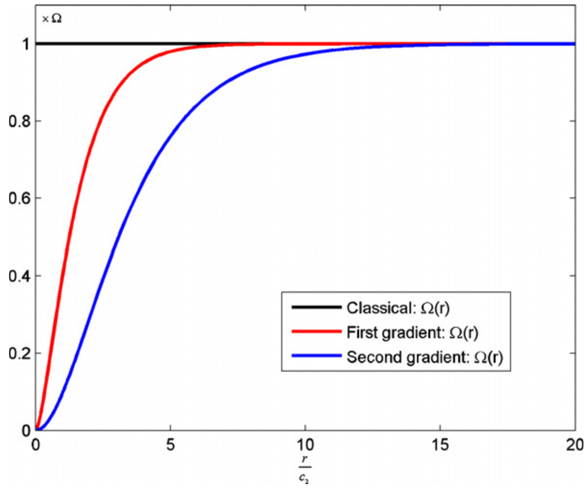

Fig. 19. The effective Frank vectors.

$$\kappa_{yx} = \frac{1}{2\mu}[\partial^2_{xy}F + \partial_z(\partial^2_{yy}f - v\Delta f)], \tag{4.38}$$

$$\kappa_{yy} = -\frac{1}{2\mu}(\partial^2_{xx}F + \partial^2_{xz}g + \partial^3_{zxy}f). \tag{4.39}$$

Substitution of Eqs. (4.17), (4.18), (4.19), (4.38) and (4.39) into Eq. (4.37), yields the expression

$$\Omega_y(r) = \Omega\left\{1 - \frac{r}{c_1^2 - c_2^2}\left[c_1 K_1\left(\frac{r}{c_1}\right) - c_2 K_1\left(\frac{r}{c_2}\right)\right]\right\}. \tag{4.40}$$

The effective Frank vectors in different theories are plotted in Fig. 19. The "modified" Frank vectors have zero value at the disclination line, and they approach to the classical Frank vector Ω as $r \to \infty$.

5. Conclusions

In this paper, disclinations in the framework of the exceptional version of second strain gradient elasticity theory (Lazar et al., 2006) have been solved. Using this theory, we have found new exact analytical solutions for the stress and strain fields of straight wedge and twist disclinations, respectively. The solutions have no singularities unlike the corresponding solutions in classical elasticity and first strain gradient theory. In addition, the double and triple stresses have been investigated, both quantities are nonsingular. Thus, singularities of the double stress which appear in first gradient theory (see, e.g., Lazar and Maugin, 2005) are regularized.

Analytical solutions of wedge and twist disclinations in the framework of second gradient theory indicate that stress and strain fields and their all first- and second-order gradients are nonsingular. Therefore, the second strain gradient theory is self-consistent and gives good physical results (Lazar et al., 2006). Furthermore, it shows that the second-order gradient theory is enough and the higher order gradient theory is not necessary.

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Appendix A. The stress function of wedge disclination under $\varepsilon = \sqrt{2}\gamma$ ($c_1 = c_2 = c$)

For $c_1 = c_2 = \gamma = c$, the stress function becomes

$$f = A\left\{r^2 \ln r + 8c^2 \ln r + 8c^2 K_0\left(\frac{r}{c}\right) + 2crK_1\left(\frac{r}{c}\right)\right\} + Br^2 + 8(A+B)c^2 \tag{A.1}$$

and the effective Frank vector is

$$\Omega_r(r) = \Omega\left\{1 - \frac{r}{c}K_1\left(\frac{r}{c}\right) - \frac{r^2}{2c^2}K_0\left(\frac{r}{c}\right)\right\}. \tag{A.2}$$

Appendix B. The stress functions of twist disclination under $\varepsilon = \sqrt{2}\gamma$ ($c_1 = c_2 = c$)

For $c_1 = c_2 = \gamma = c$, the stress functions become

$$f = -\frac{\mu\Omega}{2\pi(1-v)} zy\left\{\ln r + \frac{4c^2}{r^2} - \frac{4c}{r}K_1\left(\frac{r}{c}\right) + K_0\left(\frac{r}{c}\right)\right\}, \tag{B.1}$$

$$F = -\frac{\mu\Omega}{2\pi(1-v)} x\left\{\ln r + \frac{4c^2}{r^2} - \frac{4c}{r}K_1\left(\frac{r}{c}\right) + K_0\left(\frac{r}{c}\right)\right\}, \tag{B.2}$$

$$g = \frac{\mu\Omega v}{\pi(1-v)} z\left\{\ln r + K_0\left(\frac{r}{c}\right) + \frac{r}{2c}K_1\left(\frac{r}{c}\right)\right\}. \tag{B.3}$$

Correspondingly, the effective Frank vector is

$$\Omega_y(r) = \Omega\left\{1 - \frac{r}{c}K_1\left(\frac{r}{c}\right) - \frac{r^2}{2c^2}K_0\left(\frac{r}{c}\right)\right\}. \tag{B.4}$$

Appendix C. The solution procedure of bi-Helmholtz Equation (3.15′)

For solving Eq. (3.15′), we note

$$g = (1 - c_2^2 \Delta) f, \tag{C.1}$$

and there is

$$(1 - c_1^2 \Delta) g = Ar^2 \ln r + B' r^2. \tag{C.2}$$

The solution of Eq. (C.2) is set as follows

$$g = A \cdot K_0\left(\frac{r}{c_1}\right) + a_1 r^2 \ln r + a_2 r^2 + a_3 \ln r + a_4, \tag{C.3}$$

substitution of Eq. (C.3) into Eq. (C.2), yields the results

$$g = A\left\{K_0\left(\frac{r}{c_1}\right) + r^2 \ln r + 4c_1^2 \ln r\right\} + B' r^2 + 4(A + B')c_1^2. \tag{C.4}$$

Now, we substitute Eq. (C.4) into Eq. (C.1), and we use the following ansatz

$$f = A\left\{b_1 K_0\left(\frac{r}{c_1}\right) + b_2 K_0\left(\frac{r}{c_2}\right)\right\} + b_3 r^2 \ln r + b_4 \ln r + b_5 r^2 + b_6. \tag{C.5}$$

Finally, we obtain the stress function

$$f = A\left\{r^2 \ln r + 4(c_1^2 + c_2^2)\ln r + \frac{4}{c_1^2 - c_2^2}\left[c_1^4 K_0\left(\frac{r}{c_1}\right) - c_2^4 K_0\left(\frac{r}{c_2}\right)\right]\right\} + B' r^2 + 4(A + B')(c_1^2 + c_2^2) \tag{C.6}$$

Appendix D. The classical elastic strain of twist disclinations

The classical elastic strain of twist disclinations is expressed as follows

$$E^0_{xx} = -\frac{\Omega}{4\pi(1-v)}\frac{zy}{r^2}\left\{(1-2v)+\frac{2x^2}{r^2}\right\}, \tag{D.1}$$

$$E^0_{yy} = -\frac{\Omega}{4\pi(1-v)}\frac{zy}{r^2}\left\{(1-2v)-\frac{2x^2}{r^2}\right\}, \tag{D.2}$$

$$E^0_{xy} = \frac{\Omega}{4\pi(1-v)}\frac{zx}{r^2}\left\{1-\frac{2y^2}{r^2}\right\}, \tag{D.3}$$

$$E^0_{zx} = \frac{\Omega}{4\pi(1-v)}\frac{xy}{r^2}, \tag{D.4}$$

$$E^0_{zy} = -\frac{\Omega}{4\pi(1-v)}\left\{(1-2v)\ln r+\frac{x^2}{r^2}\right\}, \tag{D.5}$$

and they contain the "classical" singularities at $r = 0$.

Appendix E. The solution procedure of bi-Helmholtz Equation (4.14)

The equation is rewritten as

$$(1-c_1^2\Delta)(1-c_2^2\Delta)f = y\ln r = r\ln r\cdot\sin\theta, \tag{E.1}$$

where

$$\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}. \tag{E.2}$$

Notes

$$g = (1-c_2^2\Delta)f, \tag{E.3}$$

and there is

$$(1-c_1^2\Delta)g = r\ln r\cdot\sin\theta, \tag{E.4}$$

its solution is given as follows

$$g = \sin\theta\cdot\left\{r\ln r+\frac{2c_1^2}{r}-2c_1K_1\left(\frac{r}{c_1}\right)\right\}. \tag{E.5}$$

substituting Eq. (E.5) into Eq. (E.3) and using the following ansatz

$$f = \sin\theta\cdot\left\{a_0 r\ln r+\frac{a_1}{r}+a_2K_1\left(\frac{r}{c_1}\right)+a_3K_1\left(\frac{r}{c_2}\right)\right\}, \tag{E.6}$$

Finally, we obtain the stress function

$$f = y\left\{\ln r+\frac{2(c_1^2+c_2^2)}{r^2}-\frac{1}{(c_1^2-c_2^2)r}\left[2c_1^3K_1\left(\frac{r}{c_1}\right)-2c_2^3K_1\left(\frac{r}{c_2}\right)\right]\right\}. \tag{E.7}$$

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