Variational principles of second, first and intermediate kinds for non-holonomic mechanics *

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Abstract　　The differential variational principles of second kind for non-holonomic mechanics are given, from which a number of integral variational principles of second kind are set up. From the latter, the general relation of $\delta q'$-δq and the general form of integral variational principles of the first kind and intermediate kinds are derived. Thus not only all previous relations of $\delta q'$-δq and integral variational principles are unified but also the existence of the variational principles of intermediate kinds are pointed out.

Keywords: non-holonomic mechanics, variational principle.

For the non-holonomic mechanics of freedom degree n, in state space, all the generalized coordinates $q = (q_1, \cdots, q_n)$ and generalized velocities $q' = (q'_1, \cdots, q'_n)$ should be looked as basic variables and so the motion equations for q_s

$$\dot{q}_s = q'_s \quad (s = 1, \cdots, n) \tag{1}$$

should be put on a bar with the motion equations for q'_s, i.e. the form of Newton's second law of motion in state space. Here the time differential of q_s has been expressed by $\dot{q}_s$ for short. All the previous variational principles for non-holonomic mechanics[1—4] can give the motions for q' only if those of q (eq. (1)) are accepted beforehand, because only the generalized coordinates are permitted to get independent variation while the generalized velocities are not permitted so. In the present paper, the variational principles from which both the motion equations of q and those of q' can be derived together will be set up for non-holonomic mechanics. The variational principles given in the present paper will be called variational principles of second kind because both the $\delta q'$ and δq are independent, while the previous variational principles for non-holonomic systems are called the variational principles of the first kind. Besides, in non-holonomic mechanics, many different relations of $\delta q'$-δq and different integral variational principles of the first kind were suggested and even many arguments appeared[1—6]. Can all those relations and principles be unified? It will be shown in the present paper that our variational principles of the second kind for non-holonomic mechanics can lead to the general relation of $\delta' q$-δq and the general form of integral variational principles of the first kind and intermediate kinds for holonomic and non-holonomic mechanics. Thus not only all previous relations of $\delta' q$-δq and integral variational principles are unified with the result of canceling any possibility of arguments, but also the existance of the variational principles of intermediate kinds are pointed out.

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1 The differential variational principles of the second kind

Consider a mechanical system composed of N mass points subject to m independent, ideal and holonomic constraints

$$W_\alpha(r, t) = 0 \quad (\alpha = 1, \cdots, m < 3N) \tag{2}$$

and g independent, non-holonomic, ideal (in Appell-Chetaev's sense[1—3]) constraints of first degree

$$G_\beta(r, r', t) = 0 \quad (\beta = 1, \cdots, g < 3N - m), \tag{3}$$

where $r = (r_1, \cdots, r_N)$, $r' = (r'_1, \cdots, r'_N)$, with r_i and r'_i being the position vector and velocity vector of ith mass point respectively. The forces of constraint R_i of ideal constraints possess the property: for vector elements $\delta r(\delta r_1, \cdots, \delta r_N)$ satisfying

$$\sum_{i=1}^{N} \frac{\partial W_\alpha}{\partial r_i} \cdot \delta r_i = 0 \quad (\alpha = 1, \cdots, m) \tag{4}$$

and Appell-Chetaev's conditions[1—3]

$$\sum_{i=1}^{N} \frac{\partial}{\partial r'_i} g_\beta(r, r', t) \cdot \delta r_i = 0 \quad (\beta = 1, 2, \cdots, g), \tag{5}$$

we have

$$\sum_{i=1}^{N} R_i \cdot \delta r_i = 0. \tag{6}$$

This is equivalent to that the forces of constraint R_i exerted upon ith mass point can be expressed as[3]

$$R_i = \sum_{\alpha=1}^{m} \mu_\alpha(t) \frac{\partial W_\alpha}{\partial r_i} + \sum_{\beta=1}^{g} \zeta_\beta(t) \frac{\partial}{\partial r'_i} G_\beta(r, r', t) \quad (i = 1, \cdots, N), \tag{7}$$

where the coefficients μ_α and ζ_β depend on temporary dynamical states. In terms of state variables, Newton's second law of motion can be written as

$$F_i + R_i - m_i \frac{\mathrm{d}}{\mathrm{d}t} r'_i = 0 \quad (i = 1, \cdots, N), \tag{8}$$

$$\dot{r}_i - r'_i = 0 \quad (i = 1, \cdots, N), \tag{9}$$

where m_i and F_i are the mass of ith mass point and the active force exerted upon it respectively.

Theorem 1 (Newtonian representation of generalized D'Alembert-Lagrange's principle (differential variational principles of second kind)). *The sufficiently smooth functions $(r(t), r'(t))$ defined on time interval $[t_0, t_1]$ is an actual motion of constrained Newtonian system represented by eqs. (2), (3) and (7)—(9), if and only if it satisfies eqs. (2), (3) and*

$$\sum_{i=1}^{N} \left(F_i - m_i \frac{\mathrm{d}}{\mathrm{d}t} r'_i \right) \cdot \delta r_i + \sum_{i=1}^{N} m_i (\dot{r}_i - r'_i) \cdot \delta r'_i = 0 \tag{10}$$

for every moment of time $t \in [t_0, t_1]$, all vector elements $\delta r'(\delta r'_1, \cdots, \delta r'_N)$ and all vector elements δr satisfying conditions (4) and (5).

Proof. We first prove the necessity of the conditions. Suppose that $(r(t), r'(t))$ is an actual motion of constrained Newtonian system represented by eqs. (2), (3), (7)—(9) on time interval $[t_0, t_1]$. Then it satisfies conditions (2) and (3). For any vector elements $\delta r'$ and any vector elements δr satisfying conditions (4) and (5), we have eq. (6) from eq. (7). Eqs. (6), (8) and (9) give (10). This completes the proof of the necessity of the conditions.

Now we prove the sufficiency of the conditions. The conditions of the theorem give (2) and (3).

Since r in (10) satisfies (4) and (5), we can use Lagrange's multiplier method with undetermined multipliers μ_α and ζ_β to obtain (8) and (9). The sufficiency of the conditions has been proven.

Now we are going to express this principle in terms of generalized state coordinates (q, q'). The number of the generalized configuration coordinates q_s on the intersection of hypersurfaces (2) is $n = 3N - m$, i.e. whenever r satisfies eq. (2), we have generalized configuration coordinates $q_i = q_i(r, t)$ $(i = 1, \cdots, n)$. For the sake of convenience, for every $t \in (t_0, t)$ we extend smoothly and invertiblly these n generalized configuration coordinates on the intersection of hypersurfaces (2) to an open field $\Sigma(t)$ in state space (r, r') containing the intersection themselves, and then in $\Sigma(t)$ we define m auxiliary configuration coordinates $q_{n+\alpha} = W_\alpha(r_1, \cdots, r_N, t)(\alpha = 1, \cdots, m)$. The generalized-velocities coordinates $q'(q'_1, \cdots, q'_{3N})$ are defined as

$$q'_s = \frac{\partial q_s}{\partial t} + \sum_{i=1}^{N} \frac{\partial q_s}{\partial r_i} \cdot r'_i \quad (s = 1, \cdots, 3N). \tag{11}$$

The inverse transformation is

$$r_i = r_i(q, t), \quad r'_i = \frac{\partial r_i}{\partial t} + \sum_{s=1}^{3N} \frac{\partial r_i}{\partial q_s} q'_s \quad (i = 1, \cdots, N). \tag{12}$$

From eqs. (10)—(12), we can easily prove the new relations

$$\frac{\partial r'_i}{\partial q'_s} = \frac{\partial r_i}{\partial q_s}, \quad \frac{d}{dt} \frac{\partial r_i}{\partial q_s} = \frac{\partial r'_i}{\partial q_s} + \sum_{k=1}^{r} (\dot{q}_k - q'_k) \frac{\partial^2 r_i}{\partial q_s \partial q_k} \quad \binom{i = 1, \cdots, N}{s = 1, \cdots, n} \tag{13}$$

corresponding to Lagrange relations and the new central equation

$$\sum_{s=1}^{n} \left(-\frac{d}{dt} \frac{\partial T}{\partial q'_s} + \frac{\partial T}{\partial q_s} + Q_s \right) \delta q_s + \sum_{s=1}^{n} (\dot{q}_s - q'_s) \delta \frac{\partial T}{\partial q'_s} = 0, \tag{14}$$

where the definitions of kinetic energy T and generalized force $Q_s \left(T = \sum_{i=1}^{N} \frac{1}{2} m_i r'^2_i \text{ and } Q_s = \sum_{i=1}^{N} F_i \right.$

$\left. \cdot \frac{\partial r_i}{\partial q_s} \right)$ are used. Due to condition (2), s and k in (13) and (14) only take 1 to n but not 1 to $3N$ (From now on, q and q' have only n components). Eq. (14) becomes

$$\sum_{s=1}^{n} \left(-\frac{d}{dt} \frac{\partial}{\partial q'_s} L + \frac{\partial}{\partial q_s} L + Q'_s \right) \delta q_s + \sum_{s=1}^{n} (\dot{q}_s - q'_s) \delta \frac{\partial L}{\partial q'_s} = 0, \tag{15}$$

where L is Lagrangian $L = T - V$ containing the contributions of all conservative active forces and Q'_s $(s = 1, \cdots, n)$ are possible non-conservative forces. Based on these results with further trivial derivation, we obtain the following statement:

Theorem 2 (Lagrange representation of generalized D'Alembert-Lagrange's principle). *The sufficiently smooth function $(q(t), q'(t))$ defined time interval $[t_0, t_1]$ is an actual motion of Lagrange system with function L and constraint conditions*

$$f_\beta(q, q', t) = G_\beta(r(q, t), r'(q, q', t), t) = 0 \quad (\beta = 1, \cdots, g) \tag{16}$$

(i.e. conditions (2) and (3)), if and only if it satisfies (16) and (15) for every moment of time $t \in [t_0, t_1]$, all $\delta q'$ and all δq satisfying conditions

$$\sum_{s=1}^{n} \frac{\partial f_\beta}{\partial q'_s} \delta q_s = 0 \quad (\beta = 1, \cdots, g). \tag{17}$$

2 Integral variational principles of second kind

By using the obvious relation $\delta \dot{q}_s = d/dt \delta q_s$, eq. (14) can be written as

$$- \frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{s=1}^{n} \frac{\partial T}{\partial q'_s} \delta q_s \right) + \delta T + \delta \sum_{s=1}^{n} (\dot{q}_s - q'_s) \frac{\partial T}{\partial q'_s} + \sum_{s=1}^{n} Q'_s \delta q_s = 0. \qquad (18)$$

Now let us extend $q(t)$ and $q'(t)$ to t_0 and t_1 continuously. By using the fundamental theorem for variational method[7], we can easily prove that (18) for time interval (t_0, t_1) is equivalent to

$$\int_{t_0}^{t_1} \left\{ \delta \left[L + \sum_{s=1}^{n} (\dot{q}_s - q'_s) \frac{\partial L}{\partial q'_s} \right] + \sum_{s=1}^{n} Q'_s \delta q_s \right\} \mathrm{d}t - \sum_{s=1}^{n} \frac{\partial L}{\partial q'_s} \delta q_s \Big|_{t_0}^{t_1} = 0, \qquad (19)$$

which is called "partial variational equation" because of the restrictions (17) on δq. Thus we have

Theorem 3 (Lagrangian representation of integral variational principle of second kind with free end-points). *The sufficiently smooth function $(q(t), q'(t))$ defined on time interval $[t_0, t_1]$ is an actual motion of Lagrange system with constraint conditions (16) (i.e. conditions (2) and (3)), if and only if it satisfies eq. (16) for every moment of time $t \in [t_0, t_1]$ and the partial variational equation (19) with all $\delta q'$ and all δq satisfying conditions satisfy condition (17).*

Under the end-point conditions

$$\delta q_s(t_0) = \delta q_s(t_1) = 0 \quad (s = 1, \cdots, n) \qquad (20)$$

or under condition (20) and

$$\delta q'_s(t_0) = \delta q'_s(t_1) = 0 \quad (s = 1, \cdots, n), \qquad (21)$$

generalized D'Alembert-Lagrange's principle remains tenable in the open interval (t_0, t_1), and so the integral variational principle also remains tenable; only the forms of eq. (19) become

$$\int_{t_0}^{t_1} \left\{ \delta \left[L + \sum_{s=1}^{n} (\dot{q}_s - q'_s) \frac{\partial L}{\partial q'_s} \right] + \sum_{s=1}^{n} Q'_s \delta q_s \right\} \mathrm{d}t = 0. \qquad (22)$$

Thus we have the following statement:

Theorem 4 (Lagrange representation of integral variational principle of second kind with half-fixed (or fixed, end points)). *The sufficiently smooth function $(q(t), q'(t))$ defined on time interval $[t_0, t_1]$ is an actual motion of Lagrange system with constraint condition (16) (i.e. conditions (2) and (3)), if and only if it satisfies eq. (16) for every moment of time $t \in [t_0, t_1]$ and the partial variational equation (22) with all $\delta q'$ (or satisfying condition (21)) and all δq satisfying conditions (17) and (20).*

By using Lagrange's multiplier (λ_β) method and considering the non-singularity of Hertz matrix $\frac{\partial^2 T}{\partial q' \partial q'}$, the above integral variational principle of the second kind gives the complete motion equations in state space: eqs. (16) and

$$- \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial q'_s} + \frac{\partial T}{\partial q_s} + Q_s + \sum_{\beta=1}^{g} \lambda_\beta \frac{\partial f_\beta}{\partial q'_s} = 0 \quad (s = 1, \cdots, n), \qquad (23)$$

$$\dot{q}_s = q'_s \quad (s = 1, \cdots, n). \qquad (24)$$

Substituting (24) into (23) gives well-known Routh equations[1—3], which also shows the correctness of our variational principles of the second kind.

3 Hamilton representation of the variational principle of the second kind

It this section we suppose that all the active forces are conservative. In view of the non-singularity of Hertz matrix $\frac{\partial^2 T}{\partial q' \partial q'}$, we can introduce the Legendre's transformation:

$$p_s = \frac{\partial}{\partial q_s'} L(q, q', t) \ (s = 1, \cdots, n), \tag{25}$$

$$H(q, p, t) = \sum_{s=1}^{n} p_s q_s' - L(q, q', t). \tag{26}$$

These two equations give

$$q_s' = q_s'(q, p, t) = \frac{\partial H}{\partial p_s} \ (s = 1, \cdots, n), \tag{27}$$

and so (19), (22), (16), (17) and (21) become

$$\int_{t_0}^{t_1} \delta\left(\sum_{s=1}^{n} p_s \dot{q}_s - H \right) \mathrm{d}t - \sum_{s=1}^{n} p_s \delta q_s \Big|_{t_0}^{t_1} = 0, \tag{28}$$

$$\int_{t_0}^{t_1} \delta\left(\sum_{s=1}^{n} p_s \dot{q}_s - H \right) \mathrm{d}t = 0, \tag{29}$$

$$u_\beta(q, p, t) = f_\beta(q, q', t) = 0 \ (\beta = 1, \cdots, g), \tag{30}$$

$$\sum_{s=1}^{n} \sum_{k=1}^{n} \frac{\partial u_\beta}{\partial p_k} \left[\frac{\partial^2 H}{\partial p \partial p} \right]_{ks}^{-1} \delta q_s = 0 \ (\beta = 1, \cdots, g), \tag{31}$$

$$\delta p_s(t_0) = \delta p_s(t_1) = 0 \ (s = 1, \cdots, n), \tag{32}$$

respectively. Thus we obtain the following two statements:

Theorem 5 (Hamilton representation of variational principle of the second kind with free end-points). *The smooth function* $(q(t), p(t))$ *defined in time interval* (t_0, t_1) *is an actual motion of Hamilton system under condition* (30) *(i. e. conditions* (2) *and* (3)), *if and only if it satisfies* (30) *for every moment of time* $t \in (t_0, t_1)$ *and the partial variational equation* (28) *with all the smooth variations* δp *and all the* δq *satisfying condition* (31).

Theorem 6 (Hamilton representation of variational principle of second kind with half-fixed (or fixed) end points). *The smooth function* $(q(t), p(t))$ *defined in time interval* (t_0, t_1) *is an actual motion of Hamilton system under condition* (30) *(i. e. conditions* (2) *and* (3)), *if and only if it satisfies* (30) *for every moment of time* $t \in (t_0, t_1)$ *and the partial variational equation* (29) *with all the smooth variations* δp *(or satisfying condition* (32)) *and all the* δq *satisfying conditions* (31) *and* (20).

Although all the variational principles given above are related to Newton system, it is easy to prove that Lagrange representation and Hamilton representation of the variational principles with the words related to Newton system deleted still exist for Lagrange systems and Hamilton systems themselves respectively. Besides, when the non-holonomic constraint conditions do not exist, our variational principles will reduce to those for holonomic mechanics, with the well-known Livens Theorem[8] and "Modified Hamilton Principles"[9, 10] being two examples.

4 Leading to integral variational principles of first kind and intermediate kinds

Since the actural motion condition (1) is a deduction of any principle of the second kind, we will get a new principle equivalent to the original one if condition (1) is put on the trial actual motion in the original principle. Then in the new principle, combining condition (1) of the trial actual motion with partial variational equation (22) we obtain a new partial variational equation

$$\int_{t_0}^{t_1} \left[\delta L + \sum_{s=1}^{n} \left(\frac{\partial L}{\partial q_s'} (\delta \dot{q}_s - \delta q_s') + Q_s' \delta q_s \right) \right]_{q' = q} \mathrm{d}t = 0. \tag{33}$$

Notice that the subscript $q' = \dot{q}$ is only put on trial actual motion, but not on any comparative motion, so $\delta q'_s$ is still independent variation (of course the end-point condition must be satisfied). Therefore the new principle still belongs to those of second kind. However, expanding δL in (33) shows that the terms containing $\delta q'_s$ are canceled out so that all selection methods of $\delta q'_s$ are equivalent. Thus in the new variational principle, imposing any linear relation (called general relation of $\delta q' - \delta q$)

$$\delta q'_s = A_s(\delta q) \ (s \in \{1, \cdots, n\}) \tag{34}$$

(here, A_s may depend on trial actual motiom q, q' and t) on $\delta q'$ is equivalent to permitting $\delta q'$ to be independent. But the new variational principle with condition (34) just belongs to those of the first kind (for the number M_s of s is n) or intermediate kinds (for $0 < M_s < n$). Therefore we have

Theorem 7 (Lagrange representation for general form of variational principle of first and intermediate kinds with half-fixed (or fixed) end points). *The sufficiently smooth function* $(q(t), q'(t))$ *defined on time interval* $[t_0, t_1]$ *is an actual motion of Lagrange system with constraint condition* (16), *if and only if it satisfies eq.* (16) *for every moment of time* $t \in [t_0, t_1]$ *and the partial variational equation* (33) *with* $(\delta q, \delta q')$ *satisfying condition* (34). *Appell-Chetaev conditions* (17) *and end-point conditions* (20) *(or plus equation* (21)).

With each specified form of A_s, a particular variational principle is obtained. It is worth noticing that as shown by the above deduction, the form of A_s can be independent of constraint condition (16). Besides, when the non-holonomic constraint conditions do not exist, the general form of the variational principles of the first kind and intermediate kinds will reduce to those for holonomic mechanics.

Take some examples as follows: (i) Let eq. (34) take the form $\delta q' = 0$. Then eq. (33) takes the

partial variational form $\int_{t_0}^{t_1} \left[\delta L(q, q', t) + \sum_{s=1}^{n} \left(\frac{\partial L}{\partial q'_s} \delta q_s + Q'_s \delta q_s \right) \right]_{q'=\dot{q}} \mathrm{d}t = 0$, whose form has not

appeared in literature. (ii) Under the condition that there are no non-conservative active forces, let eq. (34) take the form of $\delta q' = \frac{\mathrm{d}}{\mathrm{d}t} \delta q$. Then eq. (33) takes Hölder partial variational form[1—4] $\int_{t_0}^{t_1} [\delta L(q, q', t)]_{q'=\dot{q}} \mathrm{d}t = 0$. (iii) Under the condition that there are no non-conservative active forces, let (34) take Susrov's form. Then (33) takes Susrov's partial variational form[1—4]. (iv) Let (34) take the form $\delta q'_s = \delta \dot{q}_s + \left(Q'_s \big/ \frac{\partial L}{\partial q'_s} \right) \delta q_s$ ($s = 1, \cdots, n$). Then (33) becomes $\int_{t_0}^{t_1} [\delta L(q, q', t)]_{q'=\dot{q}} \mathrm{d}t = 0$.

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