Interacting Circular Inhomogeneities

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ABSTRACT

This paper presents a technique, the pseudo-dislocations method, to calculate the elastic field in a plane which contains numerous circular inclusions under far loadings. The method is as accurate and simple for interacting inhomogeneities as the widely used pseudo-tractions method for interacting defects.

1. INTRODUCTION

In the meso-scope scale, most solid materials are not homogeneous, but heterogeneous, containing many microstructures, such as inclusions, inhomogeneities, voids and cracks. The most commonly used approaches for interaction effects of microstructures are indirect methods, that is some kind of "effective medium methods". In recent years, various approximate methods have been developed for treating the direct interactions problem of multiple cracks and voids. But studies of interacting inhomogeneities other than cracks and holes, are relatively few [1-3].

This paper presents a general technique for the calculation of the stress and strain fields in an elastic media which contains numerous circular inhomogeneities under arbitrary far field loadings. We take any one inhomogeneity for consideration. First we suppose the inhomogeneity domain is homogeneous (replacing the inhomogeneity with the matrix), calculate the elastic fields in the domain due to remote fields and the other inhomogeneities perturbation. Then the matrix in the domain is replaced with the inhomogeneity, but the domain is under an unchangeable stresses state. So the surface tractions continuity condition is preserved, but the strains state will be different from that of the former. There will be pseudo displacements mismatch at the inhomogeneity-matrix interface. To eliminate this pseudo displacements mismatch, we add a perturbation displacements mismatch along the interface. We call the method "pseudo-dislocations method".

The method is as accurate and simple for interacting inhomogeneities as the widely used pseudo-tractions method for interacting defects [4], but more general, for defects can be considered as weak inhomogeneities. For convenience, the method is outlined for interacting multiple circular inhomogeneities in an infinite elastic plane. The circular inhomogeneities can be of any radii, spacing, elastic moduli, and of any kind of distribution.

2. BASIC EQUATIONS AND FORMULATIONS

The boundary value problem which will be dealt with in this work is most effectively formulated in terms of the complex stress potentials. In polar coordinates the basic equations

for two-dimensional classical linear elasticity are:

$$\sigma_r + \sigma_\theta = 2[\Phi(z) + \overline{\Phi(z)}] \quad (1)$$

$$\sigma_r + i\sigma_{r\theta} = \Phi(z) + \overline{\Phi(z)} - e^{-2i\theta}[z\overline{\Phi'(z)} + \overline{\Psi(z)}] \quad (2)$$

$$2\mu\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right) = iz[\kappa\Phi(z) - \overline{\Phi(z)} + \overline{z}\overline{\Phi'(z)} + \frac{\overline{z}}{z}\overline{\Psi(z)}] \quad (3)$$

where μ is the shear modulus, and $\kappa = 3 - 4\nu$ for plane strain or $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, ν being Poisson's ratio.

A boundary $z\overline{z} = a^2$ cuts the whole plane into two parts. Let superscripts I denotes the circular region or S^I , and 0 the remainder region or S^0 respectively. Elastic quantities in different regions are denoted by subscripts. In the usual manner, we extend the definitions of the holomorphic functions $\Phi^i(z)$ ($z \in S^0$ for $i = 0$ and $z \in S^I$ for $i = I$) in one region into the other region by the substitution

$$\Phi^i\left(\frac{a^2}{\overline{z}}\right) = -\overline{\Phi^i(\overline{z})} + \overline{z}\overline{\Phi^{i'}(\overline{z})} + \left(\frac{\overline{z}^2}{a^2}\right)\overline{\Psi^i(\overline{z})}, \quad \begin{cases} z \in S^0 & \text{for } i = 0, \\ z \in S^I & \text{for } i = I. \end{cases} \quad (4)$$

Then the elastic fields in the two regions can be expressed by

$$\sigma_r^i + \sigma_\theta^i = 2[\Phi^i(z) + \overline{\Phi^i(z)}] \quad (5)$$

$$\sigma_r^i + i\sigma_{r\theta}^i = \Phi^i(z) - \Phi^i\left(\frac{a^2}{\overline{z}}\right) + \overline{z}\left(\frac{\overline{z}}{a^2} - \frac{1}{z}\right)\overline{\Psi^i(z)} \quad (6)$$

$$2\mu_i\left(\frac{\partial u^i}{\partial \theta} + i\frac{\partial v^i}{\partial \theta}\right) = iz[\kappa_i\Phi^i(z) + \Phi^i\left(\frac{a^2}{\overline{z}}\right) - \overline{z}\left(\frac{\overline{z}}{a^2} - \frac{1}{z}\right)\overline{\Psi^i(z)}] \quad (7)$$

where $z \in S^0$ for $i = 0$, $z \in S^I$ for $i = I$.

Since $\Phi^i(z)$ and $\Psi^i(z)$ are holomorphic in the region S^i , $\Phi^0(z)$ is holomorphic in the region S^I except at the point $z = 0$, where it may have a pole of order two and therefore its principal part has the form $c_0 + c_1/z + c_2/z^2$. On the other hand, $\Phi^I(z)$ is holomorphic at $z = \infty$ and its principal part there is a constant.

In Eq. (6), letting z tend to a point $t (= ae^{i\theta})$ of the boundary from S^0 for $i = 0$ and S^I for $i = I$, the continuity condition of the surface tractions are expressed as

$$\Phi^{0+}(t) - \Phi^{0-}(t) = \Phi^{I-}(t) - \Phi^{I+}(t) \quad (8)$$

where $\Phi^{i+}(t)$ and $\Phi^{i-}(t)$ are the limiting values of $\Phi^i(z)$ as $z \rightarrow t$ from S^0 and S^I , respectively. Then we have

$$\Phi^{0+}(t) + \Phi^{I+}(t) = \Phi^{0-}(t) + \Phi^{I-}(t) \quad (9)$$

Since $\Phi^0(z)$ and $\Phi^I(z)$ are holomorphic at infinity we may conclude that $\Phi^0(z) + \Phi^I(z)$ is holomorphic over the entire plane excluding the origin $z = 0$, where $\Phi^0(z)$ may have a pole. Therefore, we have $\Phi^0(z) + \Phi^I(z) = c_0 + c_1/z + c_2/z^2$.

The differentiation of the displacement mismatch on the boundary $|z| = a$ with respect to the boundary coordinate $t = ae^{i\theta}$, that is $d[u^0 - u^I]/dt + id[v^0 - v^I]/dt$, is called dislocation here, and expressed as $f(t)$. Using Eq. (7), we have

$$\frac{1}{2\mu_0}[\kappa_0\Phi^{0+}(t) + \Phi^{0-}(t)] - \frac{1}{2\mu_I}[\kappa_I\Phi^{I-}(t) + \Phi^{I+}(t)] = f(t) \quad (10)$$

It may be further expressed as

$$\Phi^{0+}(t) + \gamma\Phi^{0-}(t) = \beta(c_0 + c_1/t + c_2/t^2) + \alpha f(t) \quad (11)$$

where γ , β and α are bi-elastic constants defined by

$$\gamma = (\mu_I + \kappa_I\mu_0)/(\mu_0 + \kappa_0\mu_I), \quad \beta = \mu_0(1 + \kappa_I)/(\mu_0 + \kappa_0\mu_I), \quad \alpha = 2\mu_0\mu_I/(\mu_0 + \kappa_0\mu_I). \quad (12)$$

3. MULTIPLE CIRCULAR INHOMOGENEITIES

Fig. 1 Arbitrarily located inhomogeneities

This section presents the pseudo-dislocations approach for multiple circular inhomogeneities. Consider an infinitely extended solid containing N distinct circular inhomogeneities under far field uniform stresses, as shown in Fig. 1. Let (x_j, y_j) and (r_j, θ_j) denote the rectangular Cartesian and pole coordinate system with origin O_j at the j th circular inhomogeneity center whose radius is a_j , ($j = 1, 2, \dots, N$). Let the distance between O_j and O_k be d_{kj} , and the angle measured from the x_j -direction to the O_jO_k -direction be denoted by ϕ_{kj} .

Let subscripts 0 and j ($j = 1, 2, \dots, N$) on elastic quantities refer to the matrix and the j th inhomogeneity, respectively. The components at the j th local coordinates are indicated with the subscripts j ($j = 1, 2, \dots, N$), and the superscripts I and 0 on them refer to the quantities in the inhomogeneity domain S_j^0 and its remaining domain S_j^I which is circumferenced by closed contour C_j , respectively.

Assuming that inhomogeneities are perfectly bounded to the matrix, the surface displacements and tractions continuity conditions have to be satisfied at each inhomogeneity-matrix interface.

We take any one inhomogeneity for consideration. First supposing the inhomogeneity domain is homogeneous with the matrix, calculate the elastic fields in the domain due to remote fields and the other inhomogeneities perturbation. Then in the domain replacing matrix with the inhomogeneity, let the inhomogeneity be under the same stresses state as before. Then the surface tractions continuity condition is preserved, but the strains will be different from that of the former. There will be pseudo displacements mismatch at the inhomogeneity-matrix interface. To eliminate this pseudo displacements mismatch, that is to keep the surface displacements continuity condition at the interface, we add a perturbation displacements mismatch along the interface. The differentiation of the displacements mismatch with respect to the interface curvilinear coordinate is called dislocations. We call the method "pseudo-dislocations method".

We suppose there exists a distribution of dislocations $f_j(t_j)$ along the interface $t_j = a_j e^{i\theta_j}$. From the last section, we know that the potential Φ_j^0 for the perturbation field should obey

$$\Phi_j^{0+}(t_j) + \gamma_j \Phi_j^{0-}(t_j) = \beta_j(c_{j0} + c_{j1}/t_j + c_{j2}/t_j^2) + \alpha_j f_j(t_j) \quad (13)$$

where γ_j , β_j and α_j are the same with (12) but with κ_I and μ_I are replaced by κ_j and μ_j respectively.

We expand the dislocation function $f_j(t_j)$ into a Fourier series as

$$f_j(t_j) = \sum_{n=-\infty}^{\infty} A_{jn} e^{in\theta_j} \quad (14)$$

The solution of Eq. (13) is

$$\Phi_j^0(z_j) = \begin{cases} \beta_j \left(\frac{c_{j1}}{z_j} + \frac{c_{j2}}{z_j^2} \right) + \alpha_j \sum_{n=-\infty}^{-1} A_{jn} \left(\frac{z_j}{a_j} \right)^n & \text{for } z_j \in S_j^0, \\ \frac{1}{\gamma_j} [\beta_j c_{j0} + \alpha_j \sum_{n=0}^{\infty} A_{jn} \left(\frac{z_j}{a_j} \right)^n] & \text{for } z_j \in S_j^I. \end{cases} \quad (15)$$

And $\Phi_j^I(z_j)$ can be expressed as

$$\Phi_j^I(z_j) = \begin{cases} c_{j0} + \frac{c_{j1}}{z_j} + \frac{c_{j2}}{z_j^2} - [\beta_j \frac{c_{j1}}{z_j} + \beta_j \frac{c_{j2}}{z_j^2} + \alpha_j \sum_{n=-\infty}^{-1} A_{jn} (\frac{z_j}{a_j})^n] & \text{for } z_j \in S_j^0, \\ c_{j0} + \frac{c_{j1}}{z_j} + \frac{c_{j2}}{z_j^2} - \frac{1}{\gamma_j} [\beta_j c_{j0} + \alpha_j \sum_{n=0}^{\infty} A_{jn} (\frac{z_j}{a_j})^n] & \text{for } z_j \in S_j^I. \end{cases} \quad (16)$$

The single valuedness of the displacements mismatch requires that the following condition be satisfied:

$$\oint_{C_j} f_j(t_j) dt_j = 0 \quad (17)$$

From it we have

$$A_{j-1} = 0 \quad (18)$$

According to the behaviour of $\Phi_j^0(z_j)$, $\Phi_j^I(z_j)$ and $\Psi_j^I(z_j)$ at infinity and near $z_j = 0$, We obtain:

$$c_{j1} = 0, \quad c_{j2} = 0, \quad (\beta_j c_{j0} + \alpha_j A_{j0})/\gamma_j = 2\text{Rec}_{j0} \quad (19)$$

From equation (3), the pseudo-dislocations on the j th inhomogeneity-matrix interface $t_j = a_j e^{i\theta_j}$, produced by stresses at infinity are

$$\begin{aligned} & (d[u_j^0 - u_j^I]/dt_j + i d[v_j^0 - v_j^I]/dt_j)_{P0} = \\ & \frac{1}{2\mu_0} [\kappa_0 \Phi_0(t_j) - \overline{\Phi_0(t_j)} + \overline{t_j} \Phi_0'(t_j) + \frac{\overline{t_j}}{t_j} \overline{\Psi_0(t_j)}] \\ & - \frac{1}{2\mu_j} [\kappa_j \Phi_0(t_j) - \overline{\Phi_0(t_j)} + \overline{t_j} \Phi_0'(t_j) + \frac{\overline{t_j}}{t_j} \overline{\Psi_0(t_j)}] \end{aligned} \quad (20)$$

where $\Phi_0 = \Gamma$, $\Psi_0 = \Gamma'$ and $\Gamma = (\sigma_x^\infty + \sigma_y^\infty)/4$, $\Gamma' = (\sigma_y^\infty - \sigma_x^\infty)/2 + i\sigma_{xy}^\infty$.

Similarly, the pseudo-dislocations on the j th inhomogeneity-matrix interface, produced by the perturbation fields of k th inhomogeneity are

$$\begin{aligned} & (d[u_j^0 - u_j^I]/dt_j + i d[v_j^0 - v_j^I]/dt_j)_{Pk} = \\ & \frac{1}{2\mu_0} [\kappa_0 \Phi_k(t_j) - \overline{\Phi_k(t_j)} + \overline{t_j} \Phi_k'(t_j) + \frac{\overline{t_j}}{t_j} \overline{\Psi_k(t_j)}] \\ & - \frac{1}{2\mu_j} [\kappa_j \Phi_k(t_j) - \overline{\Phi_k(t_j)} + \overline{t_j} \Phi_k'(t_j) + \frac{\overline{t_j}}{t_j} \overline{\Psi_k(t_j)}] \end{aligned} \quad (21)$$

$\Phi_k(z_j)$ and $\Psi_k(z_j)$ are the transformed stress potentials of $\Phi_k^0(z_k)$ and $\Psi_k^0(z_k)$ from the $x_k O_k y_k$ local coordinate system changing to the $x_j O_j y_j$ system. The formulae of the transform are

$$\Phi_k(z_j) = \Phi_k^0(z_k), \quad \Psi_k(z_j) = \Psi_k^0(z_k) + \overline{z_{ojk}} \Phi_k'(z_j) \quad (22)$$

where z_{ojk} is the coordinate of the point O_j in the $x_k O_k y_k$ system. Obviously, $z_k = z_j + z_{ojk}$, and $z_{ojk} = d_{jk} e^{i\phi_{jk}}$.

We express $\Phi_k^0(z_k)$ and $\Psi_k^0(z_k)$ in terms of a series as

$$\begin{aligned} \Phi_k^0(z_k) &= \alpha_k \sum_{m=-\infty}^{-2} A_{km} \left(\frac{z_k}{a_k}\right)^m \\ \Psi_k^0(z_k) &= \alpha_k \left(\frac{a_k}{z_k}\right)^2 \left[\sum_{m=-\infty}^{-2} (1-m) A_{km} \left(\frac{z_k}{a_k}\right)^m + \frac{1}{\gamma_k} \sum_{m=1}^{\infty} \overline{A_{km}} \left(\frac{z_k}{a_k}\right)^{-m} + \frac{2}{\alpha_k} \text{Rec}_{k0} \right] \end{aligned} \quad (23)$$

and then

$$\begin{aligned}\Phi_k(z_j) &= \alpha_k \sum_{m=-\infty}^{-2} A_{km} \left(\frac{z_j + z_{ojk}}{a_k} \right)^m \\ \Psi_k(z_j) &= \alpha_k \sum_{m=-\infty}^{-2} (1-m) A_{km} \left(\frac{z_j + z_{ojk}}{a_k} \right)^{m-2} + \frac{\alpha_k}{\gamma_k} \sum_{m=1}^{\infty} \bar{A}_{km} \left(\frac{z_j + z_{ojk}}{a_k} \right)^{-m-2} \\ &\quad + 2 \operatorname{Re} c_{k0} \left(\frac{z_j + z_{ojk}}{a_k} \right)^{-2} + \alpha_k \sum_{m=-\infty}^{-2} m A_{km} \left(\frac{z_j + z_{ojk}}{a_k} \right)^{m-1} \left(\frac{\bar{z}_{ojk}}{a_k} \right)\end{aligned}\quad (24)$$

Let $z_j = t_j = a_j e^{i\theta_j}$ in (24), and substitute it into (21), the pseudo-dislocations on the j th inhomogeneity-matrix interface produced by k th inhomogeneity are expressed in terms of a series with complex unknowns A_{km} .

On inhomogeneity-matrix interfaces, to keep the displacement continuity conditions, the added perturbation distribution of dislocations must eliminate the pseudo-dislocations, produced by remote loadings and the perturbation fields of all other inhomogeneities. That is

$$f_j(t_j) + \sum_{\substack{k=0 \\ k \neq j}}^N (d[u_j^0 - u_j^I]/dt_j + i d[v_j^0 - v_j^I]/dt_j)_{Pk} = 0, \quad j = 1, \dots, N \quad (25)$$

Substituting (14), (21) and (24) into (25), and comparing the coefficients of $e^{in\theta_j}$, we obtain a system of algebraic equations for unknowns A_{jn} ,

$$-A_{jn} = \sum_{\substack{k=1 \\ k \neq j}}^N \left\{ \alpha_k \eta_j \sum_{m=-\infty}^{\infty} [B_{nm}^{jk} A_{km} + C_{nm}^{jk} \bar{A}_{km}] \right\} + D_n^j \quad (26)$$

where

$$\begin{aligned}B_{nm}^{jk} &= \begin{cases} \frac{\delta_j}{\eta_j} \bar{I}_{n,-m}^{jk} & \text{for } n \geq 0, m \leq -2, \\ \frac{1}{\gamma_k} I_{-n-2,m+2}^{jk} & \text{for } n \leq -2, m \geq 1, \\ \frac{1}{(2\gamma_k - \beta_k)} I_{-n-2,2}^{jk} & \text{for } n \leq -2, m = 0, \\ 0 & \text{otherwise,} \end{cases} \\ C_{nm}^{jk} &= \begin{cases} -I_{0,-m}^{jk} & \text{for } n = 0, m \leq -2, \\ (1-m) I_{-n-2,2-m}^{jk} - (1+n) I_{-n,-m}^{jk} & \text{for } n \leq -2, m \leq -2, \\ \times \left[1 - \frac{n}{n+m+1} \left(\frac{a_j}{d_{jk}} \right)^{-2} \right] & \text{for } n \leq -2, m \leq -2, \\ \frac{1}{(2\gamma_k - \beta_k)} I_{-n-2,2}^{jk} & \text{for } n \leq -2, m = 0, \\ 0 & \text{otherwise,} \end{cases} \\ D_n^j &= \begin{cases} (\delta_j - \eta_j) \Gamma & \text{for } n = 0, \\ \eta_j \bar{\Gamma}' & \text{for } n \leq -2, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}\quad (27)$$

with

$$\begin{aligned}\delta_j &= \frac{\kappa_0}{2\mu_0} - \frac{\kappa_j}{2\mu_j}, \quad \eta_j = \frac{1}{2\mu_0} - \frac{1}{2\mu_j} \\ I_{n,m}^{jk} &= \frac{(-1)^n (m+n-1)!}{n!(m-1)!} \left(\frac{a_j}{d_{jk}} \right)^n \left(\frac{a_k}{d_{jk}} \right)^m e^{i(n+m)\phi_{jk}}\end{aligned}\quad (28)$$

As seen in (26)-(28), the coefficients A_{jn} are of the order of $(a/d)^{|n+1|+1}$, where $a/d = O(a_j/d_{jk}) = O(a_k/d_{jk})$. Neglecting high order Fourier coefficients, A_{jn} , $n < -M-1$, and $n > M-1$, Eqns (26) become $2M \times N$ complex equations for the $2M \times N$ complex unknowns, $A_{j-M-1}, \dots, A_{j-2}, A_{j0}, \dots, A_{jM-1}$, $j = 1, \dots, N$. Then the equations can be easily solved, and the solution of the original multiple inhomogeneities problem can be obtained by superposition.

4. NUMERICAL EXAMPLES AND DISCUSSION

Application of the proposed pseudo-dislocations method to several typical problems involving interactions of circular inhomogeneities and holes is presented in this section. The method can handle arbitrary distributions of circular inhomogeneities and holes of different sizes, positions, and elastic moduli, and can also handle some kinds of periodic ones.

Fig. 2 Two circular inhomogeneities

To provide an estimation of the accuracy of the present solution, numerical results from the proposed method are compared with existing solutions in literatures.

For two circular holes with radius a and center space d , under far uniform tension, the calculated results as weak inhomogeneities are shown in Table 1, which coincide with those by Horri and Nemat-Nasser [4]. The accuracy of the solution within the indicated figures does not change for N greater than for those indicated. We can see from the table, that this method is as accurate and effective for inhomogeneities as the pseudo-tractions method for defects.

Table 1. The maximum and minimum hoop stresses around the hole.

		Present results				Results in ref. [4]	
d/a	θ_1	σ_θ	(N)	σ_θ	(N)	σ_θ	(N)
2.4	95.41	2.6188	20	2.618770	28	2.6188	22
	180	-0.9048		-0.9047527		-0.9048	
3	94.60	2.6500	13	2.649992	18	2.6500	13
	180	-0.8962		-0.8961552		-0.8962	

For other cases, such as two inhomogeneities of equal sizes and identical elastic moduli under far uniform loadings, the calculated results are identical with those given by Yu and Sendekyj [1].

The present method has high efficiency and accuracy. The solution converges quickly to desired degree of accuracy. It may provide an efficient tool for dealing with micromechanical problems of heterogeneous solid materials, such as for numerically obtaining the effective moduli and directly verifying various kinds of effective medium methods.

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Interacting Circular Inhomogeneities

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