Journal of Applied Mechanics

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Cruciform Rigid Line Problem in an Infinite Plate

Y. Z. Chen[1]

1 Introduction

Recently there has been a resurgence of interest in the rigid line problem. Many works in this field were presented by Wang, Zhang, and Chou (1985), Hasebe and Takeuchi (1985), Dunders and Markenscoff (1989), Dunders (1989), Ballarini (1990), Chen and Hasebe (1992a), and Markenscoff and Li (1996). Characteristics of the stress field near the tip of a rigid line inhomogeneity can be found from an earlier investigation by England (1971). England analyzed the singular behavior of a wedge with the wedge angle $\alpha = \pi$ when the two edges are fixed. The analysis invariably leads to the conclusion that the stresses are singular with the asymptotic expression $\sigma_{ij} \approx O(r^{-1/2})$, where r denotes the distance between the rigid line tip and a point in the vicinity of the tip. The stress singularity coefficient (abbreviated as SSC) was defined and the leading term of stresses was also introduced by Hasebe and Takeuchi (1985).

For the simple reason that the definition used for SSC has the same expression with the crack problem, we use the following definition:

$$K_{1R} - iK_{2R} = 2\sqrt{2\pi} \lim_{z\to a} \sqrt{z-a}\,\phi'(z) \qquad (1)$$

where the rigid line tip is located at the point $z = a$ and $\phi'(z)$ is a complex potential defined by Muskhelishvili (1953).

In this note the cruciform rigid line problem is studied. The objective of this paper is to provide the singular integral equation approach for the problem. Secondly, numerical results are present to show the interaction between rigid branches. In addition, some particular features in the rigid line problem are compared with those in the crack problem.

2 Analysis

The following analysis depends on the complex variable function method in plane elasticity (Muskelishvili, 1953). In this method the stresses $(\sigma_x, \sigma_y, \sigma_{xy})$, the displacements (u, v), and the resultant force function (X, Y) are expressed in terms of two complex potentials $\phi(z)$, $\psi(z)$ such that

$$\sigma_x + \sigma_y = 4\,\mathrm{Re}\phi'(z)$$

$$\sigma_y - \sigma_x + 2i\sigma_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \qquad (2)$$

$$f = -Y + iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \qquad (3)$$

$$2G(u + iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \qquad (4)$$

where G is the shear modulus of elasticity, $\kappa = (3 - \nu)/(1 + \nu)$ for the plane stress problem, and ν is the Poisson's ratio. In addition, the following two derivatives in a specified direction are useful (Chen and Hasebe 1992b):

$$J_1\left(z, \frac{\overline{dz}}{dz}\right) = \lim_{\Delta z\to 0} \frac{1}{\Delta z}\{(f)_{z+\Delta z} - (f)_z\}$$

$$= \phi'(z) + \overline{\phi'(z)} + \frac{\overline{dz}}{dz}(z\overline{\phi''(z)} + \overline{\psi'(z)})$$

$$= N + iT \qquad (5)$$

$$J_2\left(z, \frac{\overline{dz}}{dz}\right) = 2G \lim_{\Delta z\to 0} \frac{1}{\Delta z}\{(u + iv)_{z+\Delta z} - (u + iv)_z\}$$

$$= \kappa\phi'(z) - \overline{\phi'(z)} - \frac{\overline{dz}}{dz}(z\overline{\phi''(z)} + \overline{\psi'(z)}). \qquad (6)$$

Physically, the J_1 value represents the traction $N + iT$ applied along the interval z, $z + \Delta z$ in Fig. 1.

In the analysis, the solution of the cruciform rigid line problem depends on the potential for a single rigid line problem. The appropriate complex potential for the single rigid line problem has been obtained previously by Chen and Hasebe (1992a), which is as follows:

[1] Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, P. R. China. (Permanent address: Division of Engineering Mechanics, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212013, P. R. China.)

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME Applied Mechanics Division, Dec. 5, 1996; final revision, Aug. 14, 1997. Associate Technical Editor: J. N. Reddy.

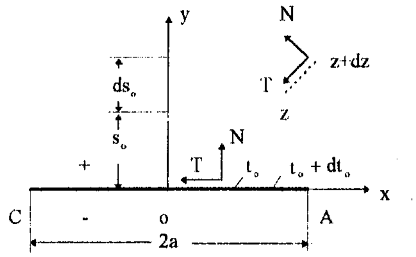

Fig. 1 A single rigid line problem in an infinite plate

$$\phi(z) = -\frac{1}{2\pi}\int_{-a}^{a} \text{Log}\,(z - t)h(t)dt$$

$$\psi(z) = \frac{\kappa}{2\pi}\int_{-a}^{a} \text{Log}\,(z - t)h(t)dt - \frac{1}{2\pi}\int_{-a}^{a}\frac{th(t)dt}{t - z} \quad (7)$$

where $h(t)$, $t \in (-a, a)$ takes the complex value in general. Physically, the function $h(t)$ represents the body force density. In this note only the normal mode case with the remote stresses σ_x^∞, σ_y^∞ is considered. In this case, the function $h(t)$ is an odd one and takes the real value.

Substituting (7) into (5) and (6), letting $z \to t_0^+$ and $z \to t_0^-$ $t_0 \in (-a, a)$ (Fig. 1), and using some results obtained previously (Chen and Hasebe, 1992b) along the line CA we can get

$$[J_1(t_0, 1)]^+ - [J_1(t_0, 1)]^- = [N(t_0) + iT(t_0)]^+$$
$$- [N(t_0) + iT(t_0)]^- = i(\kappa + 1)h(t_0), \quad |t_0| < a \quad (8)$$

$$[J_2(t_0, 1)]^+ = [J_2(t_0, 1)]^- = J_2(t_0, 1) = \frac{\kappa}{\pi}\int_{-a}^{a}\frac{h(t)dt}{t - t_0},$$
$$|t_0| < a. \quad (9)$$

Equation (8) shows that the J_1 contribution has a jump when a moving point is passing through the line CA in Fig. 1. On the contrary, Eq. (9) shows that the J_2 contribution is continuous in the same condition.

The J_2 contribution along the interval $\overline{is_0, i(s_0 + ds_0)}$ in Fig. 1 can be obtained by simply substituting $z = is_0$, $dz = ids_0$, $d\bar{z}/dz = -1$, in (6), and it can be expressed in the form

$$J_2 = \frac{\kappa}{2\pi}\int_{-a}^{a}\frac{2is_0h(t)dt}{t^2 + s_0^2} - \frac{1}{2\pi}\int_{-a}^{a}\frac{2th(t)dt}{(t + is_0)^2}$$

(for a point along the y-axis in Fig. 1). (10)

Since the rigid line is in equilibrium, from Eq. (8) we obtain the following constraint equation:

$$\int_{-a}^{a} h(t)dt = 0. \quad (11)$$

From Eqs. (2) and (7), the SSC at the right tip A in Fig. 1 can be evaluated from

$$K_{1R} = -(2\pi)^{1/2}\lim_{t\to a}\sqrt{|t - a|}h(t). \quad (12)$$

The original stress field for the cruciform rigid line problem (Fig. 2(*a*)) can be considered as a superposition of the uniform stress field (Fig. 2(*b*)) and the perturbation field (Fig. 2(*c*)). Clearly, the original stress field has the following solution (Muskhelishvili, 1953):

$$2G(u + iv) = (\kappa - 1)\Gamma z - \Gamma_1\bar{z} \quad (13)$$

where

$$\Gamma = (\sigma_x^\infty + \sigma_y^\infty)/4, \quad \Gamma_1 = (\sigma_y^\infty - \sigma_x^\infty)/2. \quad (14)$$

Actually, the J_2 contribution caused by the uniform stress field should be canceled by the J_2 contribution caused by the perturbation field. Secondly, the cruciform rigid line problem is considered as a superposition of two single rigid line problems. With this in mind, letting the body force density $h_1(s_1)$, $(h_2(s_2))$ be placed along the horizontal portion (the vertical portion) of the cruciform configuration, and considering the J_2 contributions as shown by (9) and (10), we obtain the following system of singular integral equations:

$$\frac{2\kappa}{\pi}\int_0^a\frac{th_1(t)dt}{t^2 - s_1^2} + \frac{2}{\pi}\int_0^b\frac{t(s_1^2 - t^2)h_2(t)dt}{(t^2 + s_1^2)^2} = (1 - \kappa)\Gamma + \Gamma_1,$$
$$(0 < s_1 < a) \quad (15)$$

$$\frac{2\kappa}{\pi}\int_0^b\frac{th_2(t)dt}{t^2 - s_2^2} + \frac{2}{\pi}\int_0^a\frac{t(s_2^2 - t^2)h_1(t)dt}{(t^2 + s_2^2)^2} = (1 - \kappa)\Gamma - \Gamma_1,$$
$$(0 < s_2 < b). \quad (16)$$

In the studied case, since the functions $h_1(s_1)$, $(h_2(s_2))$ are odd real functions, the condition like Eq. (11) is satisfied automatically. It is of interest to point out that substituting $\kappa = -1$ into Eqs. (15) and (16) will yield the singular integral equation for the cruciform crack problem (Boiko and Karpenko, 1981).

After solving the integral equation, the SSCs at two tips A and B (Fig. 2) can be evaluated by

$$K_{1R,A} = -(2\pi)^{1/2}\lim_{t\to a}\sqrt{|t - a|}h_1(t)$$
(for the tip A in Fig. 2) (17)

$$K_{1R,B} = -(2\pi)^{1/2}\lim_{t\to b}\sqrt{|t - b|}h_2(t)$$
(for the tip B in Fig. 2). (18)

3 Numerical Example

In the numerical solution, a semi-open quadrature rue is most suitable to solve the singular integral Eq. (15) and (16). The quadrature rule suggested by Boiko and Karpenko (1981) is used to solve the singular integral equation.

In the numerical solution we let

$$h_1(t) = \sqrt{\frac{t}{a - t}}H_1(t), \quad (0 \le t \le a)$$

$$h_2(t) = \sqrt{\frac{t}{b - t}}H_2(t), \quad (0 \le t \le b). \quad (19)$$

Finally, by using (17), (18), and (19) the SSCs at two rigid line tips A and B can be evaluated by

$$K_{1R,A} = -(2\pi a)^{1/2}H_1(a), \quad K_{1R,B} = -(2\pi b)^{1/2}H_2(b). \quad (20)$$

Under the following conditions (1) the plane stress condition

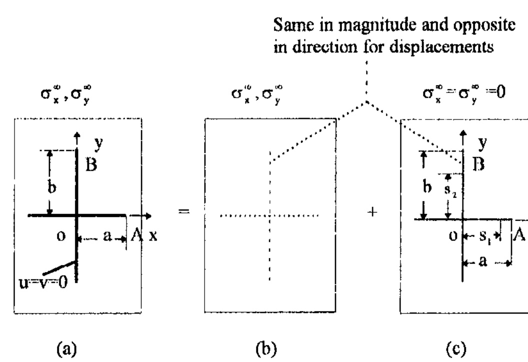

Fig. 2 Utilization of the superposition principle (*a*) the original stress field, (*b*) the uniform stress field, (*c*) the perturbation stress field

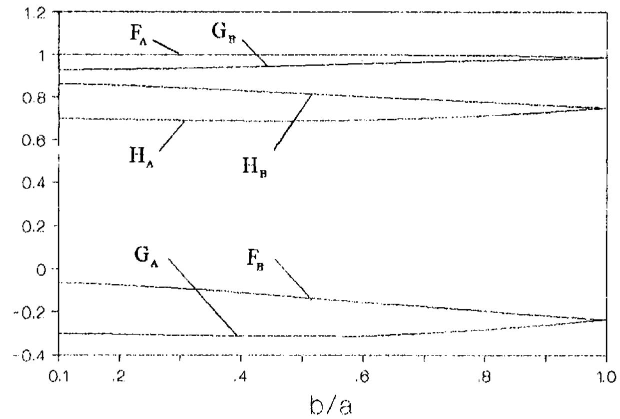

Fig. 3 Normalized SSCs at two rigid line tips A and B (see Fig. 2 and Eqs. (21), (22), (23), (24))

and $\nu = 0.3$ are assumed, and (b) the remote stresses σ_x^∞, σ_y^∞ are applied at infinity. The SSCs at two rigid line tips are expressed by

$$K_{1R,A} = (F_A(b/a)\sigma_x^\infty + G_A(b/a)\sigma_y^\infty)\frac{\kappa+1}{4\kappa}\sqrt{\pi a} \quad \text{(for the tip A in Fig. 2)} \quad (21)$$

$$K_{1R,B} = (F_B(b/a)\sigma_x^\infty + G_B(b/a)\sigma_y^\infty)\frac{\kappa+1}{4\kappa}\sqrt{\pi b} \quad \text{(for the tip B in Fig. 2).} \quad (22)$$

Finally, the calculated $F_A(b/a)$, $G_A(b/a)$, $F_B(b/a)$, $G_B(b/a)$ values are plotted in Fig. 3.

In the case of $\sigma_x^\infty = \sigma_y^\infty = p$, the SSCs can be expressed by

$$K_{1R,A} = H_A(b/a)p\frac{\kappa+1}{4\kappa}\sqrt{\pi a}, \quad H_A = F_A + G_A \quad \text{(for the tip A in Fig. 2)} \quad (23)$$

$$K_{1R,B} = H_B(b/a)p\frac{\kappa+1}{4\kappa}\sqrt{\pi b}, \quad H_B = F_B + G_B \quad \text{(for the tip B in Fig. 2).} \quad (24)$$

The calculated results for F_A, F_B, G_A, G_B, H_A, H_B values are listed plotted in Fig. 3. From Fig. 3 we see that, in the remote stress σ_x^∞ case, F_A values are dominant and $|F_A| > |F_B|$. In the meantime, in the remote stress σ_y^∞ case, G_B values are dominant and $|G_B| > |G_A|$.

4 Remarks

A comparison has been made between the rigid inclusion problem with the cavity problem (Dunders, 1989). Similarly, we also make a comparison for two problems. In the cruciform crack case with the remote stress σ_x^∞, the stress intensity factors at the tips A and B can be expressed by (Chen, 1993)

Table 1 Comparison results for the cruciform rigid line problem and the cruciform crack problem

	$b/a =$		
	0.1	0.2	1.0
F_A	1.000	1.000	0.985
F_B	−0.064	−0.075	−0.235
f_A	0.003	0.001	−0.217
f_B	0.997	0.998	1.081

$$K_{1A} = f_A(b/a)\sigma_x^\infty\sqrt{\pi a}, \quad K_{1B} = f_B(b/a)\sigma_x^\infty\sqrt{\pi b}. \quad (25)$$

The calculated F_A, F_B (for the cruciform rigid line problem) and f_A, f_B (for the cruciform crack problem) values are listed in Table 1. From Table 1 we see that in the cruciform rigid line case F_A values are dominant and $|F_A| > |F_B|$, and in the cruciform crack case f_B values are dominant and $|f_B| > |f_A|$.

Acknowledgment

The research project is support by National Natural Science Fund of China.

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A Modified Element-by-Element Preconditioner for Elastostatics

M. Barboteu[2], P. Alart[2], and F. Lebon[2]

In this paper we present in detail a modified element-by-element strategy. This kind of method is well known to be effective for large-scale problems because of its implicit parallelism. Numerical experiments described in this paper confirm the efficiency of this solver.

1 Introduction

Hughes, Levit, and Winget (1983) and also Hughes and Winget (1985) have introduced an element-by-element precon-

[2] Laboratoire de Mécanique et Génie Civil, UMR 5508 Université Montpellier 2-CNRS, Pl. E. Bataillon, 34095 Montpellier Cedex 5, France.

Contributed by the Applied Mechanics Division of The American Society of Mechanical Engineers for publication in the ASME Journal of Applied Mechanics. Manuscript received by the ASME Applied Mechanics Division, Feb. 27, 1997; final revision, Aug. 11, 1997. Associate Technical Editor: W. K. Liu.