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On the shear instability of saturated soil

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Abstract

In this paper, the dynamic instability of simple shear of saturated soil is discussed. The governing equations are obtained based on mixture theory in which the inertia effect and the compressibility of grains are considered. Perturbation method is used to analyze and it is shown that two types of instability may exist. One of them is dominated by pore-pressure-softening, while the other by strain-softening. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Saturated soil; Instability; Perturbation; Strain-hardening

1. Introduction

The issue of the inception of localized shearing in granular materials subjected to undrained deformation has been the object of both theoretical as well as experimental research. Theoretical contributions are related mainly to stability and bifurcation analysis of diffused and localized failure models [1–3]. Typically, the stability problem is formulated by considering small perturbations in field variables (e.g. displacement and pore pressure). Classical continuum approach leads, in this case, to the ordinary diffusion equation for the perturbation in pore pressure [1,2]. Alternatively, the localization in fluid-infiltrated soil may be considered as a bifurcation problem [4]. The experimental evidence on deformation instabilities and the failure models comes from conventional triaxial [5,6] as well as plain strain biaxial tests [7]. The results indicated that the uniform response is often followed by the onset of a diffused, non-homogeneous deformation model, after which distinct shear bands form. However, this problem is often discussed under inertia-free undrained conditions [1,8,9]. In view of the above, an attempt of this paper is to discuss the instability of saturated soil so as to obtain a comprehensive and precise picture of the instability phenomenon.

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Nomenclature

σ_{ex}	effective stress in x-direction
σ_{ey}	effective stress in y-direction
τ	shear stress
γ	shear strain
p	pore pressure
ρ_w	density of water
ρ_s	density of grains
v_{wx}, v_{wy}	velocities of water in x- and y-directions
v_{sx}, v_{sy}	velocities of grains in x- and y-directions
g	gravity acceleration
μ	viscosity of water
k	Darcy permeability
C_s	material parameter
E_r	compressible modulus of grains
α	frequency of perturbation
β	wave number of perturbation
R_0	strain-hardening coefficient
Q_0	pore-pressure-softening coefficient
H_0	strain-ratio-hardening coefficient
t_c	characteristic time
l_c	characteristic length

2. Formulation of problem

Consider a sample of saturated soil, which is subjected to a partial drained deformation under simple shear. The pore water is assumed to be incompressible, while grains are compressible. x-axis is in horizontal direction, while y-axis is in vertical direction. Shear loading is applied in x-axis. The deformation can only occur in x-direction but may have a gradient in the other direction.

3. The constitutive relations

The skeleton of soil is taken as visco-plastic, so the constitutive relations may be expressed as follows under shear loading [9,10]:

$$\begin{aligned} \sigma_{ex} &= f_1\left(\gamma, \dot{\gamma}, p\right), \\ \sigma_{ey} &= f_2\left(\gamma, \dot{\gamma}, p\right), \\ \tau &= f_3\left(\gamma, \dot{\gamma}, p\right), \end{aligned} \tag{1}$$

in which σ_{ex}, σ_{ey} are the effective stresses in x- and y-directions, respectively, τ the shear stress, γ the shear strain, $\dot{\gamma}$ the shear strain ratio, p the pore pressure and is equal to $\sigma - \sigma_{ey}$, σ is the total stress and is a constant, which means, p denotes σ_{ey}.

4. The governing equations

Here, the momentum equations of water and grains are given as follows according to mixture theory [11]:

$$\begin{aligned}
&\rho_w \frac{\partial v_{wx}}{\partial t} + \frac{\partial p}{\partial x} = -Kn(v_{wx} - v_{sx}),\\
&\rho_w \frac{\partial v_{wy}}{\partial t} + \frac{\partial p}{\partial y} = -Kn(v_{wy} - v_{sy}) - \rho_w g,\\
&(1-n)\rho_s \frac{\partial v_{sx}}{\partial t} + (1-n)\frac{\partial p}{\partial x} + \frac{\partial \sigma_{ex}}{\partial x} - \frac{\partial \tau}{\partial y} = Kn^2(v_{wx} - v_{sx}),\\
&(1-n)\rho_s \frac{\partial v_{sy}}{\partial t} + (1-n)\frac{\partial p}{\partial y} + \frac{\partial \sigma_{ey}}{\partial y} - \frac{\partial \tau}{\partial x} = Kn^2\left(v_{wy} - v_{sy}\right) - (1-n)\rho_s g,
\end{aligned} \tag{2}$$

where v_{wx}, v_{wy} are velocities of pore water in two directions and v_{sx}, v_{sy} the velocities of solid phase in two directions, n the porosity, K the obstraction coefficient and $K = \mu/k$, where k is the physical permeability and μ is the viscosity.

Differentiate the last two equations of Eq. (2) in the x- and y-directions, respectively, and then add them together, the first governing equation is obtained as follows:

$$(1-n)\rho_s \frac{\partial^2 \gamma}{\partial t^2} - \frac{\partial^2 \tau}{\partial y^2} = -Kn^2 \frac{\partial \gamma}{\partial t}, \tag{3}$$

in which γ is the shear strain.

The mass balance equation for solid phase is as follows considering that $\partial/\partial x = 0$:

$$\frac{1}{1-n}\frac{\partial n}{\partial \mathrm{t}} = \frac{\partial v_{sy}}{\partial y} + \frac{1}{\rho_s}\frac{\partial \rho_s}{\partial t}. \tag{4}$$

The mass balance equation for the fluid phase is as follows considering that $\partial/\partial x = 0$:

$$-\frac{\partial}{\partial y}\left[n\left(v_{wy} - v_{sy}\right)\right] = \frac{1-n}{\rho_s}\frac{\partial \rho_s}{\partial t} + \frac{n}{\rho_w}\frac{\partial \rho_w}{\partial t} + \frac{\partial v_{sy}}{\partial y}. \tag{5}$$

Assume $\rho_w =$ constant, then the equations above are simplified as follows:

$$-\frac{\partial \varepsilon_{sv}}{\partial t} = \frac{1}{1-n}\frac{\partial n}{\partial t} - \frac{1}{\rho_s}\frac{\partial \rho_s}{\partial t} \tag{6}$$

$$\text{as } \frac{\partial \varepsilon_{sv}}{\partial t} = -\frac{\partial v_{sy}}{\partial y} \tag{7}$$

and

$$-\frac{\partial q}{\partial x}=\frac{1}{1-n}\frac{\partial n}{\partial t}-\frac{n}{\rho_{s}}\frac{\partial \rho_{s}}{\partial t}, \tag{8}$$

in which

$$q=n\left(v_{wy}-v_{sy}\right). \tag{9}$$

The relative specific discharge is related to the pore-fluid pressure gradient according to Darcy's law

$$q=-\frac{k}{\mu}\frac{\partial p}{\partial y}. \tag{10}$$

Assume that changes in porosity are mainly due to shear (dilatancy law), we will have

$$-\frac{1}{1-n}\frac{\partial n}{\partial t}=C_{s}\frac{\partial \gamma}{\partial t}, \tag{11}$$

in which C_s is a material parameter.

Assume that changes in solid density (grain compressibility) are due to changes in pore pressure, we will have the following equation [12]:

$$-\frac{1}{\rho_{s}}\frac{\partial \rho_{s}}{\partial t}=\frac{1}{E_{r}}\frac{\partial p}{\partial t}, \tag{12}$$

in which E_r is the compressible modulus of grains, which means grains are compressible.

The second governing equation is obtained by instituting Eqs. (10)–(12) into Eq. (8)

$$-\frac{k}{\mu}\frac{\partial^{2} p}{\partial y^{2}}=C_{1}\frac{\partial \gamma}{\partial t}-\frac{n}{E_{r}}\frac{\partial p}{\partial t}, \tag{13}$$

in which $C_1 = C_s/n$.

Now, the governing equations can be rewritten as follows:

$$\begin{aligned}&-\frac{1}{Kn}\frac{\partial^{2} p}{\partial y^{2}}=C_{1}\frac{\partial \gamma}{\partial t}-\frac{\partial p}{E_{r}\partial t},\\&(1-n)\rho_{s}\frac{\partial^{2}\gamma}{\partial t^{2}}-\frac{\partial^{2}\tau}{\partial y^{2}}=-Kn^{2}\frac{\partial \gamma}{\partial t}.\end{aligned} \tag{14}$$

The solutions of these equations are difficult to seek for because of the non-linear. It has been shown by experiments and numerical results that the soil deformation develops from uniform to non-homogeneous, which means from stable modes to unstable modes.

The aim of this paper is to seek for the condition of instability. The perturbation method is therefore adopted here. Hence, a smooth developing deformation state γ_0, τ_0, p_0 is taken as the base state, which is a solution of Eq. (14). When perturbation has been applied to the governing equations, we will be able to analyze the factors and condition of instability.

5. Perturbation analysis [13,14]

We study the solutions in next form

$$\begin{aligned} \gamma &= \gamma_0 + \gamma'; \quad |\gamma'| \ll |\gamma_0|, \\ p &= p_0 + p'; \quad |p'| \ll |p_0|, \\ \tau &= \tau_0 + \tau'; \quad |\tau'| \ll |\tau_0|, \end{aligned} \tag{15}$$

where γ_0, p_0, τ_0 is a solution of Eq. (14), and

$$\begin{aligned} \gamma' &= \gamma^{\bullet} \mathrm{e}^{\alpha t + \mathrm{i}\beta y}, \\ p' &= p^{\bullet} \mathrm{e}^{\alpha t + \mathrm{i}\beta y}, \\ \tau' &= \tau^{\bullet} \mathrm{e}^{\alpha t + \mathrm{i}\beta y}, \end{aligned} \tag{16}$$

where α, β are respectively, the frequency and the wave number.

Differentiating the constitutive relations (1), we may obtain

$$\mathrm{d}\tau = R_0 \, \mathrm{d}\gamma - Q_0 \, \mathrm{d}p + H_0 \, \mathrm{d}\dot{\gamma}, \tag{17}$$

where

$$R_0 = \left(\frac{\partial \tau}{\partial \gamma}\right)_0, \quad Q_0 = -\left(\frac{\partial \tau}{\partial p}\right)_0, \quad H_0 = \left(\frac{\partial \tau}{\partial \dot{\gamma}}\right)_0. \tag{18}$$

Therefore

$$\tau^{\bullet} = R_0 \gamma^{\bullet} - Q_0 p^{\bullet} + \alpha H_0 \gamma^{\bullet}. \tag{19}$$

The parameters R_0, Q_0, H_0 may be determined by experiments and it needs much more work.

Institute Eqs. (15), (16), and (19) into Eq. (14), a homogeneous system of equations is obtained as follows:

$$\begin{aligned} &[(1-n)\rho_{\mathrm{s}}\alpha^2 + \beta^2(R_0 + \alpha H_0) + Kn^2\alpha]\gamma^{\bullet} - \beta^2 Q_0 p^{\bullet} = 0, \\ &E_{\mathrm{r}} C_1 \alpha \gamma^{\bullet} + \left(\frac{E_{\mathrm{r}}\beta^2}{Kn} - \alpha\right) p^{\bullet} = 0. \end{aligned} \tag{20}$$

As we all know, the determinant of the coefficients should be equal to zero, if the system has solutions, which leads to

$$(1-n)\rho_s\alpha^3 + \left(\frac{\rho_s E_r(1-n)}{Kn}\beta^2 + H_0\beta^2 + Kn^2\right)\alpha^2 + A_1\alpha + \frac{E_r R_0}{Kn}\beta^4 = 0, \tag{21}$$

where $A_1 = (E_r H_0/Kn)\beta^4 + E_r n\beta^2 + R_0\beta^2 - E_r C_1 Q_0\beta^2$.

It is a spectral equation and if α has a positive real root, instability is possible. Now, we give the dimensionless form of Eq. (21) by using the next dimensionless variables

$$\alpha = \frac{1}{\rho_s k_1}\bar{\alpha}, \quad \beta^2 = \frac{1}{\rho_s R_0 k_1^2}\bar{\beta}^2, \quad k_1 = \frac{1}{K}, \quad A = \frac{R_0}{E_r}, \quad B = \frac{H_0}{\rho_s k_1 E_r}, \quad C = C_1 Q_0. \tag{22}$$

Then, the spectral Eq. (21) may be reduced to the following form:

$$n(1-n)A\bar{\alpha}^3 + [(1-n)\bar{\beta}^2 + nB\bar{\beta}^2 + An^3]\bar{\alpha}^2 + [(n^2 + nA - nC)\bar{\beta}^2 + AB\bar{\beta}^4]\bar{\alpha} + \bar{\beta}^4 = 0. \tag{23}$$

It is obvious this equation has two extreme situations:

(i) For long wavelength($\beta \to 0$), Eq. (23) has two solutions

$$\bar{\beta} = 0,\ \bar{\alpha} = 0 \quad \text{or} \quad \bar{\alpha} = -\frac{n^2}{1-n}. \tag{24}$$

It shows that the deformation is always stable.

(ii) For short wavelength ($\beta \to \infty$), Eq. (23) has only one solution, which is

$$\bar{\beta} \to \infty,\ \bar{\alpha} = -\frac{1}{AB} \tag{25}$$

It is again always stable. This solution is not coincided with that of Rice [1], which is that the shortest the wavelength is, the fastest the deformation develops. The reason may be that the inertia is considered here. If the inertia terms $\partial v_{wx}/\partial t, \partial v_{wy}/\partial t, \partial v_{wx}/\partial t, \partial v_{wy}/\partial t$ in Eq. (2) are neglected, by perturbation analysis, the value of α will be obtained as follows:

$$\alpha = R_0/[k_n(C_1 Q_0 - 1)/\beta^2 - H_0]. \tag{26}$$

It is obvious the shortest wavelength develops the fastest under this condition.

Nevertheless, we can see there is a negative term $-nC\bar{\beta}^2\bar{\alpha}$ in Eq. (23), which may lead to instability. It must occur at special wave numbers. Therefore, it is of interest to seek for the wave number $\bar{\beta}_m$ for which the corresponding $\alpha_m > 0$ is a maximum. In addition to the spectral Eq. (23), $\bar{\alpha}_m$ and $\bar{\beta}_m$ must satisfy the next equation

$$\frac{d\bar{\alpha}}{d\bar{\beta}^2} = 0, \tag{27}$$

which leads to

$$\bar{\beta}_m^2 = -\frac{(1+n+nB)\bar{\alpha}_m^2 + [n^2 + n(A-C)]\bar{\alpha}_m}{2(AB\bar{\alpha}_m + 1)}. \tag{28}$$

Keeping $\bar{\beta}_{\mathrm{m}}^2 > 0$ in mind, we arrive at an important inequality to determine the limit of the $\bar{\alpha}_{\mathrm{m}}$ value

$$0 < \bar{\alpha}_{\mathrm{m}} < \frac{n(C-A)-n^2}{nB+n+1} = \bar{\alpha}_{\mathrm{m}}^{\bullet}. \tag{29}$$

Hence, for qualitative discussion, the value of $\bar{\alpha}_{\mathrm{m}}^{\bullet}$ can be used to represent the point of intersection $\bar{\alpha}_{\mathrm{m}}$. The characteristic time can be expressed as

$$t_{\mathrm{c}} \sim \frac{1}{\alpha_{\mathrm{m}}} \sim \frac{k_1\rho}{\bar{\alpha}_{\mathrm{m}}} \sim \frac{\rho k_1(nB+n+1)}{n(C-A)-n^2} = \frac{nH_0+(n+1)\rho_{\mathrm{s}}E_{\mathrm{r}}k_1}{n(C_1Q_0E_{\mathrm{r}}-R_0-nE_{\mathrm{r}})}.$$

It is obvious that the characteristic time is affected by compressible modulus of grains, strain-rate-hardening, obstruction, strain-hardening and pore-pressure-softening. The characteristic length l_{c} is related to t_{c} by

$$l_{\mathrm{c}}^2/t_{\mathrm{c}} \sim \alpha_{\mathrm{m}}/\beta_{\mathrm{m}}^2 \sim R_0k_1(\bar{\beta}_{\mathrm{m}}^2/\bar{\alpha}_{\mathrm{m}}).$$

The length is the pattern one rather than diffusion one.

Combining both the spectral Eq. (23) and the extreme condition (28), the equation to determine $\bar{\alpha}_{\mathrm{m}}$ may be obtained as follows:

$$f_1 = f_2, \tag{30}$$

in which

$$f_1 = 4An[(1-n)\bar{\alpha}_{\mathrm{m}} + n^2](AB\bar{\alpha}_{\mathrm{m}} + 1), \tag{31}$$

$$f_2 = \left\{(1+n+nB)\bar{\alpha}_{\mathrm{m}} + [n^2 + n(A-C)]\right\}^2. \tag{32}$$

From Fig. 1, for the region $\bar{\alpha}_{\mathrm{m}} > 0$, it may be seen that the left branch of function f_2 and the right branch of f_1 must have an intersection between 0 and $\bar{\alpha}_{\mathrm{m}}^{\bullet}$ as long as

$$\frac{A - 2\sqrt{An} + n}{C} < 1. \tag{33}$$

It is the criterion for the existence of a solution $\bar{\alpha}_{\mathrm{m}}$ and therefore is what we desired. If $n \ll A$, this criterion can be simplified to

$$\frac{A}{C} = \frac{R_0}{E_{\mathrm{r}}C_1Q_0} < 1. \tag{34}$$

This means the condition of instability is that pore-pressure-softening overcomes the strain-hardening. It is very interesting that whether instability occurs or not is not related to the

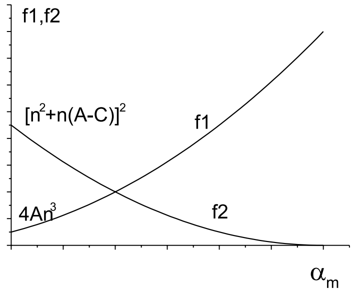

Fig. 1. Plots of the functions f_1 and f_2, defined in Eqs. (31) and (32).

permeability k and the strain-rate-hardening H_0. However, these factors influence instability markedly in some other aspects, which will be discussed later. Next, three interesting special cases: small permeability, no strain-hardening and no pore-pressure-softening will be discussed.

(i) *Permeability is small*, $K \to \infty$. In this case, the spectral equation (21) becomes

$$(1-n)\rho_s\bar{\alpha}_m^2 + H_0\beta^2\bar{\alpha}_m + (R_0 - Q_0E_rC_1)\beta^2 = 0. \tag{35}$$

We can see, if $R_0 - Q_0E_rC_1 < 0$, namely $A/C < 1$, $\bar{\alpha}_m$ has a positive real root and instability will occur. It is important to appreciate that the same formal criterion (34) can be used whether the instability is under permeable condition or not.

(ii) *No strain-hardening*, $R_0 = 0$. The spectral equation (21) will become the next form on this condition

$$\frac{(1-n)\rho_s}{E_r}\alpha^2 + \left[\frac{\rho_2(1-n)}{Kn}\beta^2 + \frac{H_0}{E_r}\beta^2 + \frac{Kn^2}{E_r}\right]\alpha + \left[\frac{H_0}{Kn}\beta^4 + n\beta^2 - C_1Q_0\beta^2\right] = 0. \tag{36}$$

The condition that $\bar{\alpha}$ has positive real root is as follows in this case:

$$\frac{H_0}{Kn}\beta^2 + n - C_1Q_0 < 0. \tag{37}$$

In this criterion, permeability, strain-rate-hardening and pore-pressure-softening play the role.

(iii) *No pore-pressure-softening*, $Q_0 = 0$. Now, we turn to discuss the second mode of instability in which there is no pore-pressure-softening, in this case, we can formulate the spectral equation as follows:

$$(1-n)\rho_s\alpha^3 + \left(\frac{\rho_sE_r(1-n)}{Kn}\beta^2 + H_0\beta^2 + Kn^2 + R_0\beta^2\right)\alpha^2 + A_1\alpha + \frac{E_rR_0}{Kn}\beta^4 = 0. \tag{38}$$

Although H_0 and K must be positive, R_0 might be negative. Therefore, $R_0 < 0$ may be another possible cause of instability. Eq. (38) can be rewritten in the next form

$$(1-n)\rho_s\alpha^3 + \left(\frac{\rho_s E_r(1-n)}{Kn}\beta^2 + H_0\beta^2 + Kn^2\right)\alpha^2 + A_1\alpha = |R_0|\beta^2\alpha + \frac{E_r|R_0|}{Kn}\beta^4, \tag{39}$$

where $A_1 = (E_r H_0/Kn)\beta^4 + E_r n\beta^2$.

It is easy to see that there must be a solution $\alpha > 0$; therefore, deformation must be unstable. It is very simple to show that no maximum in α exists and α is a monotonically increasing function of β, with

$$\lim_{\beta\to 0}\alpha \to 0 \text{ and } \lim_{\beta\to\infty}\alpha \to \frac{|R_0|}{H_0},$$
$$\lim_{\beta\to\infty} t = t_{\min} = \frac{H_0}{|R_0|}.$$

This implies the shorter the wavelength, the earlier the occurrence of instability.

6. Practical criterion

Now we concentrate on the instability mode dominated by pore-pressure-softening, but turn to practical considerations.

It is especially useful that criterion (34) implies a strain criterion. Recalling $[R_0]$ = stress/strain, we can easily deduce that the inequality (34) is equivalent to a strain criterion. It is desirable to establish a criterion connecting state parameters and material constants on each side of the inequality.

If the constitutive relation of the soil concerned is formulated explicitly, the critical strain is easy to obtain. For instance, supposing Q_0 = constant, and rate-constant stress and strain relationship behaves as

$$\tau = G\gamma^m, \tag{40}$$

where G is the shear modulus, m is a constant. Then, the practical criterion is obtained as follows by the inequality (34):

$$\gamma_0 > \frac{m\tau_0}{E_r C_1 Q_0}. \tag{41}$$

7. Conclusions

It has been shown that there may exist two types of possible instability of saturated soil under simple shear; one is dominated by pore-pressure-softening while the other by strain-softening.

The criterion for the first mode of instability combines pore-pressure-softening, strain-hardening and compressible modulus of grains and implies a practical critical strain. The critical strain can be obtained simply when the constitutive relation has an explicit expression. The analysis of this paper (e.g., Eq. (41)) may be applied to saturated sand, clay and rock, but these results are not suited for some soils such as loess because of their special characteristics.

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