AN ANALYSIS OF COLLECTIVE DAMAGE FOR SHORT FATIGUE CRACKS BASED ON EQUILIBRIUM OF CRACK NUMERICAL DENSITY

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Abstract—Collective damage of short fatigue cracks was analyzed in the light of equilibrium of crack numerical density. With the estimation of crack growth rate and crack nucleation rate, the solution of the equilibrium equation was studied to reveal the distinct feature of saturation distribution for crack numerical density. The critical time that characterized the transition of short and long-crack regimes was estimated, in which the influences of grain size and grain-boundary obstacle effect were investigated. Furthermore, the total number of cracks and the first order of damage moment were discussed.
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1. INTRODUCTION

IN THE past two decades, different characteristics of fatigue damage have been observed in the primary and the final stages of fatigue failure. In the primary stage of fatigue failure, the length of most cracks is comparable with the grain size. One distinct growth behaviour of short cracks has been reported as the crack growth rates with deceleration and acceleration patterns, which has been attributed to the interaction of crack tip plastic zone with microstructural barriers to plastic flow [1–4]. In the recent years, another distinct phenomenon of short cracks has been revealed in the literature [5–10], namely that in short-crack regime fatigue process presents collective damage characteristics with the gradual evolution of crack numerical density, showing the number of short fatigue cracks increases with increasing number of fatigue cycles. The method based on the equilibrium of crack numerical density [11] is a possible approach to describe and analyze the collective evolution process of dispersed short cracks [12]. The basic consideration of the method is that the total number of short cracks is composed of cracks due to crack nucleation and crack growth.

It is known that the short-crack regime takes up a large portion of total fatigue life and the approach to life prediction of this regime is still a problem remained for solution. One essential aspect of this problem is to define the transition point between short and long-crack regimes, and then to determine the critical values of loading cycles and crack length. This is an important step toward the prediction of total fatigue life. The main objective of this paper is to present a new method to evaluate the collective behaviour of dispersive short-fatigue-cracks by using the concept of the equilibrium of crack numerical density, in which the crack numerical density denotes the number of cracks within unit area and unit crack length. After setting the expressions for crack growth rate and crack nucleation rate, the solution of the equilibrium equation and the evolution characteristics of crack population are studied. Three cases of transition criteria are discussed, with each showing the variation of critical time with grain size and microstructural obstacle effect. The total number of cracks and the first order of damage moment (i.e. the total length of cracks) are also investigated, which are two factors available for the description of the extent of short-crack damage.

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2. BASIC EQUATIONS

In order to evaluate the collective damage of short fatigue cracks, the equilibrium equation of crack numerical density[11] is used in the following analysis. Its non-dimensional form is[12]:

$$\frac{\partial}{\partial t} n(c, t) + \frac{\partial}{\partial c} [A(c) \cdot n(c, t)] = N_g \cdot n_N(c) \quad (1)$$

where $n(c, t)$ is the crack numerical density, with $n(c, t)dc$ being the number of cracks with length between c and $c + dc$ at time t ; $A(c)$ is crack growth rate; and $n_N(c)$ is crack nucleation rate. All of these physical quantities are dimensionless, and the non-dimensional coefficient

$$N_g = \frac{n_N^* \cdot d}{n^* \cdot A^*},$$

with n_N^* being the characteristic crack nucleation rate, A^* the characteristic crack growth rate, n^* the characteristic crack numerical density, and d the characteristic dimension of the material concerned (e.g. the grain diameter). Note that

$$\frac{d}{A^*}$$

is equivalent to the characteristic time. Equation (1) describes the equilibrium of crack numerical density in phase space. The second term at the left side describes the flow of crack numerical density in phase space, which is attributed to crack growth; and the term at the right side describes the contribution to crack numerical density made by crack nucleation.

If $n(c, 0) = 0$ and the threshold value of crack length for growth is zero, the theoretical solution of eq. (1) is[11]:

$$n(c, t) = \frac{1}{A(c)} \int_{\eta(c, t)}^c N_g \cdot n_N(c') dc' \quad (2)$$

The lower integral boundary $\eta(c, t)$ is of such a definition that for a crack with an initial length of $\eta(c, t)$ at $t = 0$, its length will advance to c at time t under the growth rate of $A(c)$.

Figure 1 is a schematic illustration showing the deceleration-acceleration pattern of short-crack growth[1]. The curves imply that before the crack tip reaches the first grain boundary, crack growth rate decreases with the increasing of crack length. If crack length is relatively large, the crack growth rate will tend to LEFM regime. Thus, in general, we may construct the

Fig. 1. Schematic of crack growth rate versus crack length in short crack regime. (Grain 1 < Grain 2 < Grain 3 < Grain 4).

following expression for $A(c)$:

$$A(c) = \begin{cases} 1 - (1 - A_d) \cdot c & (c \leq 1) \\ \bar{d} \cdot c & (c > 1) \end{cases} \quad (3)$$

where A_d is the crack growth rate when the normalized crack length $c = 1$, $\bar{d}$ is the ratio of non-dimensional average grain size d to non-dimensional initial crack growth rate A_0 , i.e. the normalized grain size. If c is normalized by grain diameter, A_d is the crack growth rate at the time the crack tip reaches the first grain boundary. Equation (3) shows the effects of crack length c , grain size d , and grain-boundary obstacle effect A_d on crack growth rate. The value of A_d is within the range from 0 to 1, which reflects a number of experimental measurements[1–4]. If $A_d = 0$, eq. (3) has the meaning that short cracks stop propagating when crack tips reach grain boundaries. This is the extreme condition that the largest grain-boundary obstacle effect prevails.

In experiments [7, 13], it was shown that the length of most cracks at initiation is less than the average grain size, and there was almost not a crack at nucleation with length larger than $2 \cdot d$. Note that the crack length at initiation also depends on the time interval used for observation. Thus, the crack nucleation rate n_N is constructed as follows:

$$n_N = \begin{cases} 1 - c/2 & (c \leq 2) \\ 0 & (c > 2) \end{cases} \quad (4)$$

After setting above estimations for $A(c)$ and $n_N(c)$, the distribution of $n(c, t)$ versus c is readily calculated from eq. (1). Figure 2 shows the variation of $n(c, t)$ with c for different time stages. Figure 2 also shows that there is a saturation distribution in the evolution process of collective short cracks (dash line), i.e. with the progress of fatigue process, the distribution of crack numerical density gradually tends to a saturation curve from small to large value of crack size. It is obvious that the saturation curve presents the stable distribution of crack numerical density. Referring to eq. (2), we may write the saturation curve as:

$$n_0(c) = \frac{1}{A(c)} \int_0^c N_g \cdot n_N(c') dc' \quad (5)$$

For the explicitness in the following, we write:

$$\alpha = \int_0^\infty N_g \cdot n_N(c) dc \quad (6)$$

and

$$D_0 = \int_0^\infty n(c, t) dc \quad (7)$$

Fig. 2. Variation of $n(c, t)$ with c for different time stages, the dash line representing the saturation curve. ($A_d = 0$, $d = 1$).

It is clear that α is the number of cracks produced within unit time duration, and D_0 is the number of total cracks at time t . Regarding the equilibrium of crack numerical density, we have

$$D_0 = \int_0^{\infty} n(c, t) dc = \alpha \cdot t \quad (8)$$

Assuming that at time t , the distribution of crack numerical density in the area of $c < c^*(t)$ would converge to the saturation curve, as such, the number of cracks in non-saturation area is defined as D_0^* , where c^* is the length of the crack (at time t) whose initial length is zero at $t = 0$. Thus from eq. (3), the evolution of c^* can be expressed as:

$$\dot{c}^* = A(c) = \begin{cases} 1 - (1 - A_d) \cdot c^* & (c^* \leq 1) \\ \bar{d} \cdot c^* & (c^* > 1) \end{cases} \quad (9)$$

The solution of eq. (9) is:

$$c^* = \begin{cases} \frac{1}{1-A_d} \{1 - \exp[-(1 - A_d) \cdot t]\} & (t \leq t_0) \\ \beta \cdot \exp(\bar{d} \cdot t) & (t > t_0) \end{cases} \quad (10)$$

where $\beta = A_d^{\bar{d}/(1-A_d)}$, and

$$t_0 = \frac{1}{\bar{d}} \ln \frac{1}{\beta},$$

which is the time when $c^* = 1$.

3. TOTAL NUMBER OF CRACKS, MAXIMUM CRACK LENGTH AND CRITICAL TIME

From eq. (2), it is possible to derive the evolutionary features of crack numerical density. However, the crack numerical density $n(c, t)$ cannot be used directly to determine the termination of short-crack regime. The boundary of short-crack regime and the start of long-crack regime should be indicated by the appearance of a crack with critical length. Thus we need to know the maximum crack length $c_{\max}(t)$ as a boundary value of the evolution system. It is stipulated that the numerical-density-equilibrium analysis of short crack is not valid when $c_{\max}(t) \geq c_{\text{cr}}$, where c_{cr} is the critical crack length, beyond which the fatigue damage enters into long-crack regime. In relation to the stipulation of criteria from different viewpoints, the corresponding value of $c_{\max}(t)$ can be identified. The following three cases are discussed.

3.0.1. *Case I.* Assume that the upper boundary of the saturation area c^* be the maximum crack length $c_{\max}(t)$ (i.e. $c_{\max} = c^*$). The important premise of Case I is that only the crack numerical density at the region of $c < c^*$ may distribute stably. The fulfill of the criterion $c^* = c_{\max} = c_{\text{cr}}$ is equivalent to the statement that a crack with its initial length of zero develops to a crack with critical length c_{cr} . The evolution of $c_{\max}$ can be shown as:

$$c_{\max} = c^* = \begin{cases} \frac{1}{1-A_d} \{1 - \exp[-(1 - A_d) \cdot t]\} & (t \leq t_0) \\ \beta \cdot \exp(\bar{d} \cdot t) & (t > t_0) \end{cases} \quad (11)$$

Substituting eq. (8) into (11), one obtains:

$$c_{\max} = \begin{cases} \frac{1}{1-A_d} [1 - \exp(-\frac{1-A_d}{\alpha} \cdot D_0)] & (t \leq t_0) \\ \beta \cdot \exp(\frac{\bar{d} \cdot D_0}{\alpha}) & (t > t_0) \end{cases} \quad (12)$$

eq. (12) shows the variation of $c_{\max}$ with A_d and $\bar{d}$. Because it is certain that $c_{\text{cr}} > 1$, then the critical time t_{cr} of the termination of short crack regime can be derived from eq. (11):

$$t_{\text{cr}} = t |_{c_{\max}=c_{\text{cr}}} = \frac{1}{\bar{d}} \ln \frac{c_{\text{cr}}}{\beta} \quad (13)$$

Fig. 3 shows the relationship between t_{cr} and the parameters of A_d and $\bar{d}$ for this case. It is seen

Fig. 3. t_{cr} as a function of A_d and $\bar{d}$ for case I.

that the value of t_{cr} increases with the decrease in the values of A_d and $\bar{d}$, suggesting that the tolerance of collective short-crack damage is enhanced if the material is of small grain size and large obstacle effect of grain boundary.

3.0.2. *Case II.* The criterion of c_{max} is set as

$$\int_{c_{max}}^{\infty} n(c) dc = \frac{1}{n^*},$$

which implies that there must be at least one crack with the length larger than c_{max} . Here n^* is the characteristic crack numerical density and the situation of $n^* = 1$ is considered. Then the criterion can be rewritten as

$$\int_{c_{max}}^{\infty} n(c) dc = 1.$$

Let t^* be the time when $c^* = c_{max} \geq c_{cr}$. If $c_{cr} \geq 2$, then at time t^* , the number of cracks in non-saturation area is:

$$\begin{aligned} D_0^* &= \alpha \cdot t^* - \int_0^{c_{cr}} \left[\frac{1}{A(c)} \int_0^c N_g \cdot n_N(\hat{c}) d\hat{c} \right] dc \\ &= \alpha \cdot t^* - N_g \cdot \left(\int_0^1 + \int_1^2 + \int_2^{c_{cr}} \right) \cdot \left[\frac{1}{A(c)} \int_0^c n_N(\hat{c}) d\hat{c} \right] dc \\ &= \alpha \cdot t^* - N_g \cdot \left\{ \frac{24 \cdot A_d - 15 \cdot A_d^2 + 4(4A_d - 3) \ln A_d - 10}{16(1 - A_d)^3} + \frac{5}{8\bar{d}} + \frac{1}{\bar{d}} \ln \frac{c_{cr}}{2} \right\} \end{aligned} \quad (14)$$

In the following, the circumstances of $D_0^* < 1$, and $D_0^* > 1$ are separately discussed.

(a) If $D_0^* < 1$, from the equilibrium of crack numerical density, t_{cr} can be derived:

$$\begin{aligned} t_{cr} &= t^* + \frac{1}{\alpha} (1 - D_0^*) \\ &= \frac{N_g}{\bar{d} \cdot \alpha} \ln \frac{c_{cr}}{2} + \zeta \end{aligned} \quad (15)$$

where

$$\zeta = \frac{1}{\alpha} \left[1 + N_g \frac{24A_d - 15A_d^2 + 4(4A_d - 3) \ln A_d - 10}{16(1 - A_d)^3} + \frac{5N_g}{8\bar{d}} \right]$$

From eq. (15), the relationship between $c_{\max}$ and D_0 is therefore obtained:

$$c_{\max} = 2 \cdot \exp \left[\frac{\bar{d}}{N_g} (D_0 - \zeta) \right] \quad (16)$$

Fig. 4 shows the variation of t_{cr} with A_d and $\bar{d}$ for $c_{\text{cr}} = 10$. It is observed that the short-crack regime may last longer with smaller values of A_d and $\bar{d}$, which respectively reflects stronger grain-boundary obstacle effect and smaller grain size.

(b) If $D_0^* > 1$ at time t^* , then t_{cr} has another form:

$$\tilde{t}_{\text{cr}} = t^* - \frac{1}{\alpha} (D_0^* - 1 + \tilde{D}_0) \quad (17)$$

where

$$\tilde{D}_0 = \int_{c^*|_{t=\tilde{t}_{\text{cr}}}}^{c_{\text{cr}}} \left(\frac{1}{A(c)} \int_0^c N_g \cdot n_N(c') dc' \right) dc - (D_0^*|_{t=\tilde{t}_{\text{cr}}} - 1) \quad (18)$$

However, in the present numerical calculation, we examine the trend of the response for D_0^* and do not observe the occurrence of $D_0^* > 1$.

3.0.3. *Case III.* Assume that the criterion of $c_{\max}$ be to satisfy $n(c_{\max}, t) \leq n_{\text{cr}}$, where

$$n_{\text{cr}} = \left[\frac{1}{A(c)} \int_0^c N_g \cdot n_N(c') dc' \right]_{c=c_{\text{cr}}}$$

The fulfill of the criterion means that the crack numerical density at $c = c_{\text{cr}}$ reaches a pre-set critical value. Note that in general, c_{cr} is always several times greater than the grain size. Consequently for $c_{\text{cr}} \geq 2$, referring to the definition of η , we have:

$$\eta(c_{\text{cr}}, t) = \begin{cases} c_{\text{cr}} \exp(-\bar{d} \cdot t) & \left(t \leq \frac{1}{\bar{d}} \ln c_{\text{cr}} \right) \\ \frac{1}{1-A_d} \{1 + \tilde{\beta} \exp[(1-A_d)t]\} & \left(t > \frac{1}{\bar{d}} \ln c_{\text{cr}} \right) \end{cases} \quad (19)$$

where

$$\tilde{\beta} = -A_d \cdot c_{\text{cr}}^{-(1-A_d)/\bar{d}}.$$

Fig. 4. t_{cr} as a function of A_d and $\bar{d}$ for case II.

eq. (19) indicates that if

$$t < \frac{1}{1-A_d} \ln \frac{-1}{\bar{d}},$$

then the crack numerical density would not be in saturation condition at $c = c_{cr}$. Thus, from eq. (2), the evolution of crack numerical density at c_{cr} is therefore obtained:

$$\begin{aligned} n(c_{cr}, t) &= \left[\frac{1}{A(c)} \int_{\eta}^c N_g \cdot n_N(c') dc' \right]_{c=c_{cr}} \\ &= \begin{cases} \frac{N_g}{c_{cr}} \cdot \frac{(2-\eta)^2}{4} & (t \geq \frac{1}{\bar{d}} \ln \frac{c_{cr}}{2}) \\ 0 & (t < \frac{1}{\bar{d}} \ln \frac{c_{cr}}{2}) \end{cases} \end{aligned} \quad (20)$$

If

$$t > \frac{1}{\bar{d}} \ln \frac{c_{cr}}{2},$$

according to eq. (20), we may derive:

$$\eta_{cr} = \eta(c_{cr}, t) |_{n=n_{cr}} = 2(1 - \sqrt{c_{cr} \cdot n_{cr}/N_g})$$

By substituting the above expression into eq. (19), t_{cr} can be solved as:

$$t_{cr} = \begin{cases} \frac{1}{\bar{d}} \ln \left(\frac{c_{cr}}{2} \frac{1}{1 - \sqrt{c_{cr} \cdot n_{cr}/N_g}} \right) & (c_{cr} \cdot n_{cr} \leq N_g/4) \\ \frac{1}{1-A_d} \ln \left[\frac{2(1-A_d)(1 - \sqrt{c_{cr} \cdot n_{cr}/N_g}) - 1}{\bar{d}} \right] & (c_{cr} \cdot n_{cr} > N_g/4). \end{cases} \quad (21)$$

Figure 5 shows t_{cr} as a function of A_d and $\bar{d}$ for this case. By comparison with Figs 3 and 4, Fig. 5 illustrates the same trend of t_{cr} varying with A_d and $\bar{d}$. It is seen that the extent of influence of grain-boundary obstacle effect on t_{cr} is greater than that of grain size, and if A_d and $\bar{d}$ are small, t_{cr} is more sensitive to them. In fact, since grain-boundary obstacle effect A_d is a factor relatively difficult to control, people may pay more attention to the influence of grain size $\bar{d}$.

Fig. 5. t_{cr} as a function of A_d and $\bar{d}$ for case III.

Equation (15 and 21) both show that t_{cr} is linear proportional to $1/\bar{d}$. Considering that short-crack regime takes up a large part of fatigue life, we assume that fatigue life is also linear proportional to $1/\bar{d}$, namely

$$N_f = f_1 \cdot \frac{1}{\bar{d}} + f_2 \quad (22)$$

where N_f is fatigue life, $\bar{d}$ is grain size, f_1 and f_2 are parameters related to fatigue stress range. Figure 6 shows the agreement between the theoretical estimation obtained from eq. (22) and experimental data from ref. [14]. It is worth pointing out that eq. (22) is proposed for the circumstance that short-crack regime takes up a large part of fatigue life.

From eq. (21), the relationship between c_{max} and D_0 is obtained:

$$D_0 = \begin{cases} \frac{\alpha}{\bar{d}} \ln \left(\frac{c_{max}}{2} \frac{1}{1 - \sqrt{c_{max} \cdot n_{cr} / N_g}} \right) & (c_{max} \cdot n_{cr} \leq N_g/4) \\ \frac{\alpha}{1-A_d} \ln \left[\frac{2(1-A_d)(1 - \sqrt{c_{max} \cdot n_{cr} / N_g}) - 1}{\beta_1} \right] & (c_{max} \cdot n_{cr} > N_g/4) \end{cases} \quad (23)$$

where

$$\tilde{\beta} = A_d \cdot c_{max}^{(1-A_d)/\bar{d}}$$

and $c_{max} > 2$. Figure 7 shows the comparison of the relationship between D_0 and c_{max} for the three cases. Evidently, the three curves almost increase exponentially with the enlargement of D_0 . There are two implications in Fig. 7. First, when the value of c_{max} is between 7 and 10, the resultant values of D_0 for the three cases are within a narrow range. Secondly, if c_{max} is large, D_0 is less sensitive to c_{max} (for example, $D_0|_{c_{max}=9}$ and $D_0|_{c_{max}=10}$ are nearly the same). Noting that $c_{cr} = c_{max}$ and $D_0 = \alpha t$, thus we may state that, when c_{cr} is around 10, the estimation of critical time t_{cr} is less dependent on the adoption of anyone of the criteria discussed. Here t_{cr} describes the critical time that a few long cracks start to dominate the fatigue damage.

4. THE FIRST ORDER OF DAMAGE MOMENT

The first order of damage moment is defined as:

$$D_1 = \gamma \int_0^\infty n(c, t) \cdot c \, dc \quad (24)$$

Fig. 6. Fatigue life versus $1/\bar{d}$ for two stress levels.

Fig. 7. Relation between D_0 and $c_{\max}$ for the three cases.

where γ is a non-dimensional coefficient. D_1 is used to describe the extent of fatigue damage, which corresponds to the expectation of total crack length. From eq. (1), the evolution equation of $c \cdot n(c, t)$ can be written as:

$$\frac{\partial[c \cdot n(c, t)]}{\partial t} + \frac{\partial[A(c) \cdot n(c, t) \cdot c]}{\partial c} = N_g \cdot c \cdot n_N + A(c) \cdot n(c, t) \quad (25)$$

eq. (25) has the same form with eq. (1) if $c \cdot n(c, t)$ is regarded as one variable. Referring to the equilibrium of evolution system, one may obtain:

$$\begin{aligned} D_1 &= \gamma \int_0^t dt' \int_0^\infty [N_g \cdot c \cdot n_N + A(c) \cdot n(c, t')] dc \\ &= \gamma \cdot \tilde{\alpha} \cdot t + \gamma \cdot D_1^* \end{aligned} \quad (26)$$

where

$$\tilde{\alpha} = \int_0^\infty N_g \cdot c \cdot n_N(c) dc \quad (27)$$

and

$$D_1^* = \int_0^t dt' \int_0^1 [A(c) - \bar{d}c] \cdot n(c, t') dc + \frac{\bar{d}}{\gamma} \int_0^t D_1(t') dt' \quad (28)$$

eq. (26) can also be expressed as:

$$D_1(t) - \bar{d} \int_0^t D_1(t') dt' = \gamma \cdot \tilde{\alpha} \cdot t + \gamma \cdot \Theta(t) \quad (29)$$

where

$$\Theta(t) = \int_0^t dt' \int_0^1 [A(c) - \bar{d} \cdot c] \cdot n(c, t') dc \quad (30)$$

Let $\bar{A}_d = 1 - A_d$, $\chi = \bar{A}_d + \bar{d}$,

$$\alpha_1 = \int_0^1 N_g \cdot n_N(c) dc,$$

and

$$\alpha_2 = \int_0^1 N_g \cdot c \cdot n_N dc.$$

Then, substituting eq. (2 and3) into (30), one gains:

$$\begin{aligned} \Theta(t) &= \int_0^t dt' \int_0^1 (1 - \chi \cdot c) \cdot n(c, t') dc \\ &= \int_0^t dt' \int_0^1 n(c, t') dc - \chi \int_0^t dt' \int_0^1 c \cdot n(c, t') dc \end{aligned} \quad (31)$$

Considering the equilibrium of crack numerical density for cracks with length between 0 and 1, one may write the total number of cracks as:

$$\int_0^1 n(c, t') dc = \alpha_1 \cdot t' - \int_0^{t'} A_d \cdot n(1, \tau) d\tau \quad (32)$$

Using the same method, one finds:

$$\int_0^1 c \cdot n(c, t') dc = \alpha_2 \cdot t' + \int_0^1 A(c) \cdot n(c, t') dc - \int_0^{t'} A_d \cdot [c \cdot n(c, \tau)]_{c=1} d\tau \quad (33)$$

Substituting eq. (32 and33) into eq. (31), one gets:

$$\Theta(t) = -\bar{A}_d \cdot \Theta(t) + \frac{(1 - \bar{d})\alpha_1 - \chi \cdot \alpha_2}{2} t^2 - A_d(1 - \chi - \bar{d}) \int_0^t dt' \int_0^1 n(1, \tau) d\tau \quad (34)$$

Note that from eq. (20),

$$\begin{aligned} \int_0^t dt' \int_0^1 n(1, \tau) d\tau &= \int_0^t dt' \int_0^1 \left[\frac{1}{\bar{A}_d} \int_{\eta(1, \tau)}^1 N_g \cdot n_N(c) dc \right] d\tau \\ &= \begin{cases} \xi + \xi_2 \cdot t + \xi_3 \cdot t^2 + \xi_4 \cdot e^{\bar{A}_d \cdot t} + \xi_5 \cdot e^{2\bar{A}_d \cdot t} & (t \leq t_0) \\ \frac{3N_g}{8\bar{A}_d} t^2 & (t > t_0) \end{cases} \end{aligned} \quad (35)$$

where

$$\begin{aligned} \xi_1 &= \frac{8(2\bar{A}_d - 1) + A_d}{16 \cdot \bar{A}_d^4} N_g, & \xi_2 &= -\frac{4(2\bar{A}_d - 1) + A_d}{8 \cdot \bar{A}_d^3} N_g \\ \xi_3 &= \frac{3 \cdot \bar{A}_d^2 - 4 \cdot \bar{A}_d + 1}{8 \cdot \bar{A}_d^2 \cdot A_d} N_g, & \xi_4 &= \frac{2 \cdot \bar{A}_d - 1}{2 \cdot \bar{A}_d^4} N_g, & \xi_5 &= \frac{A_d}{16 \cdot \bar{A}_d^4} N_g \\ t_0 &= \frac{1}{\bar{A}_d} \ln \frac{1}{\bar{A}_d} \end{aligned}$$

By substituting eq. (35) into (34), $\Theta(t)$ can be finally solved:

$$\Theta(t) = \begin{cases} \xi_1^* + \xi_2^* \cdot t + \xi_3^* \cdot t^2 + \xi_4^* \cdot e^{\bar{A}_d \cdot t} + \xi_5^* \cdot e^{2\bar{A}_d \cdot t} & (t \leq t_0) \\ \xi_6^* \cdot t^2 & (t > t_0) \end{cases} \quad (36)$$

where

$$\begin{aligned} \xi_1^* &= -A_d \cdot q \cdot \xi_1, & \xi_2^* &= -A_d \cdot q \cdot \xi_2, & \xi_3^* &= \xi - A_d \cdot q \cdot \xi_3, \\ \xi_4^* &= -A_d \cdot q \cdot \xi_4, & \xi_5^* &= -A_d \cdot q \cdot \xi_5, & \xi &= \frac{(1 - \bar{d})\alpha_1 - \gamma \cdot \alpha_2}{2(1 + \bar{A}_d)}, \\ q &= \frac{1 - \gamma - \bar{d}}{1 + \bar{A}_d}, & \xi_6^* &= \xi - \frac{3}{8} q \cdot N_g \end{aligned}$$

If the initial condition is $D_1(0) = 0$, the solution of eq. (29) will be:

$$D_1(t) = \gamma \cdot \bar{d} \cdot e^{\bar{d} \cdot t} \left\{ \int_0^t [\Theta(t') + \tilde{\alpha} \cdot t'] e^{-\bar{d} \cdot t'} dt' - \frac{\Theta(0)}{\bar{d}} \right\} + \gamma [\tilde{\alpha} \cdot t + \Theta(t)] \quad (37)$$

Therefore $D_1(t)$ is obtained by substituting eq. (36) into eq. (37),

$$D_1(t) = \begin{cases} (\tilde{\xi}_1 + \tilde{\xi}_2 \cdot t + \tilde{\xi}_3 \cdot e^{\bar{A}_d \cdot t} + \tilde{\xi}_4 \cdot e^{2\bar{A}_d \cdot t} + \tilde{\xi}_5 \cdot e^{\bar{d} \cdot t}) \cdot \gamma & (t \leq t_0) \\ -\frac{2\gamma\tilde{\xi}_6}{\bar{d}} t + \frac{2\tilde{\xi}_6 + \tilde{\alpha} \cdot \bar{d}}{\bar{d}^2} \gamma \cdot (e^{\bar{d} \cdot t} - 1) & (t > t_0) \end{cases} \quad (38)$$

where

$$\begin{aligned} \tilde{\xi}_1 &= -\frac{\xi_2^* + \tilde{\alpha}}{\bar{d}} - \frac{2 \cdot \xi_3^*}{\bar{d}^2}, & \tilde{\xi}_2 &= -\frac{2 \cdot \xi_3^*}{\bar{d}}, & \tilde{\xi}_3 &= \xi_4^* \left(1 + \frac{\bar{A}_d}{\bar{A}_d - \bar{d}} \right), \\ \tilde{\xi}_4 &= \xi_5^* \left(1 + \frac{\bar{d}}{2 \cdot \bar{A}_d - \bar{d}} \right), & \tilde{\xi}_5 &= -\left(\frac{\xi_2^* + \tilde{\alpha}}{\bar{d}} + \frac{2 \cdot \xi_3^*}{\bar{d}^2} + \xi_4^* \frac{\bar{A}_d}{\bar{A}_d - \bar{d}} + \xi_5^* \frac{\bar{A}_d}{2 \cdot \bar{A}_d - \bar{d}} \right) \end{aligned}$$

eq. (38) is the theoretical solution to the evolution equation of the first order of damage moment, and the relation between

$$\frac{D_1}{\gamma}$$

and t is illustrated in Fig. 8. Since D_1 is often used to assess damage extent but hard to survey directly, and D_0 is relatively easy to determine, Fig. 8 may provide a way to link together experiments and theoretical analysis. Figure 9 shows the influence of material parameters (A_d and $\bar{d}$) on the first order of damage moment, which indicates that the influence of A_d is relatively weak compared with that of $\bar{d}$. Since ξ_6 is considerably small in comparison with $\alpha \cdot \bar{d}$, it is easy to find that, when $t > t_0$, eq. (38) reduced to

$$D_1(t) \doteq \frac{\tilde{\alpha} \cdot \gamma}{\bar{d}} [\exp(\bar{d} \cdot t) - 1] \quad (39)$$

The above equation corresponds to the resultant expression for D_1 from a numerical simulation in ref.[12], in which

Fig. 8. Relation between D_1/γ and t at $A_d=0$ and $\bar{d}=1$.

Fig. 9. Variation of D_1/γ with A_d and $\bar{d}$ at $t = 3.0$.

$$D_1(t) = \frac{\omega}{\kappa} [\exp(\kappa \cdot t) - 1] \quad (40)$$

where ω and κ are material parameters. Equation (39 and 40) suggest that $\dot{D}_1$ is linearly proportional to D_1 :

$$\dot{D}_1 = g_1 \cdot D_1 + g_2 \quad (41)$$

where g_1 and g_2 are material parameters. Equations (39)–(41) express the relationship between damage evolution rate and damage extent, thus reflect the tendency of damage evolution, which is a result of equilibrium equation of crack numerical density and is independent of material properties and experimental environments. From eq. (39), it also can be seen that fatigue damage may develop faster with larger grain size, and the impact of grain size d on D_1 has the form of $[\exp(d) - 1]/d$. Note that in eqs (39)–(41), the influence of A_d is omitted, which suggests that D_1 is independent of grain-boundary obstacle effect.

5. CONCLUSIONS

The concept of the equilibrium of crack numerical density is applied to describe the collective evolution of short fatigue cracks, and the following conclusions are drawn:

1. There exists a saturation distribution of crack numerical density during the development of fatigue process. On the basis of this, the critical time that characterizes the termination of short-crack regime can be obtained upon the stipulation of a criterion.
2. The value of the critical time t_{cr} is influenced by the normalized grain size $\bar{d}$ and the obstacle effect of grain boundary A_d . t_{cr} becomes larger with the decrease of $\bar{d}$ and increases with the reduction in A_d . The total fatigue life N_f is linearly proportional to $1/\bar{d}$.
3. The total number of cracks D_0 increases linearly with time t (relevant to the number of fatigue cycles), whereas the first order of damage moment D_1 increases exponentially with t . When $t > t_0$, $\dot{D}_1$ is linearly proportional to D_1 . The influence of grain-boundary obstacle effect on D_1 can be omitted, and the influence of grain size d has the form of $[\exp(d) - 1]/d$.

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