Modification of Bertrand's Theorem and Extended Runge-Lenz Vector *

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(Received 16 March 1999)

It is shown that for a particle with suitable angular momenta in the screened Coulomb potential or isotropic harmonic potential, there still exist closed orbits rather than ellipse, characterized by the conserved aphelion and perihelion vectors, i.e. extended Runge-Lenz vector, which implies a higher dynamical symmetry than the geometrical symmetry O_3. The closeness of a planar orbit implies the radial and angular motional frequencies are commensurable.

PACS: 03. 65. −w, 03. 65. Ge

In classical mechanics, the maximum number of functional independent conserved quantities of a closed system with N degrees of freedom is $2N-1$.[1] For a system with independent conserved quantities no fewer than N is called integrable.[2] An integrable classical system with $N+\Lambda$ independent conserved quantities $(0\le\Lambda\le N-1)$ is called Λ-fold degenerate, and there exist Λ linear relations to the frequencies $\nu_i(i=1,2,\cdots,N)$ of the system.[3] A classical system for $\Lambda=N-1$ is called a completely degenerate system, and there remains only one independent frequency, which implies the existence of closed orbits. For example, for a particle in a central potential, besides the Hamiltonian, the angular momentum $\boldsymbol{L}$ is also conserved, and the particle in a general central potential is 1-fold degenerate and moves in a plane perpendicular to $\boldsymbol{L}$ while the planar orbits are in general not closed. However, the orbit of a particle in the attractive Coulomb potential $V(r)=-k/r$ is always closed for any continuous negative energy $E<0$ and angular momentum $\boldsymbol{L}$, i.e., an ellipse, of which the length of semi-major axis is $a=1/(2|E|)$ for $m=k=1$ and the eccentricity is $e=\sqrt{1-2|E|/L^2}$. The period of motion is $T=1/\nu=\pi|E|^{-3/2}/\sqrt{2}=2\pi a^{3/2}$ from Kepler's law, where ν is frequency. The closeness of orbits is guaranteed by the existence of an additional conserved quantity the Runge-Lenz vector $\boldsymbol{R}=\boldsymbol{p}\times\boldsymbol{L}-\boldsymbol{r}/r$.[4] In fact, the direction of $\boldsymbol{R}$ is just that of the major axis of elliptic orbit and the magnitude of $\boldsymbol{R}$ is the eccentricity, $|\boldsymbol{R}|=e$. It is seen that $\boldsymbol{R}\cdot\boldsymbol{L}=0$ and $\boldsymbol{R}^2=2H\boldsymbol{L}^2+1$, so the number of independent conserved quantities is 5, and the hydrogen atom is a completely degenerate system. The existence of Runge-Lenz vector implies that the Coulomb potential has a higher dynamical symmetry O_4 than its geometric symmetry O_3.[5] A similar situation exists for an isotropic harmonic oscillator.

Concerning the closeness of orbits, there is a famous Bertrand's theorem,[6] which says that the only central forces that result in closed orbits for all bound particles follow the inverse square law and Hooke's law. In the derivation of Bertrand's theorem, a power-law central potential $W(r)=ar^\nu$ was assumed. If the restriction of a simple power law is relaxed, the Bertrand's theorem needs modification. In this letter we will show that for another type of central potential, the screened Coulomb potential or isotropic harmonic potential $V(r)=W(r)-\lambda/r^2$, where $W(r)$ is the Coulomb potential or isotropic harmonic potential, there still exist closed orbits (rather than elliptic orbits) for suitable angular momenta. The properties of these closed orbits and the corresponding conserved quantity (extended Runge-Lenz vector) will be investigated.

For the valence electron in an alkali atom $V(r)$ may be approximately expressed as the screened Coulomb potential $(e=m=1)$:

$$V(r)=-\frac{1}{r}-\frac{\lambda}{r^2},\quad (0<\lambda\ll 1). \tag{1}$$

In this case, the orbit equation is $(u=1/r)$

$$\mathrm{d}\theta=-\frac{\mathrm{d}u}{\sqrt{2E/L^2+2u/L^2-\kappa^2u^2}}, \tag{2}$$

where $\kappa=\sqrt{1-2\lambda/L^2}$ and $0\le\kappa\le 1$. Integrating Eq. (2), we get[7]

$$u=\frac{1}{r}=\frac{1}{L^2\kappa^2}\{1+\sqrt{1+2EL^2\kappa^2}\cos[\kappa(\theta-\theta_0)]\}, \tag{3}$$

where $\sqrt{1+2EL^2\kappa^2}=\sqrt{1+2E(L^2-2\lambda)}\ge 0$ In general, the orbit is not closed. The precession of a non-closed orbit on an invariant torus is shown in Fig. 1. In fact, all the precessing orbits lie between the perihelion circle with a radius $r_\mathrm{p}=[1/(2|E|)](1-\sqrt{1-2\kappa^2L^2|E|})$ and the aphelion circle with a radius $r_\mathrm{a}=[1/(2|E|)](1+\sqrt{1-2\kappa^2L^2|E|})$. Using the Newton's second law of motion for the screened Coulomb

*Supported by the National Natural Science Foundation of China under Grant No. 19675002, Nonlinear Science Project of China, Post-Doctoral Foundation, and Doctoral Program Foundation of Education Committee of China.

potential (1), $\dot{\boldsymbol{p}} = -(r+2\lambda)\boldsymbol{r}/r^4$, so that at the aphelion and perihelion points, i.e. at $\dot{r} = 0$

$$\frac{\mathrm{d}\boldsymbol{R}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{p}\times\boldsymbol{L} - \left(1+\frac{2\lambda}{r}\right)\frac{\boldsymbol{r}}{r}\right] = 0. \qquad (4)$$

where $\boldsymbol{R}$ is in the opposite direction of the radial vector $\boldsymbol{r}$ and its magnitude is

$$|R'| = \sqrt{2(H-\lambda/r^2)L^2 + (1+2\lambda/r)^2}.$$

However, for irrational values of κ, a particle starting from any point never reture to the initial point, i.e., the orbit is not closed.

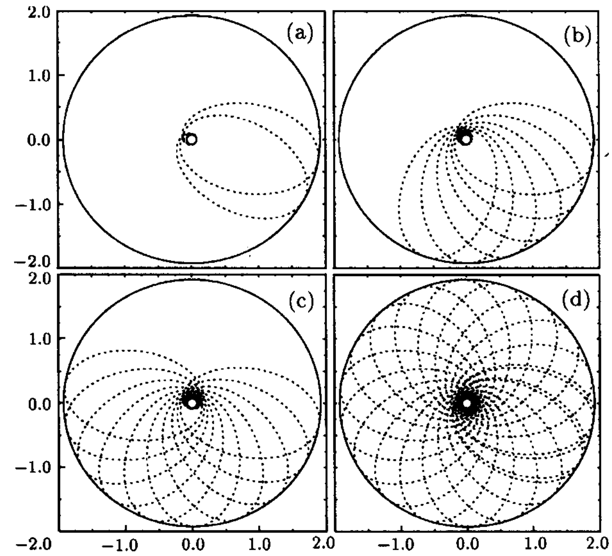

Fig. 1. Precession of a non-closed orbit of a particle on the $r-\theta$ plane in screened Coulomb potential of Eq.(1) with $\lambda = 0.2$, $E = -0.5$, and $\kappa = 1/2 + \sqrt{2}/100$.

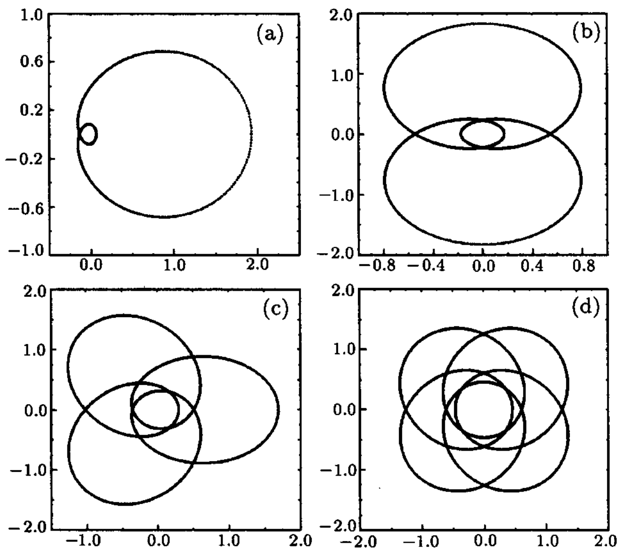

Fig. 2. Closed orbits of a particle on the $r-\theta$ plane in the screened Coulomb potential of Eq. (1) with $\lambda = 0.2$ and $E = -0.5$; (a) $\kappa = (1/2)$ or $L = (2/3)\sqrt{6\lambda}$, (b) $\kappa = (2/3)$ or $L = (3/5)\sqrt{10\lambda}$, (c) $\kappa = (3/4)$ or $L = (4/7)\sqrt{14\lambda}$, (d) $\kappa = (4/5)$ or $L = (5/3)\sqrt{2\lambda}$.

It is interesting to note that there exist infinite numbers of closed orbits corresponding to rational values of $\kappa = \sqrt{1-2\lambda/L^2}$. Some illustrative examples are displayed in Fig. 2. The geometry of closed orbits depends only on the value of κ, i.e. angular momentum L, but is irrelevant to the energy E. The directions of each aphelion vector and perihelion vector expressed by θ_a and θ_p respectively keep constant during the motion, i.e.,

$$\theta_a - \theta_0 = (2n+1)\frac{\pi}{\kappa},$$
$$\theta_p - \theta_0 = 2n\frac{\pi}{\kappa}, \quad n = 0, 1, 2, \cdots. \qquad (5)$$

Of course, the length of aphelion (perihelion) vector increases (decreases) with increasing E. The closeness of a planar orbit implies that the radial frequency ω_r and angular frequency ω_θ are commensurable. It can be shown that

$$\frac{\omega_r}{\omega_\theta} = \kappa. \qquad (6)$$

For $\lambda = 0$, i.e. $\kappa = 1$, and $\omega_r/\omega_\theta = 1$, the closed orbit becomes an ellipse and $\boldsymbol{R}'$ is reduced to the famous Runge-Lenz vector $\boldsymbol{R} = \boldsymbol{p}\times\boldsymbol{L} - \boldsymbol{r}/r$. However, while $\mathrm{d}\boldsymbol{R}/\mathrm{d}t = 0$ holds at every point along the closed orbit for the Coulomb potential, $\mathrm{d}\boldsymbol{R}'/\mathrm{d}t = 0$ for $\lambda \neq 0$ holds only at the aphelion and perihelion points, which implies that the original dynamical symmetry O_4 for the Coulomb potential is partially broken.

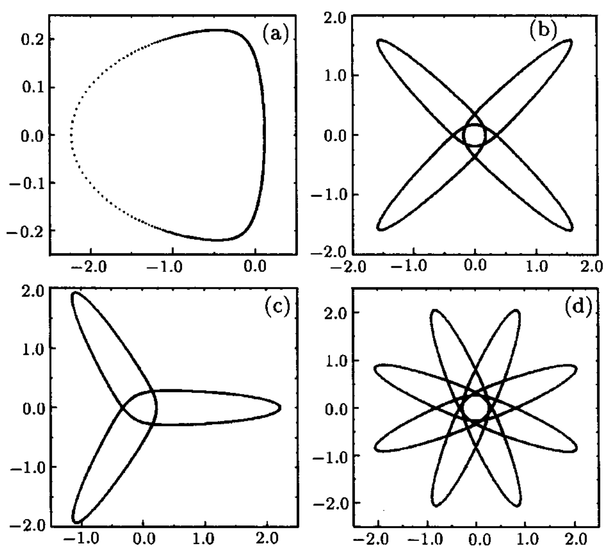

Fig. 3. Closed orbits of a particle on the $r-\theta$ plane in the screened isotropic potential of Eq. (7) with $\lambda = 0.2$ and $E = 5$.

Next, we consider the particle in the screened isotropic harmonic potential:

$$V(r) = r^2 - \frac{\lambda}{r^2}. \qquad (7)$$

The orbit equation is

$$d\theta = -\frac{du}{\sqrt{2E/L^2 - 2/(L^2u^2) - \kappa^2 u^2}}, \quad (8)$$

where $\kappa = \sqrt{1 - 2\lambda/L^2}$. Integrating Eq. (8), we get

$$u^2 = \frac{1}{r^2} = \frac{1}{L^2\kappa^2}\{E + \sqrt{E^2 - 2L^2\kappa^2}\cos[2\kappa(\theta - \theta_0)]\}. \quad (9)$$

Similarly, closed orbits still exist for rational number values of κ. The closed orbits for $\kappa = 1/2$, $2/3$, $3/4$, and $4/5$ are displayed in Figs. 3(a)−3(d), respectively. The direction of the aphelion and perihelion vectors can be expressed by

$$\theta_a - \theta_0 = (n + \frac{1}{2})\frac{\pi}{\kappa},$$
$$\theta_p - \theta_0 = n\frac{\pi}{\kappa},\ n = 0, 1, 2, \cdots, \quad (10)$$

and

$$\frac{\omega_r}{\omega_\theta} = 2\kappa. \quad (11)$$

Finally, it is worthwhile to mention that there exists intimate relation between the conserved quantities responsible for the closeness of classical orbits on one side the quantum mechanics raising and lowering operators on the other side.[11,12] For a classical particle in a power law central potential $W(r)$, for and only for the Coulomb potential or isotropic harmonic oscillator, the orbits are closed for any negative energy and positive angular momentum. In quantum mechanics, it was shown[8–10] that only for the Coulomb potential and isotropic harmonic potential the radial Schrödinger equation can be factorized and from the factorization one can construct both energy and angular momentum raising and lowering operators, which is equivalent to the conserved quantities responsible for the closeness of classical orbits.[11] But for the screened Coulomb potential (1) or isotropic harmonic potential (7), the orbit is not closed in general. However, the closed orbits still exist for any negative energy, but only for suitable discrete angular momentum

$$L = \sqrt{2\lambda/(1 - \kappa^2)}$$

(κ being rational number), which implies the original dynamical symmetry of $W(r)$ is partially broken. Correspondingly, in quantum mechanics, it can be shown[7] that in this case only the energy (but not angular momentum) raising and lowering operators can be constructed from the factorization of radial Schrödinger equation. The dynamical symmetry of the screened Coulomb potential and isotropic potential needs further investigation.

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