ELSEVIER

Theoretical and Applied Fracture Mechanics 33 (2000) 191–198

theoretical and applied fracture mechanics

www.elsevier.com/locate/tafmec

An elastoplastic constitutive relation of whisker-reinforced composite for meso damage: Part I – formulation

H.Q. Liu *, N.G. Liang

Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, People's Republic of China

Abstract

An elastoplastic constitutive relation is developed for meso damage of whisker-reinforced composites. A model is constructed that includes orientation distribution of whiskers and slip systems as well as interface and crystal sliding. Evolution of damage will be addressed. Given in Part I is the formulation while examples will be illustrated in Part II.

1. Introduction

Continuous Damage Mechanics for homogeneous material has been developed in [1]. Using the intrinsic theory of thermo-dynamics, the damage variable was brought into the framework of Continuum Mechanics [2], whereby the isotropic damage constitutive equation [3] was extended to include anisotropy [4].

Recently, many efforts have been made to develop relationships between the material microstructure and macroperformance of composites. Self-consistent scheme and equivalent inclusion method were used as a basis for the mechanical analysis of fiber or particle reinforced composites. As an extension of Eshelby's method, the work in [5] has been widely adopted for composites. An approximate analysis of two neighboring microcracks was introduced to account for interaction among randomly oriented and located microcracks [6]. From a parallel fiber bundle model for long-fiber-reinforced composites, damage was analyzed using the probability distribution of fiber rupture strengths [7].

Since the microscopic description of whisker reinforcing and crystal sliding mechanisms was introduced [8], the mesoscopic material model and constitutive relations for polycrystalline metals and whisker-reinforced composites have been established [9–11]. Thus, the evolution of the yield surface and deformation-induced anisotropy was predicted [10–12]. Recently, the fiber-breaking mechanism has been applied to the elastic damage constitutive relation for predicting the damage-induced anisotropy and damage-rate effect of brittle fiber-reinforced composite material [13].

Experimental results show that damage of composites is associated with multiple physical mechanisms. The response is sensitive to the interface-linking state between the matrix and reinforcement [14]. Anisotropy however should be investigated in combination with the internal stress redistribution due to damage [15] and damage rate [13,16]. It is still a challenge to formulate a constitutive relation and to explain the complex nature of damage behavior.

* Corresponding author.
E-mail address: hliu@lnm.imech.ac.cn (H.Q. Liu).

0167-8442/00/$ - see front matter
PII: S 0 1 6 7 - 8 4 4 2 (0 0) 0 0 0 1 4 - 8

Part I of this work is concerned with the microstructure and damage mechanism of matrix and whiskers. A model is constructed followed by a discussion of the crystal and interface sliding criteria and damage evolution of the constituents. This leads to the derivation of a constitutive equation that accounts for damage.

2. Material model

Whisker-reinforced metal–matrix composites (MMCs) are composed of metallic matrix and whiskers. The overall elastoplastic damage behavior has been associated with the microstructure and damage of the matrix, whiskers, and interface. To develop a material model that has characteristics consistent with the composite material, it is assumed that:

- Deformation of the matrix is decomposed into an elastic part $\dot{\mathbf{E}}_{\mathrm{me}}$ in the crystal grains and a plastic part $\dot{\mathbf{E}}_{\mathrm{ms}}$ caused by crystal sliding. The total strain rate of the matrix is compatible with macro strain rate $\dot{\mathbf{E}}$, i.e.

$$\dot{\mathbf{E}} = \dot{\mathbf{E}}_{\mathrm{me}} + \dot{\mathbf{E}}_{\mathrm{ms}}. \tag{1}$$

- Local stresses $\tilde{\mathbf{S}}_{\mathrm{g}}$ in a crystal grain and $\tilde{\tau}_{\mathrm{s}}$ in a slip system are proportional to the average stress $\mathbf{S}_{\mathrm{m}}$ and resolved shear stress τ_{s} of the matrix, i.e.

$$\tilde{\mathbf{S}}_{\mathrm{g}} = \tilde{\mathbf{c}}_{\mathrm{g}} : \mathbf{S}_{\mathrm{m}}, \tag{2}$$

$$\tilde{\tau}_{\mathrm{s}} = \tilde{c}_{\mathrm{s}} \tau_{\mathrm{s}}, \tag{3}$$

$$\tau_{\mathrm{s}} = \mathbf{P}_{\mathrm{s}} : \mathbf{S}_{\mathrm{m}}, \tag{4}$$

where $\tilde{\mathbf{c}}_{\mathrm{g}}$ and $\tilde{c}_{\mathrm{s}}$ are stress heterogeneity factors of the crystal grain and slip system. Note that

$$\mathbf{P}_{\mathrm{s}} = \frac{1}{2}(\mathbf{m} \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{m}) \tag{5}$$

is an orientation tensor of the slip system. In Eq. (5), $\mathbf{m}$ is a unit slip vector and $\mathbf{n}$, a unit normal vector of the sliding plane.

- Local strain rate $\dot{\tilde{\varepsilon}}_{\mathrm{f}}$ in a whisker is proportional to the average strain rate $\dot{\varepsilon}_{\mathrm{f}}$ that is compatible with macro strain rate tensor, i.e.

$$\dot{\tilde{\varepsilon}}_{\mathrm{f}} = \tilde{c}_{\mathrm{f}} \dot{\varepsilon}_{\mathrm{f}}, \tag{6}$$

$$\dot{\varepsilon}_{\mathrm{f}} = \mathbf{P}_{\mathrm{f}} : \dot{\mathbf{E}}, \tag{7}$$

where $\tilde{c}_{\mathrm{f}}$ is a heterogeneity factor of the strain rate of the whisker. Here

$$\mathbf{P}_{\mathrm{f}} = \mathbf{l} \otimes \mathbf{l} \tag{8}$$

is an orientation tensor of the whisker and $\mathbf{l}$, a unit vector along the whisker.

The symbols with tilde refer to location. $\tilde{\mathbf{c}}_{\mathrm{g}}$ and $\tilde{c}_{\mathrm{s}}$ may vary with the average stress of the matrix, and $\tilde{c}_{\mathrm{f}}$ may vary with macro strain. The crystal grains, slip systems, and whiskers are incrementally linear, such that

$$\dot{\tilde{\mathbf{E}}}_{\mathrm{g}} = \tilde{\mathbf{C}}_{\mathrm{g}} : \dot{\tilde{\mathbf{S}}}_{\mathrm{g}}, \tag{9}$$

$$\dot{\tilde{\gamma}}_{\mathrm{s}} = \frac{1}{\tilde{h}_{\mathrm{s}}} \dot{\tilde{\tau}}_{\mathrm{s}}, \tag{10}$$

$$\dot{\tilde{\sigma}}_{\mathrm{f}} = \tilde{E}_{\mathrm{f}} \dot{\tilde{\varepsilon}}_{\mathrm{f}}, \tag{11}$$

where $\dot{\mathbf{E}}_{\mathrm{g}}$ and $\tilde{\mathbf{C}}_{\mathrm{g}}$ are local strain rate and compliance tensors of crystal grains, respectively. The local sliding rate and hardening modulus of slip systems are given by $\dot{\tilde{\gamma}}_{\mathrm{s}}$ and $\tilde{h}_{\mathrm{s}}$, while $\dot{\tilde{\sigma}}_{\mathrm{f}}$ and $\tilde{E}_{\mathrm{f}}$ are the local stress rate and Young's modulus of whiskers, respectively.

2.1. The matrix

Polycrystalline metal matrix consists of many irregular crystal grains. Crystal sliding is the plastic deformation mechanism of the matrix.

Elasticity of the matrix: According to the assumptions, the power stored in crystal grains can be expressed as

$$\dot{W} = \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{\mathbf{S}}_{\mathrm{g}} : \tilde{\mathbf{E}}_{\mathrm{g}} \,\mathrm{d}V = \mathbf{S}_{\mathrm{m}} : \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{\mathbf{c}}_{\mathrm{g}} : \tilde{\mathbf{E}}_{\mathrm{g}} \,\mathrm{d}V, \tag{12}$$

where V_{m} is volume fraction of the matrix. Being a power conjugate with $\mathbf{S}_{\mathrm{m}}$, the elastic strain rate tensor in crystal grains can be written as

$$\dot{\mathbf{E}}_{\mathrm{me}} = \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{\mathbf{c}}_{\mathrm{g}} : \dot{\tilde{\mathbf{E}}}_{\mathrm{g}} \,\mathrm{d}V. \tag{13}$$

Substitution of Eqs. (9) and (2) into Eq. (13) yields

$$\dot{\mathbf{E}}_{\mathrm{me}} = \overline{\mathbf{C}}_{\mathrm{me}} : \dot{\mathbf{S}}_{\mathrm{m}}$$
$$\overline{\mathbf{C}}_{\mathrm{me}} = \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{\mathbf{c}}_{\mathrm{g}} : \tilde{\mathbf{C}}_{\mathrm{g}} : \tilde{\mathbf{c}}_{\mathrm{g}} + \frac{1}{2} \tilde{\mathbf{c}}_{\mathrm{g}} : \tilde{\mathbf{C}}_{\mathrm{g}} : \left(\tilde{\mathbf{S}}_{\mathrm{m}} : \frac{\mathrm{d}\tilde{\mathbf{c}}_{\mathrm{g}}}{\mathrm{d}\mathbf{S}_{\mathrm{m}}} + \frac{\mathrm{d}\tilde{\mathbf{c}}_{\mathrm{g}}}{\mathrm{d}\mathbf{S}_{\mathrm{m}}} : \tilde{\mathbf{S}}_{\mathrm{m}} \right) \mathrm{d}V. \tag{14}$$

Thus, the elastic behavior of the heterogeneous matrix described by Eq. (14) can be represented by one of a uniform *elastic medium* with the average compliance tensor $\overline{\mathbf{C}}_{\mathrm{me}}$. $\dot{\mathbf{S}}_{\mathrm{m}}$ and $\dot{\mathbf{E}}_{\mathrm{me}}$ represent the average stress and elastic strain rate tensors of the elastic medium or matrix.

Plasticity of crystal sliding: Consider slip in the direction $\mathbf{m}$ on slip planes with the unit normal vector $\mathbf{n}$. The power dissipated by the local slip systems can be expressed as

$$\dot{W} = \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{\tau}_{\mathrm{s}} \dot{\tilde{\gamma}}_{\mathrm{s}} \, \mathrm{d}V = \left\{ \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{c}_{\mathrm{s}} \dot{\tilde{\gamma}}_{\mathrm{s}} \, \mathrm{d}V \right\} \tau_{\mathrm{s}}. \tag{15}$$

An average sliding strain rate conjugate with τ_{s} is

$$\dot{\gamma} = \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \tilde{c}_{\mathrm{s}} \dot{\tilde{\gamma}}_{\mathrm{s}} \, \mathrm{d}V. \tag{16}$$

By use of Eqs. (3) and (10), the local sliding strain rate can be derived

$$\dot{\tilde{\gamma}}_{\mathrm{s}} = \frac{1}{\tilde{h}_{\mathrm{s}}} (\tilde{c}_{\mathrm{s}} \dot{\tau}_{\mathrm{s}} + \dot{\tilde{c}}_{\mathrm{s}} \tau_{\mathrm{s}}) = \frac{1}{\tilde{h}_{\mathrm{s}}} \left(\tilde{c}_{\mathrm{s}} + \frac{\mathrm{d}\tilde{c}_{\mathrm{s}}}{\mathrm{d}\tau_{\mathrm{s}}} \tau_{\mathrm{s}} \right) \dot{\tau}_{\mathrm{s}}. \tag{17}$$

Substituting Eq. (17) into Eq. (16), the local slip systems can be represented by an *equivalent slip system* with an average hardening modulus $\overline{h}_{\mathrm{s}}$, and

$$\dot{\gamma}_{\mathrm{s}} = \frac{1}{\overline{h}_{\mathrm{s}}} \dot{\tau}_{\mathrm{s}},$$
$$\frac{1}{\overline{h}_{\mathrm{s}}} = \frac{1}{V_{\mathrm{m}}} \int_{V_{\mathrm{m}}} \frac{\tilde{c}_{\mathrm{s}}}{\tilde{h}_{\mathrm{s}}} \left(\tilde{c}_{\mathrm{s}} + \frac{\mathrm{d}\tilde{c}_{\mathrm{s}}}{\mathrm{d}\tau_{\mathrm{s}}} \tau_{\mathrm{s}} \right) \mathrm{d}V, \tag{18}$$

where $\dot{\tau}_{\mathrm{s}}$ and $\dot{\gamma}_{\mathrm{s}}$ denote resolved shear stress and sliding rates of the equivalent slip system.

An orientation distribution of equivalent slip systems can be described with a statistical orientation density ρ_{s} of all slip systems in the matrix. Without consideration of any preferred orientation, equivalent slip systems are assumed to be homogeneously distributed in the 3D space, and $\rho_{\mathrm{s}} = 1/4\pi^2$.

A slip system may be active as long as the resolved shear stress reaches its critical value $\tau_{+\mathrm{cr}}$ in $\mathbf{m}$ direction or $\tau_{-\mathrm{cr}}$ in $-\mathbf{m}$ direction. An activation criterion of the slip system can be stated as [11]

$$\begin{cases} \text{If } \tau_{\mathrm{s}} = \tau_{+\mathrm{cr}} \\ \quad \text{then} \quad \dot{\gamma}_{\mathrm{s}} > 0, \dot{\tau}_{+\mathrm{cr}} = \dot{\tau}_{\mathrm{s}} = \overline{h}_{\mathrm{s}} \dot{\gamma}_{\mathrm{s}} \text{ and} \\ \qquad \qquad \dot{\tau}_{-\mathrm{cr}} = \dot{\tau}_{+\mathrm{cr}} - 2\tau_{\mathrm{cr0}}. \\ \text{If } \tau_{\mathrm{s}} = \tau_{-\mathrm{cr}} \\ \quad \text{then} \quad \dot{\gamma}_{\mathrm{s}} < 0, \dot{\tau}_{-\mathrm{cr}} = \dot{\tau}_{\mathrm{s}} = \overline{h}_{\mathrm{s}} \dot{\gamma}_{\mathrm{s}} \text{ and} \\ \qquad \qquad \dot{\tau}_{+\mathrm{cr}} = \dot{\tau}_{-\mathrm{cr}} + 2\tau_{\mathrm{cr0}}. \\ \text{Otherwise} \quad \dot{\gamma}_{\mathrm{s}} = 0, \dot{\tau}_{\pm cr} = 0. \end{cases} \tag{19}$$

In Eq. (19), Prager's kinematics hardening rule is applied, and τ_{cr0} is an initially critical resolved shear stress.

2.2. Whiskers

Experimental results show that the interface siding between the matrix and whiskers may take place when the fiber strain reaches its critical value [17]. And the interface debonding is mainly dependent on the maximum fiber-direction strain [14]. Therefore, the interface sliding may be regarded as pseudo-plasticity of whisker [18]. Hence, the whisker behaves as in elasto-pseudo-plasticity.

Elasticity of whiskers: Consider whiskers distributed unidirectionally in the matrix. The stored power in the whiskers can be expressed as

$$\dot{W}_{\mathrm{f}} = \frac{1}{V_{\mathrm{f}}} \int_{V_{\mathrm{f}}} \tilde{\sigma}_{\mathrm{f}} \dot{\tilde{\varepsilon}}_{\mathrm{f}} \, \mathrm{d}V = \left\{ \frac{1}{V_{\mathrm{f}}} \int_{V_{\mathrm{f}}} \tilde{c}_{\mathrm{f}} \tilde{\sigma}_{\mathrm{f}} \, \mathrm{d}V \right\} \dot{\varepsilon}_{\mathrm{f}}, \tag{20}$$

where V_{f} is volume fraction of whiskers in direction $\mathbf{l}$. An average stress σ_{f} conjugate with $\dot{\varepsilon}_{\mathrm{f}}$ is

$$\sigma_{\mathrm{f}} = \frac{1}{V_{\mathrm{f}}} \int_{v_{\mathrm{f}}} \tilde{c}_{\mathrm{f}} \tilde{\sigma}_{\mathrm{f}} \, \mathrm{d}V. \tag{21}$$

Thus, the stress rate can be derived as

$$\dot{\sigma}_{\mathrm{f}} = \frac{1}{V_{\mathrm{f}}} \int_{V_{\mathrm{f}}} (\tilde{c}_{\mathrm{f}} \dot{\tilde{\sigma}}_{\mathrm{f}} + \dot{\tilde{c}}_{\mathrm{f}} \tilde{\sigma}_{\mathrm{f}}) \, \mathrm{d}V. \tag{22}$$

Substitution of Eq. (11) and Eq. (6) into Eq. (22) yields

$$\dot{\sigma}_{\mathrm{f}} = \overline{E}_{\mathrm{f}}\dot{\varepsilon}_{\mathrm{f}},$$
$$\overline{E}_{\mathrm{f}} = \frac{1}{V_{\mathrm{f}}}\int_{V_{\mathrm{f}}}\left(\tilde{c}_{\mathrm{f}}^{2}\tilde{E}_{\mathrm{f}} + \tilde{\sigma}_{\mathrm{f}}\frac{\mathrm{d}\tilde{c}_{\mathrm{f}}}{\mathrm{d}\varepsilon}\right)\mathrm{d}V. \tag{23}$$

Those whiskers in direction **l** are represented by a *fiber-bundle* with the average stiffness modulus $\overline{E}_{\mathrm{f}}$. Note that $\dot{\sigma}_{\mathrm{f}}$ and $\dot{\varepsilon}_{\mathrm{f}}$ denote the stress and strain rates of the fiber-bundle, respectively.

Statistically, a probability density ρ_{f} can be introduced to describe the orientation distribution of the fiber-bundles. The homogeneous orientation distribution density is $\rho_{\mathrm{f}} = 1/2\pi$.

Pseudo-plasticity of interface sliding: Interface sliding may take place when the fiber strain reaches its critical value $\varepsilon_{+\mathrm{cr}}$ in tension or $\varepsilon_{-\mathrm{cr}}$ in compression. Then, the behavior of the fiber-bundle becomes pseudo-plastic that can be described as

$$\dot{\sigma}_{\mathrm{f}} = \overline{E}_{\mathrm{fs}}\ \dot{\varepsilon}_{\mathrm{f}}, \tag{24}$$

where $\overline{E}_{\mathrm{fs}}$ is an average interface sliding modulus. An interface-sliding criterion can be stated as

$$\begin{cases} \text{If} \quad \varepsilon_{\mathrm{f}} = \varepsilon_{+cr} \\ \quad \text{then} \quad \dot{\varepsilon}_{+\mathrm{cr}} = \dot{\varepsilon}_{\mathrm{f}} = \dot{\sigma}_{\mathrm{f}}/\overline{E}_{\mathrm{fs}} \text{ and} \\ \quad\quad\quad\quad \dot{\varepsilon}_{-\mathrm{cr}} = \dot{\varepsilon}_{+\mathrm{cr}} - 2\varepsilon_{\mathrm{cr0}}. \\ \text{If } \tau_{\mathrm{s}} = \tau_{-\mathrm{cr}} \\ \quad \text{then} \quad \dot{\varepsilon}_{-\mathrm{cr}} = \dot{\varepsilon}_{\mathrm{f}} = \dot{\sigma}_{\mathrm{f}}/\overline{E}_{\mathrm{fs}} \text{ and} \\ \quad\quad\quad\quad \dot{\varepsilon}_{+\mathrm{cr}} = \dot{\varepsilon}_{-\mathrm{cr}} + 2\varepsilon_{\mathrm{cr0}}. \\ \text{Otherwise} \quad \dot{\varepsilon}_{\pm cr} = 0. \end{cases} \tag{25}$$

The quantity$\varepsilon_{\mathrm{cr0}} = (\varepsilon_{+\mathrm{cr0}} - \varepsilon_{-\mathrm{cr0}})/2$ is an initial critical sliding strain of the fiber-bundle [10].

3. Damage

Damage of composites is a complex process. Their failure modes are dominated by multiple mechanisms. In the following, damage is attributed to the mechanical property degradation of three elements.

3.1. Statistical expression of damage

Let D_{m} denote a volume fraction of the degraded elastic medium, D_{s} denote a percentage of the sliding net area reduction of the slip system, and D_{f} denote a percentage of the load-bearing-capacity degradation of the fiber-bundle.

According to [1], there exist

$$\mathbf{S}_{\mathrm{m}} = (1 - D_{\mathrm{m}})\mathbf{S}_{\mathrm{m}}^{(\mathrm{ef})}, \tag{26}$$

$$\tau_{\mathrm{s}} = (1 - D_{\mathrm{s}})\tau_{\mathrm{s}}^{(\mathrm{ef})}, \tag{27}$$

$$\sigma_{\mathrm{f}} = (1 - D_{\mathrm{f}})\sigma_{\mathrm{f}}^{(\mathrm{ef})}. \tag{28}$$

The effective stresses $\mathbf{S}_{\mathrm{m}}^{(\mathrm{ef})}, \tau_{\mathrm{s}}^{(\mathrm{ef})}$ and $\sigma_{\mathrm{f}}^{(\mathrm{ef})}$ satisfy Eqs. (14), (18) and (23), respectively. This gives

$$\dot{\mathbf{S}}_{\mathrm{m}}^{(\mathrm{ef})} = \overline{\mathbf{K}}_{\mathrm{me}} : \dot{\mathbf{E}}_{\mathrm{me}}, \tag{29}$$

$$\dot{\tau}_{\mathrm{s}}^{(\mathrm{ef})} = \overline{h}_{\mathrm{s}}\dot{\gamma}_{\mathrm{s}}, \tag{30}$$

$$\dot{\sigma}_{\mathrm{f}}^{(\mathrm{ef})} = \overline{E}_{\mathrm{f}}\dot{\varepsilon}_{\mathrm{f}}. \tag{31}$$

Statistically, there exists a probability distribution density of the broken strain for the fiber-bundle [13]. The effect of interface-debonding on load-bearing capacity of the fiber-bundle is taken into account. Let $\psi_{\mathrm{f}}(\varepsilon_{\mathrm{fc}})$ denote the probability distribution density of fiber breaking and interface-debonding. Here, $\varepsilon_{\mathrm{fc}}$ denotes the smaller one of broken strain and debonding strain of whiskers. D_{f} can be expressed as

$$D_{\mathrm{f}}(\varepsilon_{\mathrm{f}}^{\mathrm{m}}) = \int_{\varepsilon_{\mathrm{cr}}}^{\varepsilon_{\mathrm{f}}^{\mathrm{m}}} \psi_{\mathrm{f}}(\varepsilon_{\mathrm{fc}})\,\mathrm{d}\varepsilon_{\mathrm{fc}}. \tag{32}$$

In Eq. (32), $\varepsilon_{\mathrm{cr}}$ is the smaller critical strain of the weakest whiskers and the weakest-linking interface, while $\varepsilon_{\mathrm{f}}^{\mathrm{m}}$ is the maximum strain of the fiber-bundle reached [11].

Similarly, let $\psi_{\mathrm{s}}(\gamma_{\mathrm{sc}})$ be the probability distribution density of the failure strain γ_{sc} of the slip system. D_{s} can be expressed as

$$D_{\mathrm{s}}(\gamma_{\mathrm{s}}^{\mathrm{m}}) = \int_{\gamma_{\mathrm{cr}}}^{\gamma_{\mathrm{s}}^{\mathrm{m}}} \psi_{\mathrm{s}}(\gamma_{\mathrm{sc}})\,\mathrm{d}\gamma_{\mathrm{sc}}, \tag{33}$$

where γ_{cr} is the critical failure strain of the weakest sliding system, and $\gamma_{\mathrm{s}}^{\mathrm{m}}$, the maximum sliding strain.

3.2. Thermodynamic consideration of damage

For an isothermal and infinitesimal-strain process, independent state variables are strain and damage tensors, $\boldsymbol{E}$ and $\boldsymbol{D}$. Let f denote the specific Helmholtz' free energy. The generalized Gibbs' relation [19] can be expressed as

$$\rho_0 \dot{f} = \mathbf{S} : \mathbf{E} - \boldsymbol{\eta} \cdot \dot{\mathbf{D}}, \tag{34}$$

where ρ_0 is mass density.

As far as each element was concerned, $\mathbf{E}_{\mathrm{me}}$ and D_{me} are conjugate to $\mathbf{S}_{\mathrm{m}}$ and η_{me}; γ_{s} and D_{s} are conjugate to τ_{s} and η_{s}; ε_{f} and D_{f} are conjugate to σ_{f} and η_{f}, respectively. Neglect the interaction of damage. There exist $D_{\mathrm{me}} = D_{\mathrm{me}}(\mathbf{E}_{\mathrm{me}}, \eta_{\mathrm{me}}(\mathbf{E}_{\mathrm{me}}))$, $D_{\mathrm{s}} = D_{\mathrm{s}}(\gamma_{\mathrm{s}},\ \eta_{\mathrm{s}}(\gamma_{\mathrm{s}}))$ and $D_{\mathrm{f}} = D_{\mathrm{f}}(\varepsilon_{\mathrm{f}}, \eta_{\mathrm{f}}(\varepsilon_{\mathrm{f}}))$. That is to say, each damage variable can be determined by its respective deformation. Thus, the damage evolution of essential elements can be expressed as

$$\dot{D}_{\mathrm{me}} = \frac{\mathrm{d}D_{\mathrm{me}}}{\mathrm{d}\mathbf{E}_{\mathrm{me}}} : \dot{\mathbf{E}}_{\mathrm{me}}, \tag{35}$$

$$\dot{D}_{\mathrm{s}} = \frac{\mathrm{d}D_{\mathrm{s}}}{\mathrm{d}\gamma_{\mathrm{s}}} \dot{\gamma}_{\mathrm{s}} = \psi_{\mathrm{s}}(\gamma_{\mathrm{s}})\dot{\gamma}_{\mathrm{s}}, \tag{36}$$

$$\dot{D}_{\mathrm{f}} = \frac{\mathrm{d}D_{\mathrm{f}}}{\mathrm{d}\varepsilon_{\mathrm{f}}} \dot{\varepsilon}_{\mathrm{f}} = \psi_{\mathrm{f}}(\varepsilon_{\mathrm{f}})\dot{\varepsilon}_{\mathrm{f}}. \tag{37}$$

The damage rate of essential elements is proportional to their respective strain rate, and the proportional coefficients are material-dependent.

4. Constitutive relation

According to the basic assumption, the whisker-reinforced metal–matrix composite can be modeled with three types of essential elements. Equivalent slip systems and fiber-bundles are distributed according to their orientation distribution density ρ_{s} and ρ_{f}. The heterogeneity is embodied in the average stiffness and compliance tensors of the three elements mentioned earlier. Based on the material model, the constitutive equations are derived for elastoplasticity and damage.

4.1. Elastoplasticity

Consider a representative elementary volume (REV) with orientation distribution density ρ_{s} and ρ_{f}. The total power of REV equals a sum of those for the three types of elements, i.e.

$$\begin{aligned}\mathbf{S} : \dot{\mathbf{E}} = {} & V_{\mathrm{m}}\mathbf{S}_{\mathrm{m}} : \dot{\mathbf{E}}_{\mathrm{me}} + V_{\mathrm{m}} \int_{\Psi} \int_{\Phi} \rho_{\mathrm{s}}\tau_{\mathrm{s}}\dot{\gamma}_{\mathrm{s}}\, \mathrm{d}\Phi\, \mathrm{d}\Psi \\ & + V_{\mathrm{f}} \int_{\Omega} \rho_{\mathrm{f}}\sigma_{\mathrm{f}}\dot{\varepsilon}_{\mathrm{f}}\, \mathrm{d}\Omega,\end{aligned} \tag{38}$$

where $\mathrm{d}\Phi$ is a solid angle in direction n, $\mathrm{d}\Psi$, a plane angle in direction m, $\mathrm{d}\Omega$, a solid angle in direction l, V_{m} and V_{f} are volume fractions of the matrix and fiber-bundles such that satisfy $V_{\mathrm{m}} + V_{\mathrm{f}} = 1$. Substituting Eqs. (4) and (7) into Eq. (38), there results

$$\begin{aligned}\mathbf{S} : \dot{\mathbf{E}} = {} & V_{\mathrm{m}}\mathbf{S}_{\mathrm{m}} : \left\{ \dot{\mathbf{E}}_{\mathrm{me}} + \int_{\Psi} \int_{\Phi} \rho_{\mathrm{s}}\dot{\gamma}_{\mathrm{s}}\mathbf{P}_{\mathrm{s}}\, \mathrm{d}\Phi\, \mathrm{d}\Psi \right\} \\ & + V_{\mathrm{f}}\left\{ \int_{\Omega} \rho_{\mathrm{f}}\sigma_{\mathrm{f}}\mathbf{P}_{\mathrm{f}}\, \mathrm{d}\Omega \right\} : \dot{\mathbf{E}}.\end{aligned} \tag{39}$$

The first term on the right-hand side of Eq. (39) is the power stored in the elastic medium; the second is that dissipated by the active slip systems; and the third is for one of the fiber-bundles.According to power conjugate principle, the strain rate tensor produced by the crystal sliding can be obtained by use of Eqs. (4) and (18) and Eq. (39)

$$\begin{aligned}\dot{\mathbf{E}}_{\mathrm{ms}} &= \int_{\Psi} \int_{\Phi} \rho_{\mathrm{s}}\dot{\gamma}_{\mathrm{s}}\mathbf{P}_{\mathrm{s}}\, \mathrm{d}\Phi\, \mathrm{d}\Psi = \overline{\mathbf{C}}_{\mathrm{ms}} : \dot{\mathbf{S}}_{\mathrm{m}}, \\ \overline{\mathbf{C}}_{\mathrm{ms}} &= \int_{\Psi} \int_{\Phi} \frac{\rho_{\mathrm{s}}}{\overline{h}_{\mathrm{s}}} \mathbf{P}_{\mathrm{s}} \otimes \mathbf{P}_{\mathrm{s}}\, \mathrm{d}\Phi\, \mathrm{d}\Psi,\end{aligned} \tag{40}$$

where $\overline{\mathbf{C}}_{\mathrm{ms}}$ is an average compliance tensor related with the orientation distribution of slip systems. Substitution of Eqs. (14) and (40) into Eq. (1) yields

$$\dot{\mathbf{E}} = (\overline{\mathbf{C}}_{\mathrm{me}} + \overline{\mathbf{C}}_{\mathrm{ms}}) : \dot{\mathbf{S}}_{\mathrm{m}}. \tag{41}$$

Eq. (41) describes the stress–strain relation of the matrix.

From the third term on the right-hand side of Eq. (39), the stress tensor for the fiber-bundles can be calculated as

$$\mathbf{S}_f = \int_\Omega \rho_f \sigma_f \mathbf{P}_f \, d\Omega. \tag{42}$$

By use of Eqs. (7) and (23), and Eq. (42), the stress rate tensor for the fiber-bundles becomes

$$\begin{aligned}\dot{\mathbf{S}}_f &= \overline{\mathbf{K}}_f : \dot{\mathbf{E}},\\ \overline{\mathbf{K}}_f &= \int_\Omega \rho_f \overline{E}_f \mathbf{P}_f \otimes \mathbf{P}_f \, d\Omega,\end{aligned} \tag{43}$$

where $\overline{\mathbf{K}}_f$ is an average stiffness tensor related to the fiber orientation distribution.

Thus, the total power in Eq. (39) can be rewritten as

$$\mathbf{S} : \dot{\mathbf{E}} = V_m \mathbf{S}_m : \left\{ \dot{\mathbf{E}}_{me} + \dot{\mathbf{E}}_{ms} \right\} + V_f \mathbf{S}_f : \dot{\mathbf{E}}. \tag{44}$$

By substituting Eq. (1) into Eq. (44), the total stress in REV can be obtained

$$\mathbf{S} = V_m \mathbf{S}_m + V_f \mathbf{S}_f. \tag{45}$$

The total stress rate tensor is

$$\dot{\mathbf{S}} = V_m \dot{\mathbf{S}}_m + V_f \dot{\mathbf{S}}_f. \tag{46}$$

Eqs. (41) and (43) can be substituted into Eq. (46) to yield

$$\dot{\mathbf{S}} = \left\{ V_m (\overline{\mathbf{C}}_{me} + \overline{\mathbf{C}}_{ms})^{-1} + V_f \overline{\mathbf{K}}_f \right\} : \dot{\mathbf{E}}, \tag{47}$$

$$\overline{\mathbf{C}}_{ms} = \int_\Psi \int_\Phi \frac{\rho_s}{\overline{h}_s} \mathbf{P}_s \otimes \mathbf{P}_s \, d\Phi \, d\Psi. \tag{48}$$

In Eq. (48)

$$\overline{h}_s = \begin{cases} \infty & \text{when } \tau_{-cr} < \tau_s < \tau_{+cr}, \\ \overline{h}_s & \text{when } \tau_s \leqslant \tau_{-cr} \text{ or } \tau_s \geqslant \tau_{+cr} \end{cases}$$

and

$$\overline{\mathbf{K}}_f = \int_\Omega \rho_f \overline{E}_f \mathbf{P}_f \otimes \mathbf{P}_f \, d\Omega, \tag{49}$$

where

$$\overline{E}_f = \begin{cases} \overline{E}_f & \text{when } \varepsilon_{-cr} < \varepsilon_f < \varepsilon_{+cr}, \\ \overline{E}_{fs} & \text{when } \varepsilon_f \leqslant \varepsilon_{-cr} \text{ or } \varepsilon_f \geqslant \varepsilon_{+cr}. \end{cases}$$

Eq. (47) is the elastoplastic constitutive equation of whisker-reinforced metal–matrix composite. It depends not only on $\overline{\mathbf{C}}_{me}, \overline{h}_s, \overline{E}_f$ and $\overline{E}_{fs}$ but also on the state of slip systems and fiber-bundles. They can be determined from the crystal and interface sliding criteria in Eqs. (19) and (25).

4.2. Damage constitutive equation

By use of Eqs. (26)–(31), the average stress rates of essential elements can be derived as

$$\dot{\mathbf{S}}_m = (1 - D_{me}) \overline{\mathbf{K}}_{me} : \dot{\mathbf{E}}_{me} - \dot{D}_{me} \mathbf{S}_m^{(ef)}, \tag{50}$$

$$\dot{\tau}_s = (1 - D_s) \overline{h}_s \dot{\gamma}_s - \dot{D}_s \tau_s^{(ef)}, \tag{51}$$

$$\dot{\sigma}_f = (1 - D_f) \overline{E}_f \dot{\varepsilon}_f - \dot{D}_f \sigma_f^{(ef)}. \tag{52}$$

Substitution of Eqs. (35)–(37) into Eqs. (50)–(52), respectively, yields

$$\dot{\mathbf{S}}_m = \mathbf{K}_{me}^{(ef)} : \dot{\mathbf{E}}_{me}, \tag{53}$$

$$\dot{\tau}_s = h_s^{(ef)} \dot{\gamma}_s, \tag{54}$$

$$\dot{\sigma}_f = E_f^{(ef)} \dot{\varepsilon}_f. \tag{55}$$

Note that

$$\mathbf{K}_{me}^{(ef)} = (1 - D_{me}) \overline{\mathbf{K}}_{me} - \frac{1}{2(1 - D_{me})} \left(\frac{dD_{me}}{d\mathbf{E}_{me}} \otimes \mathbf{S}_m + \mathbf{S}_m \otimes \frac{dD_{me}}{d\mathbf{E}_{me}} \right), \tag{56}$$

$$h_s^{(ef)} = (1 - D_s) \overline{h}_s - \frac{\tau_s}{1 - D_s} \frac{dD_s}{d\gamma_s}, \tag{57}$$

$$E_f^{(ef)} = (1 - D_f) \overline{E}_f - \frac{\sigma_f}{1 - D_f} \frac{dD_f}{d\varepsilon_f}. \tag{58}$$

Irreversibility implies that as $|\varepsilon_f|$ or $|\gamma_s|$ decreases, $\dot{D}_f \equiv 0$ or $\dot{D}_s \equiv 0$. Hence, the effective moduli of the slip system and fiber-bundle can be further expressed as

$$h_s^{(ef)} = \begin{cases} (1 - D_s) \overline{h}_s - \frac{\tau_s}{1 - D_s} \Psi_s \\ \quad \text{when } |\gamma_s| \geqslant \max(|\gamma_s^m|, |\gamma_{cr}|) \text{ and } \gamma_s d\gamma_s > 0; \\ (1 - D_s) \overline{h}_s \\ \quad \text{when } |\gamma_s| < \max(|\gamma_s^m|, |\gamma_{cr}|) \text{ or } \gamma_s d\gamma_s \leqslant 0; \end{cases} \tag{59}$$

and

$$E_{\mathrm{f}}^{(\mathrm{ef})} = \begin{cases} (1-D_{\mathrm{f}})\overline{E}_{\mathrm{f}} - \frac{\sigma_{\mathrm{f}}}{1-D_{\mathrm{f}}}\Psi_{\mathrm{f}}, \\ \quad \text{when } |\varepsilon_{\mathrm{f}}| \geqslant \max(|\varepsilon_{\mathrm{f}}^{\mathrm{m}}|, |\varepsilon_{\mathrm{cr}}|) \text{ and } \varepsilon_{\mathrm{f}}\,\mathrm{d}\varepsilon_{\mathrm{f}} > 0, \\ (1-D_{\mathrm{f}})\overline{E}_{\mathrm{f}}, \\ \quad \text{when } |\varepsilon_{\mathrm{f}}| < \max(|\varepsilon_{\mathrm{f}}^{\mathrm{m}}|, |\varepsilon_{\mathrm{cr}}|) \text{ or } \varepsilon_{\mathrm{f}}\, d\varepsilon_{\mathrm{f}} \leqslant 0, \end{cases} \tag{60}$$

respectively.

When damage ceases to grow, the effective stiffness tensor Eq. (56) of the elastic medium becomes

$$\mathbf{K}_{\mathrm{me}}^{(\mathrm{ef})} = (1-D_{\mathrm{me}})\overline{\mathbf{K}}_{\mathrm{me}}. \tag{61}$$

This is the same as that described previously. It can be seen that the damage-rate effect of the uniform material has little influence and it will not be discussed here.

Therefore, substituting $\mathbf{K}_{\mathrm{me}}^{(\mathrm{ef})}, E_{\mathrm{f}}^{(\mathrm{ef})}$, and $h_{\mathrm{s}}^{(\mathrm{ef})}$ for $\overline{\mathbf{K}}_{\mathrm{m}}, \overline{E}_{\mathrm{f}}$, and $\overline{h}_{\mathrm{s}}$ into Eqs. (47)–(49), there results the elastoplastic damage meso-constitutive relation. It depends not only on the damage state but also on the damage rate effect.

The predictive capability of the derived constitutive relation will be presented in Part II [20] of this work.

Acknowledgements

This research work has been supported through the Grant No. 19872065 and No. 19732060 from the National Natural Science Foundation of China and granted by Chinese Academy of Sciences (KJ951-1-201).

References

[1] L.M. Kachanov, Time of the rupture process under creep condition, Ivz. Akad.Nauk. USSR. Otd. Tekhn Nauk 8 (1958) 26–31 (in Russian).

[2] G. Rousselier, Finite deformation constitutive rela-tions including ductile fracture damage, in: Proceedings of the IUTAM Symposium on Three-Dimensional Constitutive Relations and Ductile Fracture, Dourdan, France, North-Holland, 1980, pp. 331–355.

[3] J. Lemaitre, J.L. Chaboche, Aspect phenomeno-logique de la rupture par endommagement, J. Meca Appl. 2(3) (1978) 317–365 (in French).

[4] S. Murakami, Notion of continuum damage mechanics and its application to anisotropic creep damage theory, ASME J. Eng. Mater. Tech. 105 (1983) 99–105.

[5] T. Mori, K. Tanaka, Average stress in matrix and average energy of materials with misfitting inclusions, Acta Metallurgica et Materialia 21 (1973) 571–574.

[6] J.W. Ju, T.M. Chen, Effective elastic moduli of two-dimensional brittle solids with interacting microcracks Part I: basic formulations and Part II: evolution damage models, ASME J. Appl. Mech. 61 (1994) 349–366.

[7] D. Krajcinovic, Damage Mechanics, Elsevier, The Netherlands, 1996 (Chapter 2).

[8] N.G. Liang, P.G. Bergan, A multi-dimension composite model of elastoplastic continua under non-proportional loading condition, ACTA Machanica Sinica 6 (4) (1990) 357–366.

[9] A.L. Kalamkarov, H.Q. Liu, A new model for the multiphase fiber-matrix composite materials, Composites 29B (5) (1998) 643–653.

[10] H.Q. Liu, N.G. Liang, A physical mechanism based elastoplastic constitutive theory of whisker-reinforced metal–matrix composites: part I. Theory and part II. Prediction, in: B. Xu, M. Tokuda, X. Wang (Eds.), The Fourth International Symposium on Microstructures and Mechanical Properties of New Engineering Materials, International Academic Publishers, Beijing, 1999, pp. 63–74.

[11] N.G. Liang, H.Q. Liu, T.C. Wang, A meso elastoplastic constitutive model for polycrystalline metals based on equivalent slip systems with latent hardening, Science in China 41A (8) (1998) 887–896.

[12] H.Q. Liu, K. Hutter, On a meso-elastoplastic constitutive equation with application to deformation-induced anisotropy of a polycrystalline aggregate, Arch. Mech. 48 (1) (1996) 53–65.

[13] H.Q. Liu, N.G. Liang, M.F. Xia, Modeling and mesoscopic damage constitutive relation of brittle short-fiber-reinforced composites, Science of China 42E (5) (1999) 530–540.

[14] S.R. Swanson, Design methodology and practices, in: P.K.Mallick (Ed.), Composites Engineering Handbook, Marcel Dekker, New York, 1997, pp. 1183–1205.

[15] K.L.Reifsnider, Fatigue of composite materials, Composite Material Series, Elsevier, Amsterdam, 1991, 4.

[16] H.Q. Liu, N.G. Liang, A physical-mechanism-based anisotropic damage-rate-dependent constitutive equation, in: B. Xu, W. Yang (Eds.), Second Asia-Pacific Symposium on Advances in Engineering Plasticity and its Application, International Academic Publishers, Beijing, 1994, pp. 155–160.

[17] S. Baste, J.M. Morvan, Under load strain partition of a ceramic matrix composites using an ultrasonic method, Experimental Mechanics 1996, pp. 148–156.

[18] X.Y Liu, W.D. Yan, N.G. Liang, A pseudo-plastic engagement effect on the toughening of discontinuous

fiber-reinforced brittle composites, Metals Mater. 4 (3) (1998) 242–246.
[19] Z.B. Kuang, Foundation of nonlinear continuum mechanics, Xi'an Jiaotong University press, 1989, Chapter 4 (in Chinese).
[20] H.Q. Liu, N.G. Liang, An elastoplastic constitutive relation of whisker-reinforced composite for meso damage: Part II – failure surfaces, J. Theoret. Appl. Fracture Mech. 33 (2000) 199–206.