

橡胶楔体与刚性缺口接触大变形分析¹⁾

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摘要 利用 Knowles 与 Sternberg 提出的非线性弹性大变形应变能函数, 对橡胶楔体与刚性缺口接触问题进行了大变形渐近分析, 推导了楔体尖端场的渐近方程, 得到楔体尖端附近的应力应变场及应力的奇异性指数与橡胶楔体角度、刚性缺口角度及材料常数有关的表达式; 楔尖附近同一半径上应力分量为常数. 同时, 利用非线性有限元理论编制了大变形有限元程序, 考虑楔体尖端与缺口接触边界条件, 计算得到了与分析解一致的结论. 当缺口角度大于 π , 则变为另一个完全不同的问题. 当缺口角度等于 π , 则为两种问题的过渡阶段.

关键词 非线性弹性, 大变形, 橡胶楔体, 刚性缺口, 渐近分析, 有限元

引 言

楔体尖端受集中力或力偶作用的线弹性解表明楔体中应力具有 r^{-1} 奇异性, 当 $r \rightarrow 0$ 时, 应变趋于无穷大, 此线弹性解不再适用, 而必须用非线性弹性有限变形理论来解答, 对于橡胶材料的楔体问题, 就更是如此. 非线性弹性有限变形最关键的问题是提出一种合理的本构关系, 特别是对含奇异点的问题, 本构关系能否描述其行为, 尤为重要. 1973 年 Knowles 和 Sternberg^[1] 提出了一种本构关系, 并对裂纹尖端场进行了分析^[1,2], 但他们只考虑了收缩区. 1990 年及 1997 年高玉臣^[3,4] 从不同的思想出发, 提出了两种本构关系, 并利用分区思想对裂纹尖端场进行了分析, 得到了合理的结论. 利用不同的本构关系, 文献 [5, 6] 分别解决了受任意方向集中力拉伸橡胶楔体尖端应力应变场这一问题, 得到了一致结论: 楔体尖端几乎处于单向拉伸状态, 应力场的幅度与外力大小有关, 而与外力方向无关. 对于接触表面含有尖点的接触问题, 小变形理论同样不恰当, 特别是对于能产生很大变形的橡胶类材料. 本文利用文献 [1] 提出的本构关系, 对橡胶楔体压入刚性缺口接触大变形问题进行了分析, 得到楔体尖端的力学特性. 同时编制了对应 Knowles 和 Sternberg^[1] 提出的应变能函数的有限变形程序, 考虑接触非线性边界条件, 计算了同一问题, 有限元计算结果与本文的理论解符合很好.

1 基本方程

考虑三维弹性体区域, P 和 Q 分别表示变形前后的位置矢量, x^i ($i = 1, 2, 3$) 表示随体坐标, 则变形前后的坐标标架为

$$P_i = \frac{\partial P}{\partial x^i}, \quad Q_i = \frac{\partial Q}{\partial x^i} \quad (1)$$

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位移梯度为

$$\mathbf{F} = \mathbf{Q}_i \otimes \mathbf{P}^i \quad (2)$$

其中 \otimes 为并矢符号, Einstein 求和约定适用于本文.

Cauchy-Green 左右变形张量为

$$\mathbf{D} = \mathbf{F}^T \cdot \mathbf{F}, \quad \mathbf{d} = \mathbf{F} \cdot \mathbf{F}^T \quad (3)$$

本文用到的几个应变不变量

$$I = \mathbf{D} : \mathbf{E} = \mathbf{d} : \mathbf{E}, \quad J = \det |\mathbf{Q}_i \cdot \mathbf{P}^j| \quad (4)$$

其中 \mathbf{E} 为单位张量.

文献 [1] 中提出的变形前单位体积内变形能为

$$W = (AI + BJ + CIJ^{-2})^n \quad (5)$$

其中 A, B, C, n 为材料常数. 由此可得到 Kirchhoff 应力张量

$$\boldsymbol{\sigma} = 2 \frac{\partial W}{\partial \mathbf{D}} \quad (6)$$

Cauchy 应力则为

$$\boldsymbol{\tau} = J^{-1} \mathbf{F} \cdot \boldsymbol{\sigma} \mathbf{F}^T \quad (7)$$

由 (1)~(7), Kirchhoff 应力为

$$\boldsymbol{\sigma} = 2n(AI + BJ + CIJ^{-2})^{n-1} \left[A\mathbf{E} + \frac{B}{2} J \mathbf{D}^{-1} - C K^{-1} (I \mathbf{D}^{-1} - \mathbf{E}) \right] \quad (8)$$

将 (8) 式代入 (7) 式, 并结合 (3), (4), 可得

$$\boldsymbol{\tau} = 2n J^{-1} (AI + BJ + CIJ^{-2})^{n-1} \left(A \mathbf{d} + \frac{1}{2} B J \mathbf{E} - C \mathbf{d}^{-1} \right) \quad (9)$$

平衡方程为

$$\mathbf{Q}^i \cdot \frac{\partial \boldsymbol{\tau}}{\partial x^i} = 0 \quad (10)$$

2 渐近分析

2.1 渐近方程

为了描述物体的变形, 我们引入两个 Lagrangian 坐标系, 一个表示变形前的柱坐标 (R, Θ, Z) , 另一个是变形后的柱坐标 (r, θ, z) , 由于研究的是平面应变问题, 所以 z 方向变形为零, 即 $z = Z$, 变形前后坐标系如图 1(a), (b) 所示. 对于变形前后同一个物质点位置的对应关系, 通过大量分析, 我们设映射函数为

$$R = r^{1-\delta} f(\theta), \quad \Theta = g(\theta) \quad (11)$$

其中 δ 是小于 1 的正数, $f(\theta)$ 和 $g(\theta)$ 为待定函数.

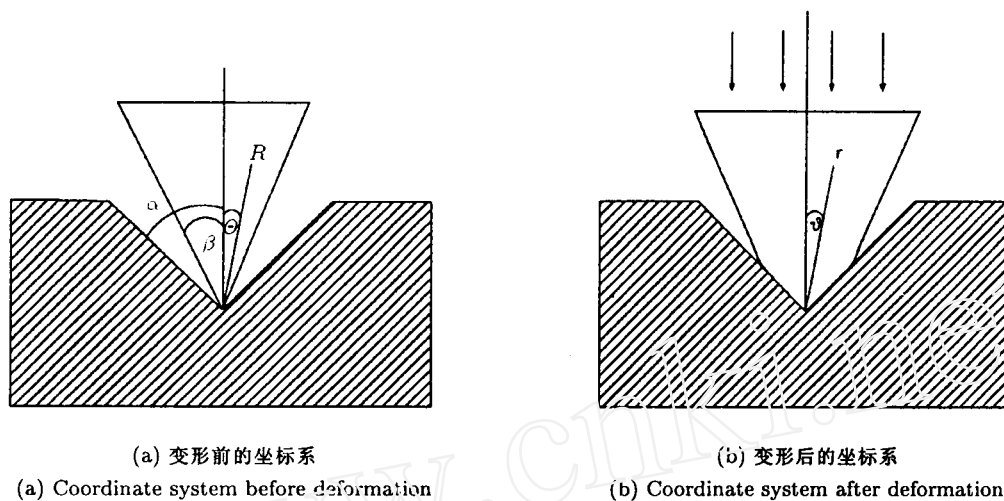


图 1 坐标系

Fig.1 Coordinate systems

令 $e_R, e_\theta, e_r, e_\theta$ 分别表示在 (R, θ) 和 (r, θ) 坐标下的单位向量, 即

$$e_R = \frac{\partial P}{\partial R}, \quad e_\theta = \frac{\partial P}{R \partial \theta} \quad (12)$$

$$e_r = \frac{\partial Q}{\partial r}, \quad e_\theta = \frac{\partial Q}{r \partial \theta} \quad (13)$$

根据方程 (1)~(3) 可得

$$\left. \begin{aligned} P_r &= \frac{\partial P}{\partial r} = r^{-\delta}(1-\delta)fe_R \\ P_\theta &= \frac{\partial P}{\partial \theta} = r^{1-\delta}(f'e_R + fg'e_\theta) \end{aligned} \right\} \quad (14)$$

$$\left. \begin{aligned} P^r &= r^\delta q^{-1}(fg'e_R - f'e_\theta) \\ P^\theta &= r^{\delta-1} q^{-1}(1-\delta)fe_\theta \end{aligned} \right\} \quad (15)$$

其中

$$q = (1-\delta)f^2g' \quad (16)$$

由方程 (1)~(3) 及 (12)~(16) 可以得到

$$\begin{aligned} d &= r^{2\delta}q^{-2}[(f^2g'^2 + f^2)e_r \otimes e_r + (1-\delta)^2f^2e_\theta \otimes e_\theta - \\ &\quad (1-\delta)ff'(e_r \otimes e_\theta + e_\theta \otimes e_r)] \end{aligned} \quad (17)$$

$$\begin{aligned} d_{-1} &= r^{-2\delta}[(1-\delta)^2f^2e_r \otimes e_r + (f^2g'^2 + f'^2)e_\theta \otimes e_\theta + \\ &\quad (1-\delta)ff'(e_r \otimes e_\theta + e_\theta \otimes e_r)] \end{aligned} \quad (18)$$

由方程 (4), (12)~(18), 有

$$I = r^{2\delta}pq^{-2}, \quad J = r^{2\delta}q^{-1} \quad (19)$$

其中 $p = f^2 g'^2 + f'^2 + (1 - \delta)^2 f^2$.

将方程 (17)~(19) 代入方程 (9), 进行量级比较, 省略高阶小量, 保留 r 的最低幂次项得

$$\begin{aligned} \tau = & -2nC^n J^{-1} I^{n-1} J^{-2(n-1)} d^{-1} = \\ & -2nC^n r^{-2\delta(n+1)} qp^{n-1} [(1-\delta)^2 f^2 e_r \otimes e_r + (f^2 g'^2 + f'^2) e_\theta \otimes e_\theta + \\ & (1-\delta) f f' (e_r \otimes e_\theta + e_\theta \otimes e_r)] \end{aligned} \quad (20)$$

根据方程 (20), 各应力分量分别为

$$\left. \begin{aligned} \tau_{rr} &= -2nC^n (1-\delta)^2 qp^{n-1} f^2 r^{-2\delta(n+1)} \\ \tau_{\theta\theta} &= -2nC^n qp^{n-1} (f^2 g'^2 + f'^2) r^{-2\delta(n+1)} \\ \tau_{r\theta} &= -2nC^n (1-\delta) qp^{n-1} f f' r^{-2\delta(n+1)} \end{aligned} \right\} \quad (21)$$

由平衡方程 (10) 可得

$$\left. \begin{aligned} \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r} (\tau_{rr} - \tau_{\theta\theta}) &= 0 \\ \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{2}{r} \tau_{r\theta} &= 0 \end{aligned} \right\} \quad (22)$$

将方程 (21) 代入 (22) 式, 并经过复杂的推导、整理、化简得下面的渐近方程

$$\left. \begin{aligned} f'' \left[1 + \frac{2}{p} (n-1) f'^2 + \frac{g'' f'}{g'} \left[1 + \frac{2}{p} (n-1) f^2 g'^2 \right] + \left(2n - \frac{\delta}{1-\delta} \right) \frac{f'^2}{f} - \right. \\ \left. 2(n-1) \frac{f'^4}{fp} - \frac{f}{1-\delta} \{ g'^2 - (1-\delta)^2 [1 - 2(n+1)\delta] \} = 0 \right. \\ \left. f'' \left[\frac{f'^2}{f^2 g'^2} - 1 \right] + 2 \frac{f' g''}{g'} + \left(1 + \frac{2}{1-\delta} \right) \frac{f'^2}{f} + \frac{f'^2}{g'^2 f} \left(1 - \delta + \frac{\delta}{1-\delta} \frac{f'^2}{f^2} \right) + \right. \\ \left. \frac{f}{1-\delta} \{ g'^2 - (1-\delta)^2 [1 - 2(n+1)\delta] \} = 0 \right\} \quad (23)$$

2.2 定解条件及方程的解

由渐近方程 (23) 可见, 方程仅包含 θ 一个自变量. 在变形过程中, 楔体尖端处的边界已与缺口的边界完全接触, 因此由接触边界条件可得

$$g(\alpha) = \beta, \quad g(-\alpha) = -\beta \quad (24)$$

其中 β 表示半个楔体角度, α 表示半个缺口角度, 如图 1 中所示.

假设楔体与缺口为光滑接触, 所以有

$$\text{在 } \theta = \pm\alpha \text{ 处, } \tau_{r\theta} = 0 \quad (25)$$

结合 (21), (25) 两式, 得下面条件

$$f'(\alpha) = f'(-\alpha) = 0 \quad (26)$$

根据定解条件 (24), (26) 两式及渐近方程 (23), 通过数值计算得

$$f = f_0, \quad g = \frac{\beta}{\alpha}\theta \quad (27)$$

其中 f_0 是一个与外载荷大小有关的表示幅值的常数, 微分方程的特征值 δ 可表示为

$$(1 - \delta)^2[1 - 2(n + 1)\delta] = \beta^2/\alpha^2 \quad (28)$$

将 (27) 式代入 (11) 式, 可以得到本文所解问题的具体变形模式

$$R = f_0 r^{1-\delta}, \quad \Theta = \frac{\beta}{\alpha}\theta \quad (29)$$

应力表达式 (21) 则变为

$$\left. \begin{aligned} \tau_{rr} &= -2nC^n(1-\delta)^2qp^{r-1}f_0^2r^{-2\delta(n+1)} \\ \tau_{\theta\theta} &= -2nC^nqp^{n-1}f_0^2g^2r^{-2\delta(1+n)} \\ \tau_{r\theta} &= 0 \end{aligned} \right\} \quad (30)$$

此时, $p = f_0^2\beta^2/\alpha^2 + (1-\delta)^2f_0^2$, $q = (1-\delta)f_0^2\beta/\alpha$.

3 有限元计算与理论分析解比较

本文对变形体中物质点的运动采用 Lagrangian 描述法, 结合本文所用应变能函数编制了非线性弹性有限元程序. 根据编制的有限元程序, 并经过算例考证后, 我们求解了本文理论部分所分析的问题. 考虑问题的对称性, 计算模型只采用了半个楔体. 采用 4 节点单元, 角度方向为 8 等分, 共 320 个单元, 361 个节点. 在计算中, 应变能函数中的常数我们取了一组值为 $A = 6.0$, $B = 12.0$, $C = 6.0$, $n = 2.5$. 对几组不同角度的楔体与缺口相压情况进行计算, 所得结果相似, 文中只给出 $\alpha = \pi/4$, $\beta = \pi/6$ 的计算结果. 计算模型如图 2 所示, 变形后楔体尖端形貌如图 3 所示. 计算所得 $r - R$, $r - \tau_{rr}$, $r - \tau_{\theta\theta}$, $\theta - \tau_{rr}$, $\theta - \tau_{\theta\theta}$, $\theta - \theta$ 关系曲线与由方程 (28)~(30) 得到的相应理论曲线比较如图 4~ 图 9.

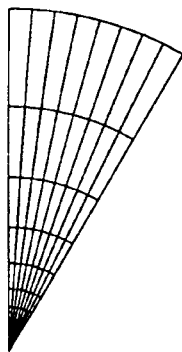


图 2 $\alpha = \pi/4$, $\beta = \pi/6$ 计算模型
Fig.2 Calculating model for $\alpha = \pi/4$, $\beta = \pi/6$

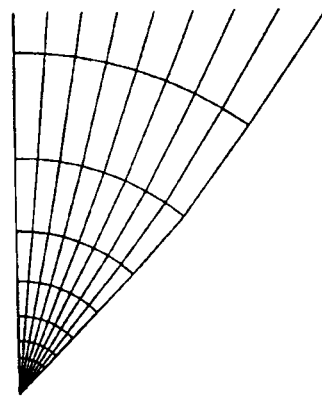


图 3 变形后楔体尖端形貌
Fig.3 Shape of the wedge tip after deformation

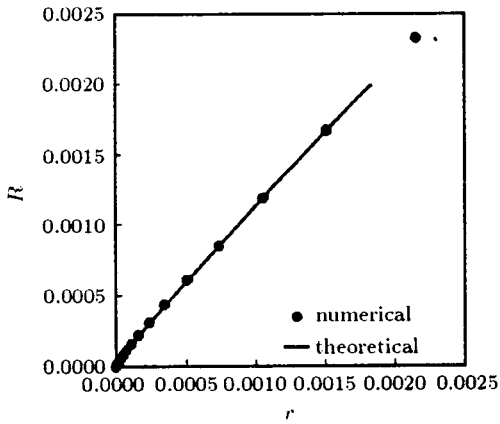


图 4 $r-R$ 曲线
Fig.4 Curves of $r-R$

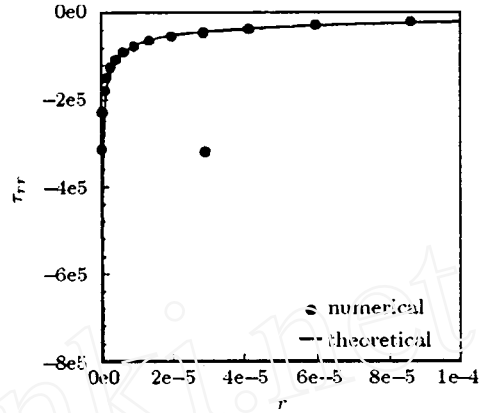


图 5 $r-\tau_{rr}$ 曲线
Fig.5 Curves of $r-\tau_{rr}$

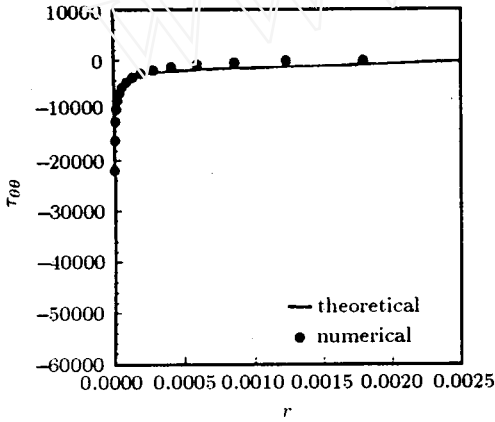


图 6 $r-\tau_{\theta\theta}$ 曲线
Fig.6 Curves of $r-\tau_{\theta\theta}$

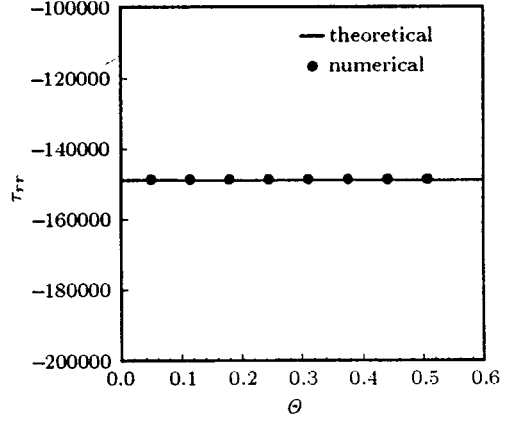


图 7 $\theta-\tau_{rr}$ 曲线
Fig.7 Curves of $\theta-\tau_{rr}$

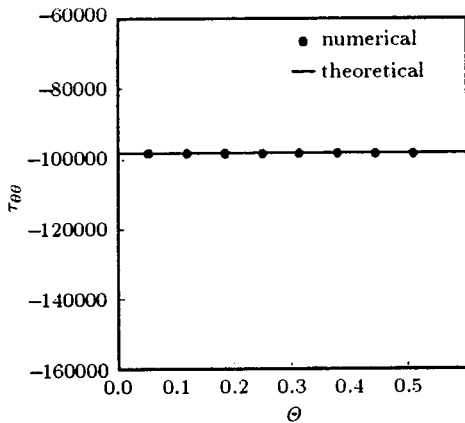


图 8 $\theta-\tau_{\theta\theta}$ 曲线
Fig.8 Curves of $\theta-\tau_{\theta\theta}$

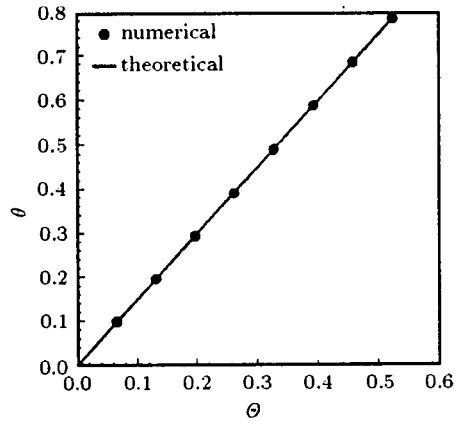


图 9 $\theta-\theta$ 曲线
Fig.9 Curves of $\theta-\theta$

4 结 论

- 1) 橡胶材料楔体与刚性缺口的接触问题, 不包含收缩区和扩张区.
- 2) 径向变形具有奇异性, 奇异指数 δ 依赖于橡胶楔体的角度与刚性缺口角度的比值.
- 3) 橡胶楔体压入缺口的部分, 压入前的角度 θ 与压入后同一物质点的角度 θ 的比值为一常数. 通过有限元计算网格看, 计算网格压入缺口部分均匀散开.
- 4) 楔体尖端应力分量 τ_{rr} 与 $\tau_{\theta\theta}$ 具有同样奇异性 $r^{-2(n+1)\delta}$, 应力分量大小都与 θ 无关.

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LARGE DEFORMATION ANALYSIS OF A RUBBER WEDGE CONTACTING WITH A RIGID NOTCH ¹⁾

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Abstract The contact problem is very complex, especially, the contact problem of large deformation. When the contact surface contains vertex, the problem must be solved by large strain theory. As for the large strain theory, the most important thing is to put out a reasonable and applicable constitutive relation. In 1973, Knowles and Sternberg gave a kind of constitutive relation for non-linear elasticity large deformation and analyzed the elastic crack tip field. In this paper, one kind of typical problem that is a rubber wedge contacting with a rigid notch is considered. Using

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the constitutive relation given by Knowles and Sternberg in 1973 and proper mapping functions between before and after deformation, the problem is solved and the theoretical analysis is given. The results show that there is no expanding sector or shrinking sector near the rubber wedge tip, not like the crack tip field in rubber-like material; When the rubber wedge is compressed into the rigid notch, the points on the same radius before deformation distribute evenly; The deformation in the direction of radius possesses singularity and the singular exponent δ depends on the ratio of the rubber wedge angle to the notch angle; The non-vanishing components of stress, τ_{rr} and $\tau_{\theta\theta}$, near the wedge tip possess the same singularity, $r^{-2(n+1)\delta}$, and both components are independent of the angle θ after deformation. Furthermore, finite element program of large deformation is made and the contact boundary conditions are added to the program, numerical calculation is done to the typical contact problem. The numerical results are consistent well with those of the theoretical analysis.

Key words non-linear elasticity, large deformation, rubber wedge, rigid notch, asymptotic analysis, finite element