The Relationship of SIF between Plate and Plane Fracture Problems and the Effect of the Plate Thickness on SIF

Yongjun Xu and Chuntu Liu

Institute of Mechanics, Chinese Academy of Sciences,
Beijing 100080, China P.R.

ABSTRACT

In the present paper, it is shown that the zero series eigenfunctions of Reissner plate cracks/notches fracture problems are analogous to the eigenfunctions of anti-plane and in-plane. The singularity in the double series expression of plate problems only arises in zero series parts. In view of the relationship with eigen-values of anti-plane and in-plane problem, the solution of eigen-values for Reissner plates consists of two parts: anti-plane problem and in-plane problem. As a result the corresponding eigen-values or the corresponding eigen-value solving programs with respect to the anti-plane and in-plane problems can be employed and many aggressive SIF computed methods of plane problems can be employed in the plate. Based on those, the approximate relationship of SIFs between the plate and the plane fracture problems is figured out, and the effect relationship of the plate thickness on SIF is given.

1. INTRODUCTION

Stress analysis in practical engineering applications inevitably encounters stress singularities caused by sudden changes in geometry, e.g. around re-entrant corners (notches) or, more severely, around crack tips. The complete eigenvalue solution and the stress intensity factor (SIF) calculation play an important role in the linear elasticity fracture mechanics (LEFM). Most of numerical methods, directly or indirectly, make use of the available eigen-solutions when applied to the SIF computation for cracks/notches problems so that singularity can be treated in a more efficient, accurate and reliable way. Recently, Xu and Yuan [1] presented an effective implementation of the imbedding method for accurate and reliable computation of complete real or complex eigen-solutions in two dimensional notch/crack singularities with multiple materials, arbitrary opening angles and various surface conditions. This algorithm is robust and may be employed by any numerical methods that make use of singular solutions. The singularity causes great difficulty to the numerical solutions that have to be invoked when analytical solutions are not available. A numerical recipe for accurate and efficient computation of stress singularity factors (SIFs) usually consists of two major ingredients, namely a powerful numerical method for the general stress

analysis and a novel approach to obtaining the desired SIFs which may include special treatment of various singularities. The underlying numerical method employed in this paper is the finite element method of lines (FEMOL) [2-5], which is a newly developed, general-purpose, semi-analytical method. In this method, a partial differential equation defined on an arbitrary domain is semi-discretized, by finite element techniques via energy theorems or variational principles, into a system of ordinary differential equations (ODEs) defined on straight or curved mesh lines. Recently, Xu and Yuan [6] exploited the orthogonal relationship between eigen-functions for anti-plane and in-plane notch problems, and a set of contour integral extraction methods (so called orthogonal integral extraction method) for SIF calculation is developed, which appears to be most powerful and efficient. Xu [7] extended the orthogonal integral extraction method to the general two-dimensional fracture problems. In the present paper, It is shown that the zero series eigenfunctions of Reissner plate cracks/notches fracture problems are analogous to the eigenfunctions of anti-plane and in-plane. The singularity in the double series expression of plate problems only arises in zero series parts. In view of the relationship with eigen-values of anti-plane and in-plane problem, the solution of eigen-values for Reissner plates consists of two parts: anti-plane problem and in-plane problems. As a result, the corresponding eigenvalues or the corresponding eigenvalue solving programs with respect to the anti-plane and in-plane problems can be employed and many aggressive SIF computed methods of plane problems can be employed in the plate problems. Based on those, the approximate relationship of SIFs between the plate and the plane fracture problems is figured out, and the effect relationship of the plate thickness on SIF is given. Two examples are given to show the two approximate relationships.
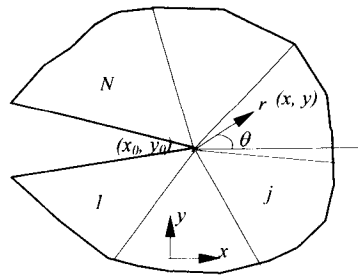
2. PRELIMINARY CONSIDERATION

Fig. 1 N-material notch problem

Figure 1 shows a two dimensional singular problem with N different materials around notch/crack tip. The body forces are assumed to be negligible. For convenience sake we omit the different material domain subscript j shown in the Fig. 1. In the polar coordinates shown in Fig. 1, for the Reissner plate, the general displacement components can be written as the following double series form

$$w^j = \sum_i \sum_n r^{\lambda_i + n} c_{ni}^j(\theta), \quad \begin{cases} \psi_r^j = \sum_i \sum_n r^{\lambda_i + n} a_{ni}^j(\theta) \\ \psi_\theta^j = \sum_i \sum_n r^{\lambda_i + n} b_{ni}^j(\theta) \end{cases} \tag{1}$$

It can be showed that the singularity maybe appear when $n = 0$ and in the first two or three

terms of i. Here we only take account of the main part of $n = 0$, the Reissner plate could be divided into two basic parts of anti-plane part and in-plane part. So the well-known Williams' displacement $w_0(r,\theta)$ and potential function $\phi_0(r,\theta)$ are

$$\text{(i)} \quad w_0(r,\theta) = \sum_{i=0,1,2,3,\dots} \alpha_i r^{\lambda_i} f_i(\theta). \qquad \text{(ii)} \quad \phi_0(r,\theta) = \sum_{i=0,1,2,3,\dots} \alpha_i r^{\lambda_i+1} F_i(\theta) \tag{2}$$

where both $f_i(\theta)$ and $F_i(\theta)$ are certain normalization eigen-functions that corresponding to the eigen-value λ_i, and the well known explicit form is

$$f_i(\theta) = A_i \cos\lambda_i\theta + B_i \sin\lambda_i\theta$$
$$F_i(\theta) = A_i \sin(\lambda_i+1)\theta + B_i \cos(\lambda_i+1)\theta + C_i \sin(\lambda_i-1)\theta + D_i \cos(\lambda_i-1)\theta \tag{3}$$

where $\{A_i, B_i\}$ and λ_i in $f_i(\theta)$ are real constants, $\{A_i, B_i, C_i, D_i\}$ and λ_i in $F_i(\theta)$ are complex constants. The corresponding displacement $(\psi_{r0_i}, \psi_{\theta0_i})$ for plane part can be derived from the stress potential function

$$\psi_{r0_i} = r^\lambda \frac{(1-\nu\lambda_i)(1+\lambda_i)F_i(\theta) + F_i'(\theta)}{\lambda E} \qquad \psi_{\theta0_i} = r^\lambda \frac{\left(1+\lambda_i-(1-\lambda_i)\lambda_i\nu+2\lambda_i^2\right)F_i'(\theta) + F_i''(\theta)}{\lambda(1-\lambda)E} \tag{4}$$

The corresponding stress components of both anti-plane and plane problems are

$$\begin{cases} \tilde{Q}_{r0_i} = G\lambda_i r^{\lambda_i-1}\alpha_i f_i(\theta) \\ \tilde{Q}_{\theta0_i} = G r^{\lambda_i-1}\alpha_i f_i'(\theta) \end{cases} \qquad \begin{cases} \tilde{M}_{r0_i} = r^{\lambda_i-1}\alpha_{iM}[F_i''(\theta)+(\lambda_i+1)F_i(\theta)] \\ \tilde{M}_{\theta0_i} = r^{\lambda_i-1}\alpha_{iM}[\lambda_i(\lambda_i+1)F_i(\theta)] \\ \tilde{M}_{r\theta0_i} = r^{\lambda_i-1}\alpha_{iM}[-\lambda_i F_i'(\theta)] \end{cases} , \quad i = 0,1,2,3,\cdots \tag{5}$$

When $r \to 0$, the stress in formula (5) will tend to infinity because there are singular terms r^{λ_i-1} in the expansions. Compared the anti-plane part and the in-plane part of Reissner plate problem shown in equations (2-5) with real anti-plane and in-plane problems[6], the following assimilated relationships can be deduced

$$w_0 \sim w, \quad \tilde{Q}_{r0} \sim \tau_{rz}, \quad \tilde{Q}_{\theta0} \sim \tau_{\theta z}$$
$$\begin{cases} \psi_{r0} \sim u_r, \quad \psi_{\theta0} \sim u_\theta \\ \tilde{M}_{r0} \sim \sigma_{rr}, \quad \tilde{M}_{\theta0} \sim \sigma_{\theta\theta}, \quad \tilde{M}_{r\theta0} \sim \sigma_{r\theta} \end{cases} \tag{6}$$

where $\{w, \tau_{rz}, \tau_{\theta z}\}$ and $\{u_r, u_\theta, \sigma_{rr}, \sigma_{\theta\theta}, \sigma_{r\theta}\}$ are the displacements and stresses corresponding to the anti-plane and in-plane problems, respectively.

3. GENERAL DEFINITION OF SINGULARITY INTENSITY FACTORS (SIFs)

For two dimensional fracture problems, the general singularity intensity factors defined as

$$K_{\text{III}} = \lim_{r \to 0}\left(\sqrt{2\pi}r^{1-\lambda}\max_\theta(\tau_{r\theta0}(r,\theta))\right), \qquad \begin{cases} K_{\text{I}} = \lim_{r \to 0}\left(\sqrt{2\pi}r^{1-\lambda}\max_\theta(\sigma_{\theta0}(r,\theta))\right) \\ K_{\text{II}} = \lim_{r \to 0}\left(\sqrt{2\pi}r^{1-\lambda}\max_\theta(\sigma_{r\theta0}(r,\theta))\right) \end{cases} \tag{7}$$

For Reissner plate fracture problems, substitute the shears and moments into Eq. (7), the SIFs are generally defined as

$$K_{III} \approx \lim_{r \to 0}\left(\sqrt{2\pi} r^{1-\lambda} \max_{\theta}\left(\frac{5}{4}\left(1 - \zeta^2\right)\tilde{Q}_{\theta 0}(r,\theta)\right)\right), \quad \begin{cases} K_I \approx -\lim_{r \to 0}\left(\sqrt{2\pi} r^{1-\lambda} \max_{\theta}\left(\frac{h}{2}\zeta \tilde{M}_{\theta 0}(r,\theta)\right)\right) \\ K_{II} \approx -\lim_{r \to 0}\left(\sqrt{2\pi} r^{1-\lambda} \max_{\theta}\left(\frac{h}{2}\zeta \tilde{M}_{r\theta 0}(r,\theta)\right)\right) \end{cases} \tag{8}$$

where $\zeta = 2z/h$, we take the maximum value as $\zeta = 1$, λ is the corresponding eigenvalue.

4. TWO RELATIONSHIPS OF SINGULARITY INTENSITY FACTORS (SIFs)

Firstly, we consider the different between zero order stresses $\{\tilde{M}_0, \tilde{Q}_0\}$ and the real stresses $\{M, Q\}$; secondly, compare the equivalent between the Reissner plate and the plane problems; we consider the same equivalent boundary conditions between the Reissner plate and the plane problems. We have the following approximate relationship with respect to SIFs between the Reissner plate and the plane fracture problems

$$K_{III\,plate} \approx \frac{3}{2h} K_{III\,anti-plane}, \quad \begin{cases} K_{I\,plate} \approx \frac{6}{h^2} K_{I\,in-plane} \\ K_{II\,plate} \approx \frac{6}{h^2} K_{II\,in-plane} \end{cases} \tag{9}$$

For the two Reissner plate notch/crack problems, the only different part is the plate thickness (h_1 and h_2). Form Eq. (9), the approximate relationship with respect to SIFs between the different pl ate thickness can be deduced

$$\frac{K_{III}(h_1)}{K_{III}(h_2)} \approx \frac{h_2}{h_1}, \qquad \frac{K_{I,II}(h_1)}{K_{I,II}(h_2)} \approx \left(\frac{h_2}{h_1}\right)^2 \tag{10}$$

5. NUMERICAL EXAMPLES

To assess the performance of the two relationships, a number of numerical examples are given in this section. The following examples are computed by the newly developed method [6-7] namely orthogonal integral extraction method.
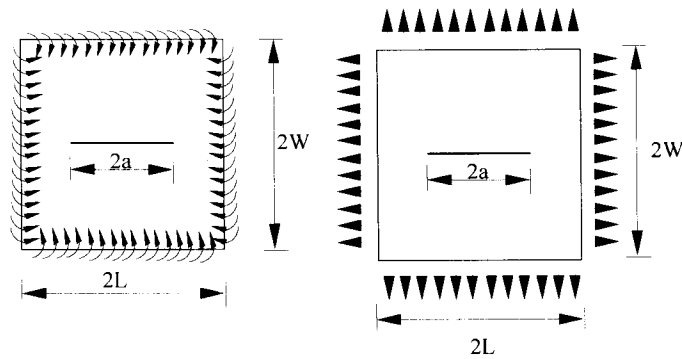
(a) Bending plate with a center crack   (b) Uniform stretch with a center crack

Fig. 2 Finite plate with a center crack

Example 1 Approximate relationship of SIFs between bending plate and uniform stretch plane

In this example, as Fig. 2 shows, we consider a finite bending plate with a center crack and a finite uniform stretch plane with a center crack. The crack length is $2a$, the plate thickness is h, and the necessary constants are $2L = 2W = 4a$, $E = 2 \times 10^6$, $v = 0.3$, $a = 1$, $h = 1$. Table 1 gives a list of the computed results of SIFs K_I and the approximate ratio of the SIFs for bending plate and plane. Where r_0 indicates the integral radius along the crack tips.

Table 1. The computed results of finite plate with a center crack

r_0	Bending	Uniform stretch	$K_{I plate}/K_{I plane}$ (Theory ≈ 6)
0.00001	9.728794	2.363948	4.115486
0.0001	9.743342	2.363946	4.121643
0.001	9.784121	2.363943	4.138899
0.01	9.865933	2.363946	4.173502
0.1	9.832484	2.363945	4.159354
0.5	10.099617	2.363951	4.272346

Example 2 Approximate effect on SIFs (K_I) of the different plate thickness

In this example, as Fig. 2(a) shows, we consider a finite bending plate with a center crack. The crack length is $2a$, and the necessary constants are $2L = 2W = 4a$, $E = 2 \times 10^6$, $v = 0.3$, $a = 1$. The three cases of the different thickness are $h_1/a = 0.1$, $h_2/a = 0.5$ and $h_3/a = 1.0$. Table 2 gives a list of the computed results of SIFs K_I, where r_0 indicates the integral radius along the crack tips. Fig. 3 gives out two ratio curves of the different SIFs K_I with respect to the different cases, the approximate values match very well with the theory values.

Table 2. The computed results of K_I for finite bending plate with a center crack

r_0 \ h/a	0.1	0.5	1.0
0.00001	793.823490	36.101453	9.728794
0.0001	793.800340	38.376926	9.743342
0.001	794.154503	42.004670	9.784121
0.01	800.501913	33.544143	9.865933
0.1	842.029973	35.785873	9.832484
0.5	929.598925	38.401774	10.099617

6. CONCLUDING REMARKS

In this present paper, the Relationship of SIF between plate and plane fracture problems and the effect of the plate thickness on SIFs are discussed. Both of the two approximate relationships are checked by two simply examples. From example 2 we know that the numerical values match very well with the theory values in the second relationship with respect to the different plate thickness. From the two relationships, we can give an approximate estimate for the unsolved problems from

the corresponding solved problems and a check to computed results and some methods based on the well known examples.
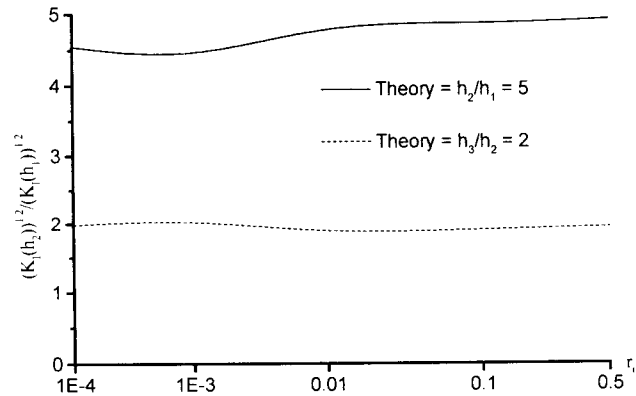
Fig. 3 $\sqrt{\dfrac{K_1(h_2)}{K_1(h_1)}}$

REFERENCES

1. Yongjun Xu and Si Yuan, *International Journal of Fracture*, Vol. 18, (1996) p. 373-381.

2. Si Yuan, *Proceedings of 1st National Conference on Analytical and Numerical Combined Methods*, Hunan (May, 1990) p. 132-136 (in Chinese).

3. Si Yuan and Jianling Gao, *Proceedings of International Conference on EPMESC*, Macau, 3 (Aug. 1-3, 1990) p. 517-526.

4. Si Yuan, *Chinese Journal of Numerical Mathematics and Applications*, 15, 1 (1993) p. 45-59.

5. Si Yuan, *The Finite Element Method of Lines: Theory and Applications*, Science Press, Beijing,-New York (1993).

6. Yongjun Xu, Complete eigenvalue solution and SIF calculation in fracture mechanics based on FEMOL, Ph.D. Thesis. Department of Civil Engineering, Beijing: Tsinghua University, 1996 (in Chinese)

7. Yongjun Xu, SIF calculation in plates, shells and three dimensional problems based on FEMOL, Postdoctoral report. Institute of Mechanics, Beijing: Chinese Academy of Sciences, 1999 (in Chinese)

Fracture and Strength of Solids IV

The Relationship of SIF between Plate and Plane Fracture Problems and the Effect of the Plate Thickness on SIF

DOI References

[] 5 ig.3 /K(h,) (h) EFERENCES . Yongjun Xu and Si Yuan, International Journal of Fracture, Vol. 18, (1996) p. 373-381.
doi:10.1007/BF00012429