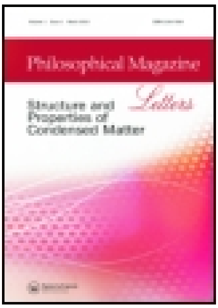

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Analysis of indentation loading curves obtained using conical indenters

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ABSTRACT

Using dimensional analysis and finite-element calculations we determine the functional form of indentation loading curves for a rigid conical indenter indenting into elastic–perfectly plastic solids. The new results are compared with the existing theories of indentation using conical indenters, including the slip-line theory for rigid-plastic solids, Sneddon's result for elastic solids, and Johnson's model for elastic-perfectly plastic solids. In the limit of small ratio of yield strength (Y) to Young's modulus (E), both the new results and Johnson's model approach that predicted by slip-line theory for rigid-plastic solids. In the limit of large Y/E, the new results agree with that for elastic solids. For a wide range of Y/E, some difference is found between Johnson's model and the present result. This study also demonstrates the possibilities and limitations of using indentation loading curves to extract fundamental mechanical properties of solids.

§ 1. INTRODUCTION

Indentation experiments have been performed for nearly one hundred years for measuring the hardness of materials (Tabor 1996). Recent years have seen increased interest in indentation because of the significant improvement in indentation equipment and the need for measuring the mechanical properties of materials on small scales. With the improvement in indentation instruments, it is now possible to monitor, with high precision and accuracy, both the load and displacement of an indenter during indentation experiments in the respective micro-Newtons and nanometre ranges (Pethica *et al.* 1983, Bhushan *et al.* 1996). In addition to hardness, basic mechanical properties of materials, such as the Young's modulus, yield strength, and work hardening exponent, may be deduced from the indentation load versus displacement curves for loading and unloading. For example, the hardness and Young's modulus may be calculated from the peak load and the initial slope of the unloading curves using the method of Oliver and Pharr (1992) or that of Doerner and Nix (1986). Finite-element methods have also been used to extract the mechanical properties of materials by matching the simulated loading and unloading curves with that of the experimentally determined ones (Bhattacharya and Nix 1988, Laursen and Simo 1992).

Recently, many attempts have been made to understand better indentation loading curves. For example, several empirical formulae have recently been proposed for the loading curves in terms of Young's modulus and hardness (Loubet *et al.* 1986, Hainsworth *et al.* 1996). Loading curves have also been discussed using energetic considerations of reversible and irreversible parts of the indentation induced deformations (Rother 1996).

In this letter, we derive simple equations which describe indentation loading curves for conical indenters indenting into elastic–perfectly plastic solids. The equations are obtained using dimensional analysis and finite-element calculations. These equations provide new insights into the indentation process and partial answers to the question: what can we learn from indentation experiments?

§ 2. Dimensional analysis

We consider a three-dimensional, rigid, conical indenter of a given angle (e.g. 136°) indenting normally into an elastic–perfectly plastic solid characterized by Young's modulus (E), Poisson's ratio (υ), and the yield strength (Y). The friction coefficient at the contact surface between the indenter and the solid is assumed zero. Unlike the cases of indenting into elastic or rigid-plastic solids, this apparently simple problem involving elastic–perfectly plastic solids has in fact no analytical solution. In general, however, the load (F) on the rigid indenter of a fixed included angle must be a function of Young's modulus (E), Poisson's ratio (υ), yield strength (Y), and the indenter displacement (h):

$$F = F(E, Y, \upsilon, h). \tag{1}$$

Among the four governing parameters, E, Y, υ and h, two of them, namely E and h, have independent dimensions. The dimensions of Y, υ and F are then given by

$$\begin{aligned} [Y] &= [E], \\ [\upsilon] &= [E]^0[h]^0, \\ [F] &= [E][h]^2. \end{aligned} \tag{2}$$

Applying the Π-theorem in dimensional analysis (Barenblatt 1987), we obtain:

$$\Pi = \Pi(\Pi_1, \upsilon), \quad \text{or equivalently}, \quad F = Eh^2\Pi\left(\frac{Y}{E}, \upsilon\right), \tag{3}$$

where $\Pi = F/Eh^2$, $\Pi_1 = Y/E$, and υ are all dimensionless.

Note that we can equally choose Y and h, instead of E and h, as the two governing parameters with independent dimensions. Dimensional analysis then yields:

$$F = Yh^2\Pi^*\left(\frac{E}{Y}, \upsilon\right). \tag{3a}$$

Obviously,

$$\Pi^*\left(\frac{E}{Y}, \upsilon\right) = \frac{E}{Y}\,\Pi\left(\frac{Y}{E}, \upsilon\right).$$

The problem of determining the indentation loading curve for a conical indenter of a given angle is now reduced to determining a function of two variables in equation (3) instead of four variables in equation (1). Furthermore, equation (3) shows that for a given material the load on the indenter (F) is proportional to the square of the indenter displacement (h).

§ 3. Finite-element analysis

Finite-element calculations using ABAQUS (Hibbitt, Karlsson & Sorensen, Inc. 1995) have been carried out to test the square-law dependence of load on displacement predicted by equation (3) and to evaluate the dimensionless function $\Pi(Y/E, \upsilon)$.

Similar to previous finite-element calculations, the large strain elasto-plastic feature of ABAQUS is used. In the finite-element model shown in figure 1 the indenter and solid were modelled as bodies of revolution to take advantage of the axisymmetry of the conical indentation. The indenter was modelled as a rigid body with half

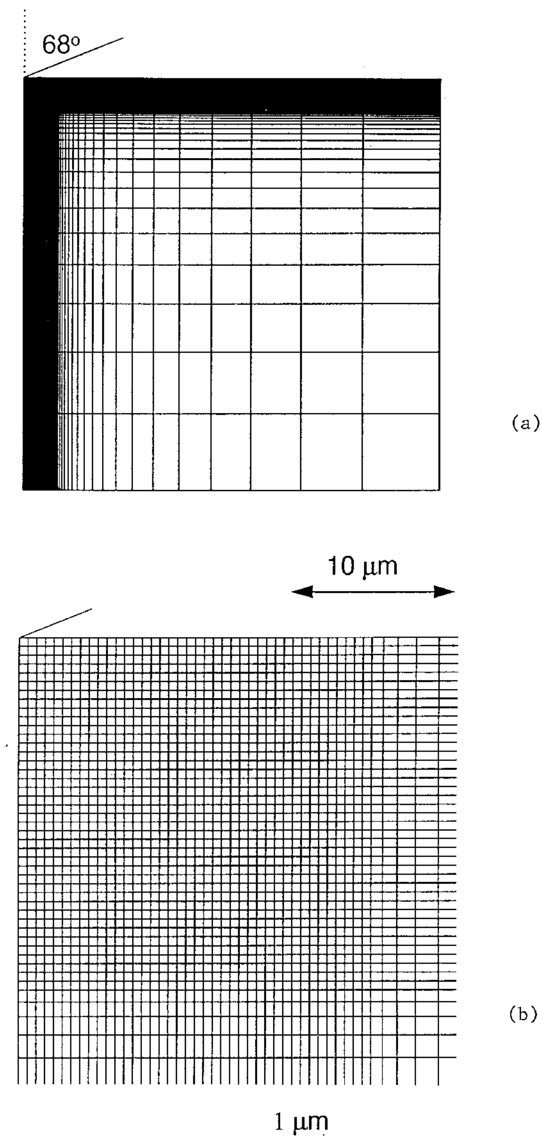
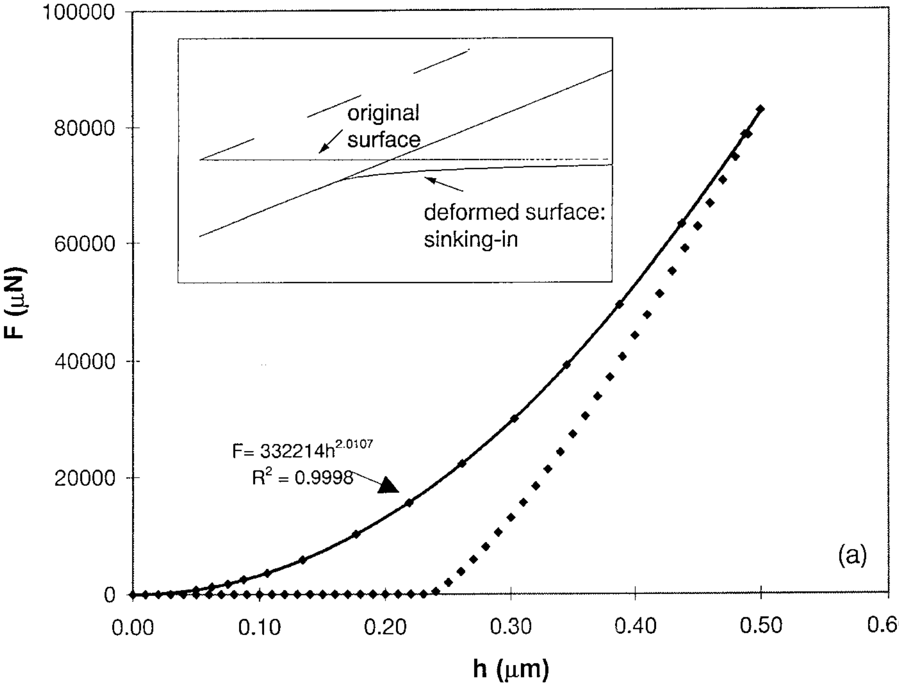


Figure 1. Finite-element model. (*a*) General view and (*b*) detailed view of the contact region.



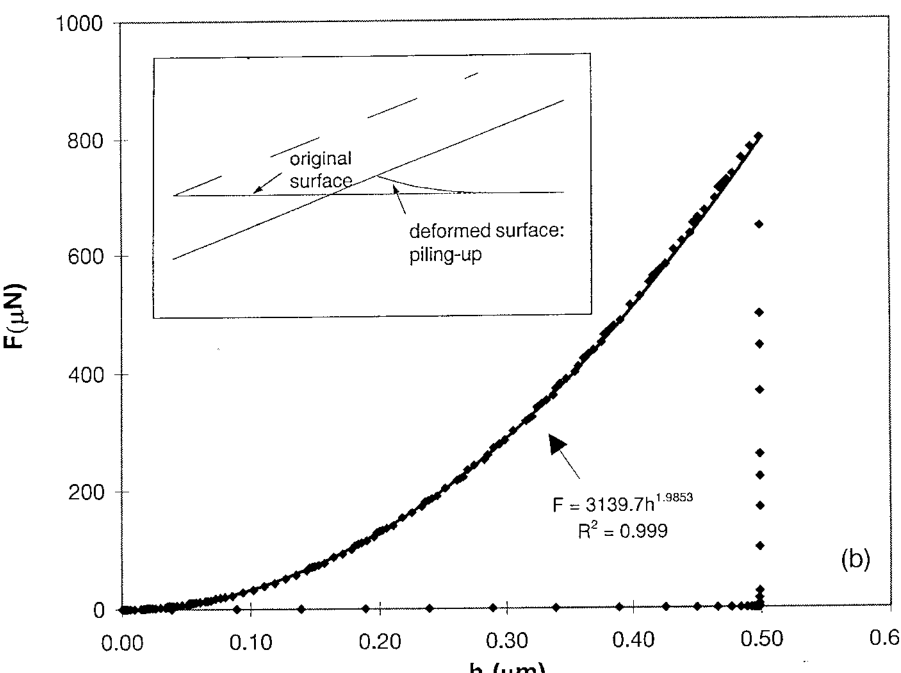


Figure 2. Indentation curves and surface profiles (inserts) obtained from finite-element analysis. (*a*) For $E = 200\,\text{GPa}$, $\upsilon = 0.3$, and $Y = 20\,\text{GPa}$ and (*b*) for $E = 200\,\text{GPa}$, $\upsilon = 0.3$, and $Y = 0.04\,\text{GPa}$.

angle 68° (following earlier work, for example, Bhattacharya and Nix (1988)). The surface of the rigid indenter was defined using the ABAQUS feature of analytic rigid surface definition. The semi-infinite elastic–perfectly plastic solid was modelled using 3600 4-node bilinear axisymmetric quadrilateral elements. A fine mesh in the vicinity of the indenter and a gradually coarser mesh away from the indenter were used to ensure a high degree of numerical accuracy and a good representation of the semi-infinite solid. Indeed, the results were shown to be insensitive to the boundary conditions at the bottom and outer boundaries of the mesh. The loading and unloading curves were essentially the same (e.g. less than 0.4% and 1% changes in peak loads and initial unloading slopes, respectively) under three types of boundary conditions: (1) the bottom and outer surface nodes were fixed, (2) the outer surface nodes were traction-free with bottom surface nodes fixed, and (3) roller boundary conditions were applied to the bottom and outer surface nodes. Therefore, the mesh used provides a good approximation to a semi-infinite solid. The yield criterion was that of Mises. The loading and unloading curves were obtained directly from the ABAQUS output of the total reaction force in the normal direction on the rigid indenter as a function of indenter vertical displacement. The calculations were performed using ABAQUS version 5.5 and 5.6 on workstations.

Figures 2(*a*) and (*b*) are typical examples of finite-element calculations for a variety of combinations of E, Y and υ. The simulated loading curves were fitted with power function: $F = ah^x$, where a and x are two fitting parameters. The exponent, x, obtained from all simulations is between 1.98 and 2.03. The simulation thus shows that the loading curves obey the square-law dependence predicted from dimensional analysis. The corresponding surface profiles under load, shown as inserts in figures 2(*a*) and (*b*), demonstrate that the well-known 'sinking-in' and 'piling-up' phenomena (see, for example, Tabor (1970)) can occur for conical indentation into elastic–perfectly plastic solids. Sinking-in occurs if Y/E is large and piling-up occurs if Y/E is small. These simulation results are expected from conical indentation theories in elastic solids of Sneddon (1963) and in rigid-plastic solids of Lockett (1963), respectively. They are also consistent with experimental observations of piling-up and sinking-in when the actual stress–strain relations approximate that of elastic–perfectly plastic solids (see, for example, Tabor (1970), Chaudhri and Winter (1988) and Bec *et al.* (1996)). In spite of the rich variations in surface profiles, however, the present calculations show that the square-law dependence of the load on displacement holds true.

To evaluate the dimensionless function $\Pi(Y/E, \upsilon)$ simulations were performed for a large number of Y and a few selected values of E and υ. Figure 3(*a*) depicts F/Eh^2 versus Y/E for two values of υ. Figure 3(*b*) is a detailed view of F/Eh^2 versus Y/E for $0 < Y/E < 0.01$. It is evident from figures 3(*a*) and (*b*) that, for a given value of υ, the quantity F/Eh^2 and Y/E lie on a single curve, as predicted by dimensional analysis (equation (3)). Thus, F/Eh^2 is a function of Y/E and υ.

§ 4. Discussion

It is instructive to consider the behaviour of $\Pi(Y/E, \upsilon)$. For small values of Y/E (i.e. < 0.002) (figure 3(*b*)), $\Pi(Y/E, \upsilon)$ becomes independent of υ and is approximately a linear function of Y/E, or equivalently,

$$F = cYh^2 \quad \text{for} \quad Y/E < 0.002, \tag{4a}$$

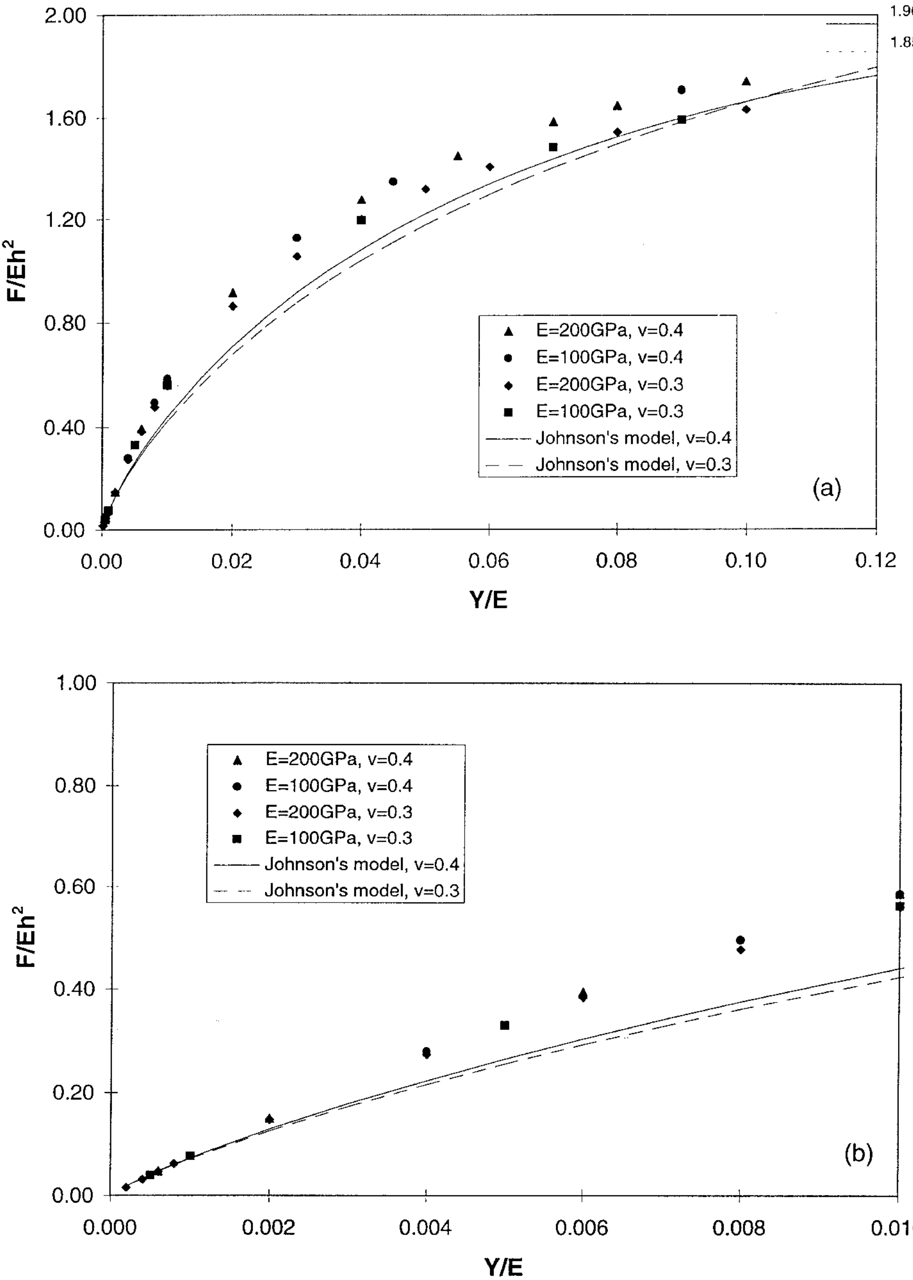


Figure 3. Scaling relationships between F/Eh^2 and Y/E. (a) For $0 < Y/E < 0.1$ and (b) for $0 < Y/E < 0.01$. Symbols: finite-element results. Lines: Johnson's model.

where c is about 74.4. Using the projected contact area determined from the finite-element output (e.g. figure 2 (b)) and equation (4 a), we obtain a relationship between the average pressure p_m, and Y:

$$p_m = 2.29Y \quad \text{for} \quad Y/E < 0.002. \tag{4 b}$$

This is in close agreement with the slip-line theory result, $p_m = 2.31Y$, for a rigid, conical indenter of 68° half angle indenting into rigid-plastic solids (Lockett 1963).

With increasing Y/E (i.e. > 0.05), $\Pi(Y/E, \upsilon)$ becomes less sensitive to Y/E. For purely elastic solids, F/Eh^2 equals 1.85 and 1.96 for $\upsilon = 0.3$ and 0.4, respectively (figure 3(*a*)). These values may be compared with the analytical results of Sneddon (1963) for conical indenters indenting into elastic solids,

$$F = \frac{2}{\pi}\frac{E}{1-\upsilon^2}h^2 \tan\theta, \tag{5}$$

where θ is the indenter half angle. When $\theta = 68^\circ$, we obtain from equation (5) $F/Eh^2 = 1.73$ and 1.88 for $\upsilon = 0.3$ and 0.4, respectively. The values obtained from the present ABAQUS calculations are, therefore, 7 and 4% *larger* than that calculated from equation (5). This agrees with previous numerical solutions and finite-element calculations of conical indentation into elastic solids by Tanaka and Koguchi (1995). This difference may be due to the fact that Sneddon's formula was based on linear elasticity, whereas ABAQUS takes into account nonlinear effects including large strain and moving contact boundaries. We note, however, that both finite-element calculations and equation (5) show that $F/Eh^2 = \Pi(Y/E, \upsilon)$ becomes $c/(1-\upsilon^2)$ for elastic solids ($Y/E \to \infty$). This suggests that $F(1-\upsilon^2)/Eh^2$ scales approximately with Y/E, when Y/E is large. Figure 4 depicts $F(1-\upsilon^2)/Eh^2$ versus Y/E. An approximate scaling relation is evident.

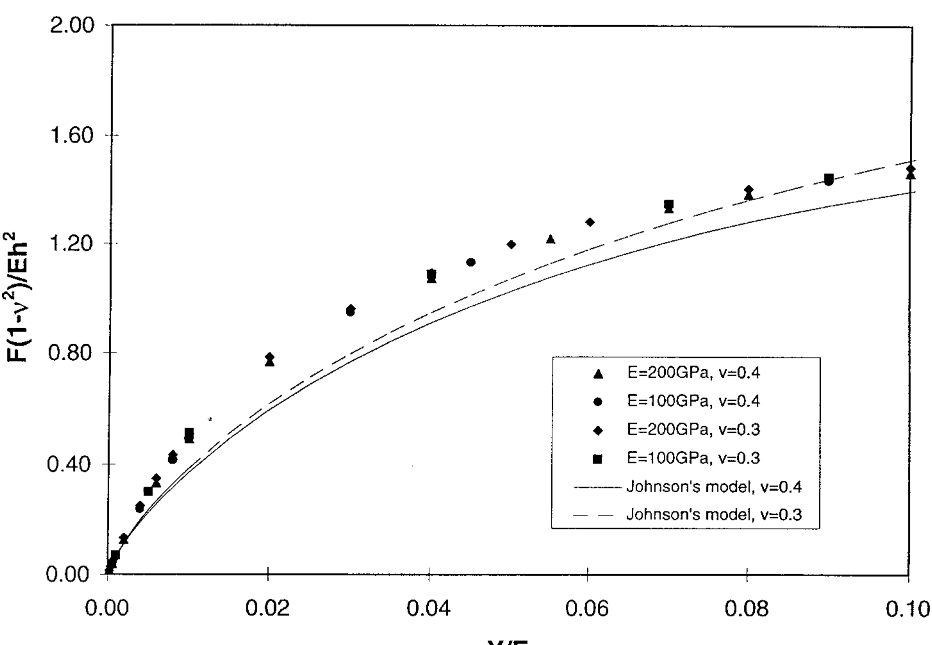


Figure 4. An approximate scaling relationship between $F(1-\upsilon^2)/Eh^2$ and Y/E. Symbols: finite-element results. Lines: Johnson's model.

It is also instructive to compare the above results with Johnson's model for conical indentation (Johnson 1970, 1985). Extending Hill's (1950) theory of expanding spherical cavities, Johnson showed that the average pressure p_m is given by

$$\frac{p_m}{Y} = \frac{2}{3}\left\{1 + \ln\left[\frac{E}{6Y(1-\upsilon)}\cot\theta + \frac{2}{3}\frac{1-2\upsilon}{1-\upsilon}\right]\right\}. \quad (6)$$

In deriving equation (6) Johnson idealized the surface profiles by neglecting piling-up and sinking-in effects. Consequently, the load F, on the indenter is given by $\pi(h\tan\theta)^2 p_m$. Equation (6) then becomes

$$\frac{F}{Eh^2} = \frac{2}{3}\pi(\tan\theta)^2\frac{Y}{E}\left\{1 + \ln\left[\frac{E}{6Y(1-\upsilon)}\cot\theta + \frac{2}{3}\frac{1-2\upsilon}{1-\upsilon}\right]\right\}. \quad (6a)$$

In figures 3 and 4, the respective F/Eh^2 and $F(1-\upsilon^2)/Eh^2$ are plotted against Y/E according to equation (6a) for $\upsilon = 0.3, 0.4$, and $\theta = 68^\circ$. As Y/E approaches 0, equation (6a) and $\Pi(Y/E, \upsilon)$ agree with each other. However, a difference between equation (6a) and the scaling function $\Pi(Y/E, \upsilon)$ is seen over a wide range of Y/E that includes most materials (e.g. $0.004 < Y/E < 0.1$). Furthermore, equation (6a) cannot be applied to elastic solids ($Y/E \to \infty$). In contrast, $\Pi(Y/E, \upsilon)$ converges to a value close to that given by Sneddon's equation based on linear elasticity.

Yoffe (1982) was the first to point out that the expanding cavity model was not valid for indentation problems based on her analysis of the stress field under indenters. We believe that the difference between Johnson's model and our new results may also be caused by the neglect of sinking-in and piling-up effects in deriving Johnson's expression (equation (6)), whereas finite element calculations include these effects. Nevertheless, Johnson's spherical cavity model provides an approximate, analytical description of conical indentation. The present results may, therefore, be considered an improvement over Johnson's equation (equation (6)) in describing indentation into elastic–perfectly plastic solids using a conical indenter.

The above analysis provides a theoretical possibility for extracting materials properties from indentation loading curves. Because the load, F, and displacement, h, can be determined experimentally, equation (3) shows that Y (or E) can be determined from loading curves, provided that E (or Y) and υ are known. In general, however, loading curves alone cannot uniquely determine both E and Y even when υ is known. Since $\Pi(Y/E, \upsilon)$ does not change significantly for typical values of $\upsilon (0.2 < \upsilon < 0.4)$, an estimated Y (or E) may be obtained from loading curves by using a typical value of υ (i.e. 0.3) if E (or Y) is known. In practice, E is usually more easily obtained than Y by a variety of measurement techniques. Equation (3) shows that Y can then be obtained from indentation loading curves without the need of measuring hardness. This is useful because hardness requires the measurement or estimation of the contact area under load, which may be difficult to obtain. Furthermore, the conversion of hardness to yield strength is not always straightforward.

§ 5. Summary

Using dimensional analysis and finite-element calculations we have determined the functional form of indentation loading curves for a rigid conical indenter indenting into elastic–perfectly plastic solids. The new result is an improvement over Johnson's spherical cavity model for conical indentation in elastic–perfectly plastic solids. It also shows that, in principle, yield strength (Y) can be obtained from indentation loading curves if the Young's modulus (E) and Poisson's ratio (υ) are known or, conversely, E can be determined if Y and υ are known. In general, however, E, υ, and Y cannot be uniquely determined from indentation loading curves alone.

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