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Variational approach to the Lane–Emden equation

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Abstract

By the semi-inverse method, a variational principle is obtained for the Lane–Emden equation, which gives much numerical convenience when applying finite element methods or Ritz method.

Keywords: Variational principle; Ritz method; Lane–Emden equation

Many problems in the literature of mathematical physics can be distinctively formulated as equations of Lane–Emden type [1,2], defined in the form

$$y'' + \frac{2}{x}y' + f(y) = 0, \quad y(0) = y_0, \quad y'(0) = 0. \tag{1}$$

Recently, Bender et al. [2] proposed a new perturbation technique based on an artificial parameter δ, the method is often called δ-method [2,3]. Wazwaz used the Adomian decomposition method [1,4] to the equation.

In this paper, we will use the Ritz's method to obtain an analytical solution of the problem. To this end, we should establish a variational principle. Rewriting (1) in the form

$$xy'' + 2y' + xf(y) = 0. \tag{2}$$

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By the semi-inverse method [5], we obtain the following variational principle:

$$J(y) = \int_0^a \left\{ -\tfrac{1}{2}x^2 y'^2 + x^2 F(y) \right\} \mathrm{d}x, \tag{3}$$

where F is a potential function defined as

$$\frac{\partial F}{\partial y} = f. \tag{4}$$

Proof. Calculating variation of the functional (3) with respect to y, we obtain the following Euler equation:

$$\frac{\partial}{\partial x}(x^2 y') + x^2 \frac{\partial F}{\partial y} = 0 \tag{5}$$

or

$$x^2 y'' + 2xy' + x^2 f = 0. \tag{6}$$

It is easy to prove that Eqs. (6) and (1) are equivalent in the case $x \neq 0$. □

Example 1. $f = y^m$.

Such case sees in the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. Its variational principle reads

$$J(y) = \int_0^a \left\{ -\frac{1}{2}x^2 y'^2 + \frac{1}{m+1}x^2 y^{m+1} \right\} \mathrm{d}x. \tag{7}$$

Example 2. $f = a\cos\omega_1 x + b\sin\omega_2 x$.

Under such condition, its variational principle can be written as

$$J(y) = \int_0^a \left\{ -\frac{1}{2}x^2 y'^2 + \frac{a}{\omega_1}\sin\omega_1 x - \frac{b}{\omega_2}\cos\omega_1 x \right\} \mathrm{d}x. \tag{8}$$

Now consider the general form of Lane–Emden equation

$$y'' + \frac{n}{x}y' + f(y) = 0, \quad y(0) = y_0, \quad y'(0) = 0. \tag{9}$$

We rewrite (9) in the form

$$x^n y'' + nx^{n-1}y' + x^n f(y) = 0. \tag{10}$$

By the semi-inverse method [5], we obtain the following functional

$$J(y) = \int_0^a \left\{ -\frac{1}{2} x^n y'^2 + x^n F(y) \right\} \mathrm{d}x. \tag{11}$$

The Ritz method, or other variational direction methods, can be applied.

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