Recovery of the Solitons Using a Lattice Boltzmann Model *

YAN Guang-wu(闫广武)[1,2], SONG Min(宋敏)[3]
[1]Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080
[2]Department of Mathematics, Jilin University, Changchun 130023
[3]Division of Foundation, Jilin Province Architect College, Changchun 130053

(Received 3 June 1998)

We formulate a lattice Boltzmann model which simulates Korteweg-de Vries equation by using a method of higher moments of lattice Boltzmann equation. Using a series of lattice Boltzmann equations in different time scales and the conservation law in time scale t_0, we obtain equilibrium distribution function. The numerical examples show that the method can be used to simulate soliton.

PACS: 47.10.+g

Lattice Boltzmann method[1,2] (LBM) has recently been introduced as a new computational tool for the study of fluid dynamics and systems governed by related partial differential equations. Unlike traditional numerical methods which search for macroscopic variables, LBM is based on the mesoscopic kinetic equation for the particle distribution function. The macroscopic quantities, such as density and velocity, are then obtained by moment integrations of distribution function. The kinetic nature of LBM introduces a number of advantages: (i) linearity of the convection operator, (ii) the lattice Boltzmann equation is a one-dimensional (1D) equation for given direction, (iii) the code is greatly simplified.

In this letter, we propose a method of higher moments on lattice Boltzmann equation for Korteweg-de Vries (KdV) equation, and use it to explore, comprehend the complex phenomena of the solitons.

Consider a 1D model, we discrete the velocity of particles into four directions. A lattice with unit spacing is used in which each node has the four nearest neighbors connected with four links. The distribution function f_α is the probability of finding a particle at time t and node x, with velocity e_α, here $\alpha = 0, 1, \cdots, 4$ ($\alpha = 0$ is a rest particle). The particle velocities e_α are 0, c, $-c$, kc, and $-kc$, $k=2$ is given four neighbor nodes. The macroscopic quantity $u(x,t)$ (particles number) is defined by

$$u(x,t) = \sum_\alpha f_\alpha(x,t), \tag{1}$$

conservational condition is

$$\sum_\alpha f_\alpha^{\rm eq}(x,t) = u(x,t). \tag{2}$$

The particle distribution function satisfies the lattice Boltzmann equations:[1,2]

$$f_\alpha(x+e_\alpha, t+1) - f_\alpha(x,t) = -\frac{1}{\tau}[f_\alpha(x,t) - f_\alpha^{\rm eq}(x,t)], \tag{3}$$

where $f_\alpha^{\rm eq}(x,t)$ is the equilibrium distribution function at time t and node x with velocity e_α and τ is the single relaxation time factor.

Using ε as the small lattice unit in physical unit, ε can play the role of the Knudsen number,[5] the lattice Boltzmann Eq. (3) in physical unit is

$$f_\alpha(x+\varepsilon e_\alpha, t+\varepsilon) - f_\alpha(x,t) = -\frac{1}{\tau}(f_\alpha - f_\alpha^{\rm eq}). \tag{4}$$

We apply Taylor expansion and Chapman-Enskog expansion[3] to Eq. (4), retaining terms up to $O(\varepsilon^5)$, under the assumption that mean free path is of the same order of ε. To discuss changes in different time scales, introduced as $t_0, \cdots, t_3$, thus, we get a series of lattice Boltzmann equations in different time scales:

$$\frac{\partial f_\alpha^{(0)}}{\partial t_0} + e_\alpha \frac{\partial f_\alpha^{(0)}}{\partial x} = -\frac{1}{\tau} f_\alpha^{(1)}, \tag{5}$$

$$\frac{\partial f_\alpha^{(0)}}{\partial t_1} - \tau(1-\frac{1}{2\tau})(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x})^2 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(2)}, \tag{6}$$

$$\frac{\partial f_\alpha^{(0)}}{\partial t_2} + (1-2\tau)(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x})\frac{\partial f_\alpha^{(0)}}{\partial t_1} + (\tau^2 - \tau + \frac{1}{6})(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x})^3 f_\alpha^{(0)} = -\frac{1}{\tau} f_\alpha^{(3)}, \tag{7}$$

$$\begin{aligned}&\frac{\partial f_\alpha^{(0)}}{\partial t_3} + (2\tau^2 - \frac{5}{2}\tau + \frac{1}{2})\frac{\partial}{\partial t_1}(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x})^2 f_\alpha^{(0)} \\ &+ (1-2\tau)\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x}) f_\alpha^{(0)} \\ &+ (-\tau^3 + \frac{3}{2}\tau^2 - \frac{7}{12}\tau + \frac{1}{24})(\frac{\partial}{\partial t_0} + e_\alpha\frac{\partial}{\partial x})^4 f_\alpha^{(0)} \\ &+ \frac{\partial f_\alpha^{(2)}}{\partial t_1} + \frac{1}{2}\frac{\partial^2 f_\alpha^{(0)}}{\partial t_1^2} = -\frac{1}{\tau} f_\alpha^{(4)}.\end{aligned} \tag{8}$$

*Supported by the Foundation of the Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, Chinese Academy of Sciences.

Above Eqs. (5)–(8) are exact equations in all dimensional model. The coefficients $\tau - 1/2$, $\tau^2 - \tau + 1/6$, and $-\tau^3 + 3\tau^2/2 - 7\tau/12 + 1/24$ in Eqs. (5)–(8) are needed in the derivation and may be used to give the feature of macroscopic equations.

The macroscopic equation may be written in the form:

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial m^0}{\partial x}+\varepsilon(\frac{1}{2}-\tau)\sum_\alpha(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x})^2 f_\alpha^{(0)}\\&+\varepsilon^2\mu\sum_\alpha(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x})^3 f_\alpha^{(0)}+\varepsilon^3\Big[(1-2\tau)\\&\cdot\sum_\alpha\frac{\partial}{\partial t_2}(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x})f_\alpha^{(0)}+(1-\frac{1}{2\tau})\\&\cdot\sum_\alpha\frac{\partial f_\alpha^{(2)}}{\partial t_1}+(2\tau^2-2\tau+\frac{1}{4})\sum_\alpha\frac{\partial}{\partial t_1}(\frac{\partial}{\partial t_0}\\&+e_\alpha\frac{\partial}{\partial x})^2 f_\alpha^{(0)}+(-\tau^3+\frac{3}{2}\tau^2-\frac{7}{12}\tau+\frac{1}{24})\\&\cdot\sum_\alpha(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x})^4 f_\alpha^{(0)}\Big]=0. \qquad (9)\end{aligned}$$

where $m^0 = \sum_\alpha f_\alpha^{(0)} e_\alpha = u^2/2$, assuming that

$$\pi^0 = \sum_\alpha f_\alpha^{(0)} e_\alpha^2 = \frac{1}{3}u^3 + \lambda u, \qquad (10)$$

$$\begin{aligned}&\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(\frac{1}{2}u^2)-\nu\frac{\partial^2 u}{\partial x^2}+\mu\sum_\alpha(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x})^3 f_\alpha^{(0)}\\&+\eta\sum_\alpha(\frac{\partial}{\partial t_0}+e_\alpha\frac{\partial}{\partial x})^4 f_\alpha^{(0)}=0 \qquad (11)\end{aligned}$$

with $\nu = \varepsilon\lambda(\tau - 1/2)$, $\mu = \varepsilon^2(\tau^2 - \tau + 1/6)$, and $\eta = \varepsilon^3(-\tau^3 + 3\tau^2/2 - 7\tau/12 + 1/24)$.

We assume

$$\begin{aligned}P^0 &= \sum_\alpha f_\alpha^{(0)} e_\alpha^3 = \xi_1 u + \frac{u^4}{4},\\ L^0 &= \sum_\alpha f_\alpha^{(0)} e_\alpha^4 = \xi_2\frac{u^2}{2} + \frac{u^5}{5}, \qquad (12)\end{aligned}$$

Eq. (11) becomes KdV equation,[4,6] therefore,

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}(\frac{u^2}{2})+\overline{\nu}\frac{\partial^3 u}{\partial x^3}=O(\varepsilon^3). \qquad (13)$$

The truncation error is

$$R = O(\varepsilon^3) = \overline{\lambda}\frac{\partial^4}{\partial x^4}(\frac{u^2}{2}) + O(\varepsilon^4),$$

where $\overline{\lambda}$ is a dissipation coefficient and the coefficient $\overline{\nu}$ is given as

$$\overline{\nu} = \xi_1\mu = \varepsilon^2(\tau^2 - \tau + \frac{1}{6})\xi_1. \qquad (14)$$

The test problem is simulated to check the performance of this model on the simulation of KdV equation[4] with initial and boundary functions

$$\begin{aligned}&u_t + uu_x + \delta^2 u_{xxx} = 0, \quad u_{t=0} = \cos(\pi x),\\&u(x+2,t) = u(x,t), \qquad (15)\end{aligned}$$

which is simulated by using 5-bit model, lattice size $N=50$, $c=1.0$, $\xi_1=1$, $\delta=0.022$, $\Delta x=0.04$, $\Delta t=0.04$, and $\lambda=0.0$. The soliton reappears, see Fig. (1).

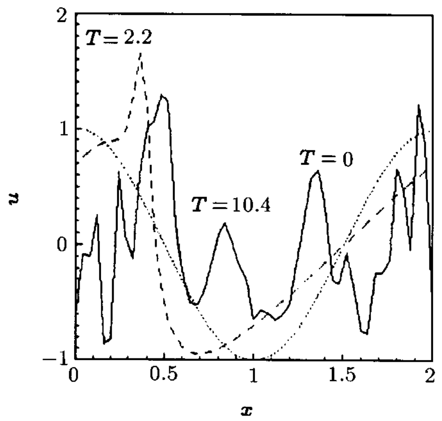

Fig. 1. Numerical results of the KdV equation. Lattice size: N=50. Parameters: c=1.0; $\xi_1=1.0$; $\delta=0.022$; $\Delta x=0.04$; $\Delta t=0.04$; $\lambda=0.0$. $T=n\Delta t$, Δt is time steps.

In 1991, Frisch[7] pointed out the possibility that the increase of the requirments of higher moments may be used to construct lattice gas model for Navier-Stokes equation.[7] In this paper, we use Chapman-Enskog expansion and multi-scale technique on distribution function to obtain $f_\alpha^{(3)}$, and KdV equation with higher order accuracy. It is a useful concept that according to conservational law in time scale t_0, the Eqs. (5)–(8) of different time scales are important result in the LBM.

REFERENCES

1 S. Y. Chen and G. D. Doolen, Annu. Rev. Fluid Mech. 30 (1998) 329.
2 F. J. Alexander, H. D. Chen, S. Y. Chen and G. D. Doolen, Phys. Rev. A 46 (1993) 1967.
3 S. Chapman and T. G. Cowling, *The Mathematical Theory of Non-uniform Gas* (Cambridge University Press, Cambridge, 1939) p. 149.
4 N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. 15 (1965) 240.
5 S. L. Hou *et al.*, J. Comput. Phys. 118 (1995) p. 329.
6 G. W. Yan, Y. S. Chen and S. X. Hu, Acta Mech. Sin. 14 (1998) 18.
7 U. Frisch, Physica, D 47 (1991) 231.