

## MODE I AND MODE II CRACK TIP ASYMPTOTIC FIELDS WITH STRAIN GRADIENT EFFECTS\*

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**ABSTRACT:** The strain gradient effect becomes significant when the size of fracture process zone around a crack tip is comparable to the intrinsic material length  $l$ , typically of the order of microns. Using the new strain gradient deformation theory given by Chen and Wang, the asymptotic fields near a crack tip in an elastic-plastic material with strain gradient effects are investigated. It is established that the dominant strain field is irrotational. For mode I plane stress crack tip asymptotic field, the stress asymptotic field and the couple stress asymptotic field can not exist simultaneously. In the stress dominated asymptotic field, the angular distributions of stresses are consistent with the classical plane stress HRR field; In the couple stress dominated asymptotic field, the angular distributions of couple stresses are consistent with that obtained by Huang et al. For mode II plane stress and plane strain crack tip asymptotic fields, only the stress-dominated asymptotic fields exist. The couple stress asymptotic field is less singular than the stress asymptotic fields. The stress asymptotic fields are the same as mode II plane stress and plane strain HRR fields, respectively. The increase in stresses is not observed in strain gradient plasticity for mode I and mode II, because the present theory is based only on the rotational gradient of deformation and the crack tip asymptotic fields are irrotational and dominated by the stretching gradient.

**KEY WORDS:** strain gradient effect, crack tip asymptotic field, plane stress, plane strain

### 1 INTRODUCTION

Many experiments have shown that materials display strong size effects when the characteristic length scale associated with non-uniform plastic deformation is on the order of microns<sup>[1~7]</sup>. The classical plasticity theories can not predict this size dependence of materials behavior at the micron scale because their constitutive models lack an internal length scale.

In order to explain the size effect, it is necessary to develop a continuum theory on a micron level. Based on the dislocation analysis, a strain gradient plasticity theory has been developed by Fleck and Hutchinson<sup>[8]</sup>, in which a material length scale was introduced

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from the dimensional grounds. While explaining experimental findings of indentation<sup>[2,4,9]</sup>, fracture<sup>[10]</sup>, it has been found necessary to introduce two length parameters<sup>[9,11]</sup>. One length refers to rotational gradients as originally proposed in connection with the torsion measurements, the other scale is related with the stretching gradients. The latter is needed to rationalize the length scale phenomena found in indentation and fracture. In 1998, Nix and Gao<sup>[12]</sup> started from the Taylor relation and gave one kind of hardening law for gradient plasticity. Motivated by the indentation hardening law, Gao, Huang et al.<sup>[13]</sup> proposed a mechanism-based theory of strain gradient plasticity. In comparison, no work conjugate of strain gradient has been defined in the alternative gradient theories<sup>[14,15]</sup>, which represent the strain gradient effects by terms relative with Laplacian of the effective strain. Retaining the essential structure of conventional plasticity and obeying thermodynamic restrictions, Acharya and Bassani<sup>[16]</sup> conclude that the only possible formulation is a flow theory with strain gradient effects represented by an internal variable, which acts to increase the current tangent-hardening modulus. In 2000, Chen and Wang<sup>[17]</sup> established a hardening law based on the incremental version of conventional  $J_2$  deformation theory, in which the effective strain gradient is only a parameter to influence the tangent modulus.

A new strain gradient deformation theory is developed by Chen and Wang in 2001<sup>[18]</sup>, which fits within the framework of general couple stress theory and involves a single material length scale  $l$ . In the theory three rotational degrees of freedom  $\omega_i$  are introduced in addition to the conventional three translational degrees of freedom  $u_i$ .  $\omega_i$  has no direct dependence upon  $u_i$ . The strain energy density is assumed to be only a function of the strain tensor and the overall curvature tensor, then Cauchy stress becomes symmetric. Using the new strain gradient theory, two typical phenomena, i.e., the thin wire torsion and micro-thin beam bending have been investigated successfully.

Though a large strain gradient exists near the crack tip, there is limited progress in applying strain gradient plasticity to the estimation of crack tip fields<sup>[19~22]</sup>. [19] investigated mode I crack-tip asymptotic fields in elastic as well as elastic-plastic materials with strain gradient effects. They showed that stresses and couple stresses near a crack tip could not have the same order of singularity. The near tip field is either stress dominated (stresses are more singular than couple stresses) or couple stresses dominated (couple stresses are more singular than stresses). In [20], mode I and mode II plane stress crack tip fields were investigated, respectively. [21] presented a finite element study as well as an asymptotic analysis for mode I and mode II crack tip fields in strain gradient plasticity. Their asymptotic solution and the finite element analysis also confirmed that stresses and couple stresses near a crack tip do not have the same order of singularity. All of them show that the stress components increase and are different from HRR solution though the crack tip is irrotational. [22] investigated mode I plane strain crack tip asymptotic field using the strain gradient theory given by [18] and found that the crack tip asymptotic field is irrotational and the stress dominated field is the same as HRR solution.

The aim of the present paper is to investigate the asymptotic fields for mode I and II crack in elastic-plastic materials with strain gradient effects, respectively, while using the new strain gradient deformation theory of plasticity<sup>[18]</sup>. We will start with a summary of the new strain gradient plasticity theory in section 2. The near tip asymptotic fields in an elastic-plastic material with strain gradient effects are given in section 3. In section 4 and section 5, mode I crack tip field and mode II crack tip asymptotic field are investigated,

respectively. Discussions are given in section 6.

## 2 THE NEW STRAIN GRADIENT DEFORMATION THEORY

A new strain gradient theory has been proposed in [18]. It fits within the framework of general couple stress theory<sup>[23]</sup>, in which three micro-rotational degrees of freedom  $\omega_i$  are introduced in addition to the conventional three translational degrees of freedom  $u_i$ .  $\omega_i$  has no direct dependence upon  $u_i$ , which is different from the material rotation vector  $\vartheta_i$ ,  $\vartheta \equiv (1/2)\text{curl}u$ .

In the general couple stress theory<sup>[23]</sup> a relative rotation tensor  $\alpha$  is defined as

$$\alpha_{ij} = e_{ijk}\omega_k - (u_{j,i} - u_{i,j})/2 = e_{ijk}(\omega_k - \vartheta_k) \quad (1)$$

The strain energy density  $W$  is assumed to depend only upon the strain tensor  $\varepsilon_{ij}$  and the curvature tensor  $\chi_{ij}$ ,  $\chi_{ij} = \omega_{i,j}$ , i.e. the relative rotation tensor  $\alpha_{ij}$  has no contribution to the strain energy density  $W$ . It follows that

$$\tau_{ij} = \partial W / \partial \alpha_{ij} = 0 \quad (2)$$

where  $\tau_{ij}$  is the anti-symmetrical part of Cauchy stress in the general couple stress theory<sup>[23]</sup>.

Then, the equilibrium equations for stress and couple stress in the body are

$$\sigma_{ij,j} = 0 \quad m_{ij,j} = 0 \quad (3)$$

The traction boundary conditions for force and moment are

$$\sigma_{ij}n_j = T_i^0 \quad \text{on } S_T \quad (4)$$

$$m_{ij}n_j = q_i^0 \quad \text{on } S_q \quad (5)$$

The additional boundary conditions are

$$u_i = u_i^0 \quad \text{on } S_u \quad (6)$$

$$\omega_i = \omega_i^0 \quad \text{on } S_\omega \quad (7)$$

The deviatoric part  $s_{ij}$  of Cauchy stress and deviatoric part  $m'_{ij}$  of couple stress are defined as the work conjugates of  $\varepsilon'_{ij}$ ,  $\chi'_{ij}$ , respectively;  $\sigma_m$  and  $m_m$  are defined as the work conjugates of  $\varepsilon_m$  and  $\chi_m$ , respectively. Then the incremental work can be given as

$$\delta W = s_{ij}\delta\varepsilon'_{ij} + m'_{ij}\delta\chi'_{ij} + \sigma_m\delta\varepsilon_m + m_m\delta\chi_m \quad (8)$$

where  $s_{ij} \equiv \sigma_{ij} - (1/3)\delta_{ij}\sigma_{kk}$  and  $m'_{ij} \equiv m_{ij} - (1/3)\delta_{ij}m_{kk}$ .

The above equation enables one to determine  $s_{ij}$ ,  $m'_{ij}$ ,  $\sigma_m$  and  $m_m$  in terms of the strain and curvature states of the solid as

$$s_{ij} = \frac{\partial W}{\partial \varepsilon'_{ij}} \quad m'_{ij} = \frac{\partial W}{\partial \chi'_{ij}} \quad \sigma_m = \frac{\partial W}{\partial \varepsilon_m} \quad m_m = \frac{\partial W}{\partial \chi_m} \quad (9)$$

According to the work by Fleck-Hutchinson<sup>[8]</sup> and Fleck, et al.<sup>[6]</sup>, it is mathematically convenient to assume that the strain energy density  $W$  depends only upon the single scalar strain measure  $E_e$ , where

$$E_e^2 = \varepsilon_e^2 + l^2 \chi_e^2 \quad (10)$$

The length scale  $l$  is a material length scale, and is required on the dimensional grounds.

An effective stress measure  $\Sigma_e$  is defined as the work conjugate of  $E_e$

$$\Sigma_e = \frac{dW(E_e)}{dE_e} \quad (11)$$

then

$$s_{ij} = \frac{2\Sigma_e}{3E_e} \varepsilon'_{ij} \quad m'_{ij} = \frac{2}{3} l^2 \frac{\Sigma_e}{E_e} \chi'_{ij} \quad \sigma_m = \frac{1}{3} \sigma_{kk} \quad m_m = \frac{1}{3} m_{kk} \quad (12)$$

$$\Sigma_e = (\sigma_e^2 + l^{-2} m_e^2)^{1/2} \quad (13)$$

where

$$\begin{aligned} \sigma_e^2 &= \frac{3}{2} s_{ij} s_{ij} & m_e^2 &= \frac{3}{2} m'_{ij} m'_{ij} \\ \varepsilon_e^2 &= \frac{2}{3} \varepsilon'_{ij} \varepsilon'_{ij} & \chi_e^2 &= \frac{2}{3} \chi'_{ij} \chi'_{ij} \end{aligned} \quad (14)$$

For the purpose of some specific calculations, the relationship between  $\Sigma_e$  and  $E_e$  can be chosen.

### 3 ASYMPTOTIC EQUATIONS

For a plane problem, the nonzero in-plane stresses and couple stresses in polar coordinate are  $\sigma_{rr}$ ,  $\sigma_{r\theta}$  ( $\sigma_{\theta r}$ ),  $\sigma_{\theta\theta}$ ,  $m_{zr}$ ,  $m_{z\theta}$ . The field equations for the equilibrium in 2D are given explicitly in polar coordinates, where the polar coordinates  $(r, \theta)$  are centered at the crack tip (Fig.1)

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \quad (15)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0$$

$$\frac{\partial m_{zr}}{\partial r} + \frac{1}{r} \frac{\partial m_{z\theta}}{\partial \theta} + \frac{m_{zr}}{r} = 0 \quad (16)$$

The relations between strains  $\varepsilon_{ij}$  and displacements  $u_i$  are

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \quad \varepsilon_{\theta\theta} = \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \quad \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) \quad (17)$$

The relations between the micro-rotation vectors  $\omega_i$  and curvature tensors  $\chi_{ij}$  are

$$\chi_{zr} = \frac{\partial \omega_z}{\partial r} \quad \chi_{z\theta} = \frac{1}{r} \frac{\partial \omega_z}{\partial \theta} \quad (18)$$

The compatibility equations of strains and curvature tensors are as follows

$$\frac{\partial^2 \varepsilon_{rr}}{\partial \theta^2} - r \frac{\partial \varepsilon_{rr}}{\partial r} - 2 \frac{\partial^2 (r \varepsilon_{r\theta})}{\partial r \partial \theta} + \frac{\partial}{\partial r} \left( r^2 \frac{\partial \varepsilon_{\theta\theta}}{\partial r} \right) = 0 \quad (19)$$

$$\frac{\partial \chi_{zr}}{\partial \theta} - \frac{\partial (r \chi_{z\theta})}{\partial r} = 0 \quad (20)$$

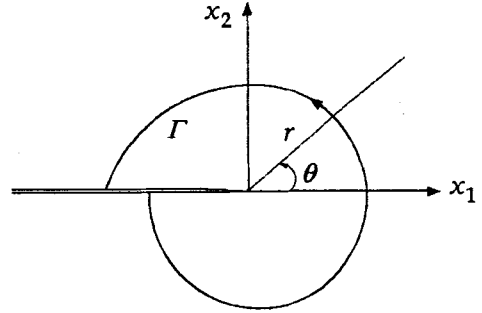


Fig.1 Schematic diagram of a crack and contour of path-independent of  $J$ -integral

Similar to the strain energy density suggested by [19], for an elastic power law hardening material with strain gradient effects, we take the strain energy density as in the following form

$$W = \frac{n}{n+1} \sigma_0 (\varepsilon_e^2 + l^2 \chi_e^2)^{(n+1)/2n} + \frac{1}{2} K \varepsilon_m^2 + \frac{1}{2} K_1 l^2 \chi_m^2 \quad (21)$$

where  $n$  is the hardening exponent,  $\sigma_0$  a measure of the tensile yield stress,  $K = E/(3-6\nu)$  the bulk modulus,  $K_1$  the bend-torsion bulk modulus,  $E$  and  $\nu$  are Young's modulus and Poisson's ratio, respectively.

The constitutive relation can be obtained from (7) and (21)

$$\varepsilon_{ij} = \frac{3}{2} \left( \frac{\Sigma_e}{\sigma_0} \right)^{n-1} \frac{s_{ij}}{\sigma_0} + \frac{1}{9K} \sigma_{kk} \delta_{ij} \quad (22)$$

$$\chi_{ij} = \frac{3}{2} \left( \frac{\Sigma_e}{\sigma_0} \right)^{n-1} \frac{m'_{ij}}{\sigma_0 l^2} + \frac{1}{9K_1 l^2} m_{kk} \delta_{ij} \quad k = r, \theta, z \quad (23)$$

Following [24,25], the asymptotic stress and couple stress fields near a crack tip can be written as

$$\sigma_{ij}(r, \theta) = \sigma_{ij}^{(0)}(\theta) r^p + O(r^p) \quad (24)$$

$$m_{z\alpha}(r, \theta) = m_{z\alpha}^{(0)}(\theta) r^p + O(r^p) \quad \alpha = r, \theta \quad (25)$$

where the power  $p$  and angular functions  $\sigma_{ij}^{(0)}(\theta)$  and  $m_{z\alpha}^{(0)}(\theta)$  are to be determined. In the following,  $\sigma_{ij}^{(0)}(\theta)$ ,  $m_{z\alpha}^{(0)}(\theta)$  are conveniently written as  $\sigma_{ij}^{(0)}$ ,  $m_{z\alpha}^{(0)}$ , respectively.

Similar to the classical HRR field<sup>[24,25]</sup>, it has been shown for a power law hardening solid that the path independence of  $J$ -integral can be expressed as

$$J = \int_{\Gamma} (W n_1 - T_i u_{i,1} - q_i \omega_{i,1}) dS = \int_{\Gamma} (W n_1 - n_j \sigma_{ji} u_{i,1} - n_j m_{ji} \omega_{i,1}) dS \quad (26)$$

then, we can obtain

$$p = \frac{-1}{n+1} \quad (27)$$

The corresponding strains and curvature tensors can be obtained via constitutive relations (22) and (23)

$$\varepsilon_{ij} = \frac{3}{2\sigma_0^n} [\Sigma_e^{(0)}(\theta)]^{n-1} s_{ij}^{(0)}(\theta) r^{np} \quad (28)$$

$$\chi_{ij} = \frac{3}{2\sigma_0^n l^2} [\Sigma_e^{(0)}(\theta)]^{n-1} m'_{ij}{}^{(0)}(\theta) r^{np} \quad (29)$$

where  $s_{ij}^{(0)}(\theta)$ ,  $m'_{ij}{}^{(0)}(\theta)$ ,  $\Sigma_e^{(0)}(\theta)$  are angular functions for  $s_{ij}$ ,  $m'_{ij}$ ,  $\Sigma_e$ , respectively. Since there are only the shear components of couple stress in the present paper, i.e.  $m_{zr}^{(0)}$  and  $m_{z\theta}^{(0)}$ , the deviatoric parts of couple stress are denoted as  $m_{z\alpha}^{(0)}$ , ( $\alpha = r, \theta$ ) and in the following, the 1st order derivative of couple stress is denoted as  $m'_{z\alpha}{}^{(0)}$ .  $\Sigma_e^{(0)}(\theta)$  is written as  $\Sigma_e^{(0)}$ .

### 3.1 Equilibrium Equations

Substituting Eqs.(24) and (25) into Eqs.(15) and (16), we obtain the equilibrium equations

$$\sigma_{r\theta}^{(0)} = -(p+1)\sigma_{rr}^{(0)} + \sigma_{\theta\theta}^{(0)} \quad (30)$$

$$\sigma_{\theta\theta}^{(0)} = -(p+2)\sigma_{r\theta}^{(0)} \quad (31)$$

$$m_{z\theta}^{(0)} = -(p+1)m_{zr}^{(0)} \quad (32)$$

### 3.2 Plane Stress Compatibility Equations

The generalized effective stress  $\Sigma_e$  for plane stress case can be written as

$$\Sigma_e^2 = \sigma_e^2 + l^{-2}m_e^2 = \sigma_{rr}^2 + \sigma_{\theta\theta}^2 - \sigma_{rr}\sigma_{\theta\theta} + 3\sigma_{r\theta}^2 + \frac{3}{2l^2}(m_{zr}^2 + m_{z\theta}^2) \quad (33)$$

When the effective couple stresses is very small, i.e.  $l^{-1}m_e \ll \sigma_e$ ,  $\Sigma_e$  becomes the same as the Von Mises stress  $\sigma_e$  and the constitutive law degenerates to the classical  $J_2$ -deformation theory.

Substituting Eqs.(24) and (25) into Eq.(33), we obtain

$$\Sigma_e^{(0)}(\theta) = \left[ (\sigma_{rr}^{(0)})^2 + (\sigma_{\theta\theta}^{(0)})^2 - \sigma_{rr}^{(0)}\sigma_{\theta\theta}^{(0)} + 3(\sigma_{r\theta}^{(0)})^2 + \frac{3}{2l^2}(m_{zr}^{(0)}m_{zr}^{(0)} + m_{z\theta}^{(0)}m_{z\theta}^{(0)}) \right]^{1/2} \quad (34)$$

Substitution of Eqs.(28) and (29) into compatibility Eqs.(19) and (20) gives the following compatibility equations

$$\begin{aligned} & (n-1)(n-2)(2\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})(\Sigma_e^{(0)})' + 2(n-1)\Sigma_e^{(0)}(2\sigma_{rr}^{(0)'} - \sigma_{\theta\theta}^{(0)'})\Sigma_e^{(0)'} + \\ & (\Sigma_e^{(0)})^2 \cdot (2\sigma_{rr}^{(0)''} - \sigma_{\theta\theta}^{(0)''}) + (n-1)\Sigma_e^{(0)}(2\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})\Sigma_e^{(0)''} - \\ & 6(np+1)[(n-1)\Sigma_e^{(0)}\sigma_{r\theta}^{(0)}\Sigma_e^{(0)'} + (\Sigma_e^{(0)})^2\sigma_{r\theta}^{(0)'}] - \\ & np(\Sigma_e^{(0)})^2(2\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)}) + np(np+1)(\Sigma_e^{(0)})^2(2\sigma_{\theta\theta}^{(0)} - \sigma_{rr}^{(0)}) = 0 \end{aligned} \quad (35)$$

$$(n-1)m_{zr}^{(0)}\Sigma_e^{(0)'} + \Sigma_e^{(0)}m_{zr}^{(0)'} - (1+np)\Sigma_e^{(0)}m_{z\theta}^{(0)} = 0 \quad (36)$$

where

$$\Sigma_e^{(0)'} = \frac{1}{2\Sigma_e^{(0)}} \left[ (2\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})\sigma_{rr}^{(0)'} + (2\sigma_{\theta\theta}^{(0)} - \sigma_{rr}^{(0)})\sigma_{\theta\theta}^{(0)'} + 6\sigma_{r\theta}^{(0)}\sigma_{r\theta}^{(0)'} + \frac{3}{l^2}(m_{zr}^{(0)}m_{zr}^{(0)'} + m_{z\theta}^{(0)}m_{z\theta}^{(0)'}) \right] \quad (37)$$

$$\begin{aligned} \Sigma_e^{(0)''} = & -\frac{1}{4(\Sigma_e^{(0)})^3} \left[ 2\sigma_{rr}^{(0)}\sigma_{rr}^{(0)'} + 2\sigma_{\theta\theta}^{(0)}\sigma_{\theta\theta}^{(0)'} - \sigma_{\theta\theta}^{(0)}\sigma_{rr}^{(0)'} - \sigma_{rr}^{(0)}\sigma_{\theta\theta}^{(0)'} + 6\sigma_{r\theta}^{(0)}\sigma_{r\theta}^{(0)'} + \right. \\ & \left. \frac{3}{l^2}(m_{zr}^{(0)}m_{zr}^{(0)'} + m_{z\theta}^{(0)}m_{z\theta}^{(0)'}) \right]^2 + \frac{1}{2\Sigma_e^{(0)}} \left\{ 2(\sigma_{rr}^{(0)'})^2 + 2(\sigma_{\theta\theta}^{(0)'})^2 + \right. \\ & (2\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})\sigma_{rr}^{(0)''} + (2\sigma_{\theta\theta}^{(0)} - \sigma_{rr}^{(0)})\sigma_{\theta\theta}^{(0)''} - 2\sigma_{rr}^{(0)'}\sigma_{\theta\theta}^{(0)'} + 6(\sigma_{r\theta}^{(0)'})^2 + \\ & \left. 6\sigma_{r\theta}^{(0)}\sigma_{r\theta}^{(0)''} + \frac{3}{l^2}[(m_{zr}^{(0)'})^2 + m_{zr}^{(0)}m_{zr}^{(0)''} + (m_{z\theta}^{(0)'})^2 + m_{z\theta}^{(0)}m_{z\theta}^{(0)''}] \right\} \end{aligned} \quad (38)$$

### 3.3 Plane Strain Compatibility Equations

The generalized effective stress  $\Sigma_e$  for plane strain case can be written as

$$\Sigma_e^{(0)}(\theta) = \left\{ \left[ \frac{3}{4}(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})^2 + 3(\sigma_{r\theta}^{(0)})^2 \right] + \frac{3}{2l^2}(m_{zr}^{(0)}m_{zr}^{(0)} + m_{z\theta}^{(0)}m_{z\theta}^{(0)}) \right\}^{1/2} \quad (39)$$

Substituting Eqs.(28) and (29) into the compatibility Eqs.(19) and (20) and combining them with Eq.(39), we obtain

$$\begin{aligned} & (n-1)(n-2)(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})\Sigma_e^{(0)'} + 2(n-1)\Sigma_e^{(0)}(\sigma_{rr}^{(0)'} - \sigma_{\theta\theta}^{(0)'})\Sigma_e^{(0)'} + \\ & (\Sigma_e^{(0)})^2 \cdot (\sigma_{rr}^{(0)''} - \sigma_{\theta\theta}^{(0)'}) + (n-1)\Sigma_e^{(0)}(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})\Sigma_e^{(0)''} - \\ & 4(np+1)[(n-1)\Sigma_e^{(0)}\sigma_{r\theta}^{(0)}\Sigma_e^{(0)'} + (\Sigma_e^{(0)})^2\sigma_{r\theta}^{(0)'}] - \\ & np(np+2)(\Sigma_e^{(0)})^2(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)}) = 0 \end{aligned} \quad (40)$$

$$(n-1)m_{zr}^{(0)}\Sigma_e^{(0)'} + \Sigma_e^{(0)}m_{zr}^{(0)'} - (1+np)\Sigma_e^{(0)}m_{z\theta}^{(0)} = 0 \quad (41)$$

where

$$\begin{aligned} \Sigma_e^{(0)'} &= \frac{1}{2\Sigma_e^{(0)}} \left[ \frac{3}{2}(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})(\sigma_{rr}^{(0)'} - \sigma_{\theta\theta}^{(0)'}) + 6\sigma_{r\theta}^{(0)}\sigma_{r\theta}^{(0)'} + \right. \\ & \left. \frac{3}{l^2}(m_{zr}^{(0)}m_{zr}^{(0)'} + m_{z\theta}^{(0)}m_{z\theta}^{(0)'}) \right] \end{aligned} \quad (42)$$

$$\begin{aligned} \Sigma_e^{(0)''} &= -\frac{1}{4(\Sigma_e^{(0)})^3} \left[ \frac{3}{2}(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})(\sigma_{rr}^{(0)'} - \sigma_{\theta\theta}^{(0)'}) + 6\sigma_{r\theta}^{(0)}\sigma_{r\theta}^{(0)'} + \right. \\ & \left. \frac{3}{l^2}(m_{zr}^{(0)}m_{zr}^{(0)'} + m_{z\theta}^{(0)}m_{z\theta}^{(0)'}) \right]^2 + \frac{1}{2\Sigma_e^{(0)}} \left\{ \frac{3}{2}(\sigma_{rr}^{(0)'} - \sigma_{\theta\theta}^{(0)'})^2 + \right. \\ & \left. \frac{3}{2}(\sigma_{rr}^{(0)} - \sigma_{\theta\theta}^{(0)})(\sigma_{rr}^{(0)''} - \sigma_{\theta\theta}^{(0)'}) + 6(\sigma_{r\theta}^{(0)'})^2 + 6\sigma_{r\theta}^{(0)}\sigma_{r\theta}^{(0)''} + \right. \\ & \left. \frac{3}{l^2}[(m_{zr}^{(0)'})^2 + m_{zr}^{(0)}m_{zr}^{(0)''} + (m_{z\theta}^{(0)'})^2 + m_{z\theta}^{(0)}m_{z\theta}^{(0)''}] \right\} \end{aligned} \quad (43)$$

The asymptotic equations for the plane stress crack tip field are Eqs.(30)~(32) and Eqs.(35),(36). The asymptotic equations for the plane strain crack tip field are Eqs. (30)~(32) and Eqs.(40),(41).

From above, we can find that the stresses and the couple stresses are coupled together through  $\Sigma_e^{(0)}$  in the asymptotic equations.

## 4 SOLUTIONS TO MODE I PLANE STRESS ASYMPTOTIC FIELDS

The traction free conditions on the crack faces are

$$\sigma_{r\theta}^{(0)}(\pm\pi) = \sigma_{\theta\theta}^{(0)}(\pm\pi) = m_{z\theta}^{(0)}(\pm\pi) = 0 \quad (44)$$

The symmetric condition for mode I crack tip field gives

$$\sigma_{r\theta}^{(0)}(0) = m_{zr}^{(0)}(0) = 0 \quad (45)$$

and due to the symmetric conditions, at  $\theta = 0$ , we have

$$\sigma_{rr}^{(0)'}(0) = 0 \quad \sigma_{\theta\theta}^{(0)'}(0) = 0 \quad (46)$$

At  $\theta = 0$ , substituting Eqs.(45), (46) into (37), we obtain

$$\Sigma_e^{(0)'}(\theta) = 0 \quad \text{at } \theta = 0 \quad (47)$$

Substituting Eq.(47) into (36), the following condition is obtained

$$m_{zr}^{(0)'}(0) = (1 + np)m_{z\theta}^{(0)}(0) \quad (48)$$

The values of  $\sigma_{rr}^{(0)}$ ,  $\sigma_{\theta\theta}^{(0)}$ ,  $m_{z\theta}^{(0)}$  at  $\theta = 0$  are unknown. Shooting method should be used to solve Eqs.(30)~(32) and Eqs.(35), (36). The unknown initial values need to be guessed to meet the crack tip face traction free conditions in Eq.(44). Since all governing equations and boundary conditions are homogeneous, we impose the normalization condition  $\sqrt{\sigma_{rr}^{(0)2} + \sigma_{\theta\theta}^{(0)2} + l^{-2}m_{z\theta}^{(0)2}}|_{\theta=0} = \sigma_0 l^{1/(n+1)}$ , where  $\sigma_0 l^{1/(n+1)}$  appears only to balance the dimension. This normalization condition can be written as

$$\begin{aligned} \frac{l^{-1}m_{z\theta}^{(0)}|_{\theta=0}}{\sigma_0 l^{1/(n+1)}} &= \cos \phi & \frac{\sigma_{rr}^{(0)}|_{\theta=0}}{\sigma_0 l^{1/(n+1)}} &= \sin \phi \cos \psi & \frac{\sigma_{\theta\theta}^{(0)}|_{\theta=0}}{\sigma_0 l^{1/(n+1)}} &= \sin \phi \sin \psi \\ 0 \leq \phi \leq \pi/2 & & 0 \leq \psi \leq 2\pi & & & \end{aligned} \quad (49)$$

For each given  $\phi$  and  $\psi$ , the Runge-Kutta method is used to integrate from  $\theta = 0$  to the crack face  $\theta = \pi$  with initial conditions given. When the numerically calculated angular distributions of stresses and couple stresses meet the traction-free conditions Eq.(44), a near tip asymptotic field is obtained. The success in choosing two parameters  $\phi$  and  $\psi$  to meet three boundary conditions in (44) simultaneously indicates that the power of stress and couple stress singularity in Eq.(27) is correct.

The entire range of  $\phi$  and  $\psi$  is discretized into  $90 \times 360$  grids, i.e., one degree per increment for  $\phi$  and  $\psi$ . The above mentioned shooting method is applied over all grid points. Two solutions are obtained for mode I, giving  $\phi = \pi/2$  and  $\phi = 0$ , respectively. One solution corresponds to a stress-dominated near tip field ( $\phi = \pi/2$ , couple stresses vanishing and the stress dominated field is the same as HRR field), while the other gives a couple stress dominated near tip field ( $\phi = 0$ , stresses vanishing and the couple stress dominated field is the same as that obtained in [19] for mode I crack tip field). The combined measure of effective stress  $\Sigma_e$  becomes the same as the Von Mises stress  $\sigma_e$  and effective couple stress  $m_e$ , respectively. However, the stress field and the couple stress field can not exist simultaneously near a crack tip in a material with the hardening law.

The angular distributions of normalized stress and couple stress are shown in Fig.2 and Fig.3 for the hardening exponent  $n = 10$ . It is observed for the stress dominated field, the stresses are consistent with the conventional HRR field, but different from the corresponding counterparts in [19]. The couple stress dominated field is the same as that in [19] for mode I crack tip field.



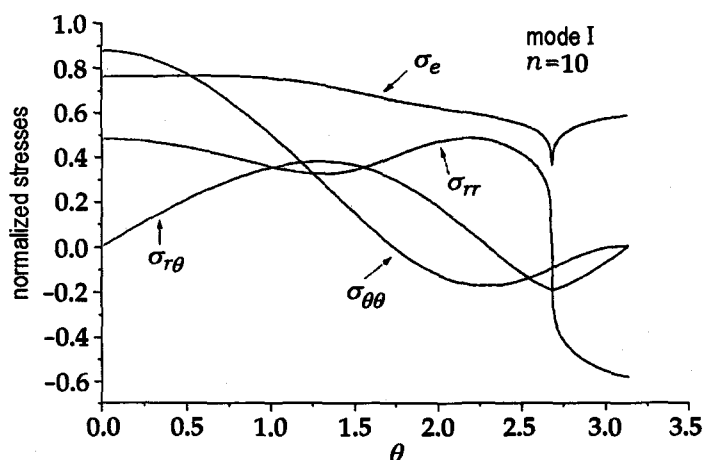


Fig.2 The angular distributions of the normalized stresses for mode I plane stress case

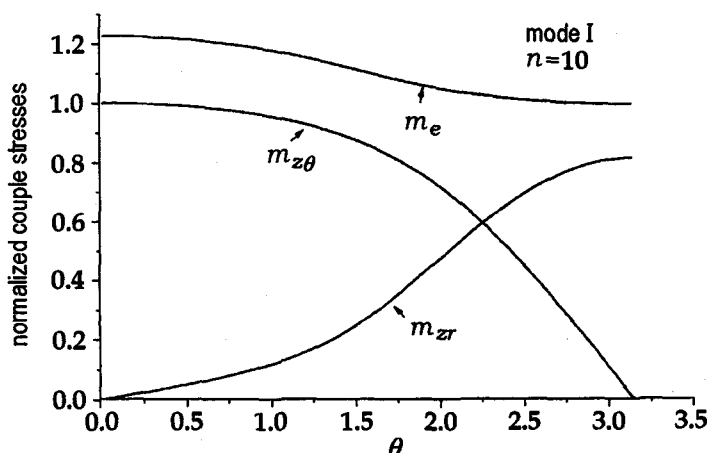


Fig.3 The angular distributions of the normalized couple stresses for mode I plane stress case

## 5 SOLUTIONS TO MODE II ASYMPTOTIC FIELDS

### 5.1 Plane Stress Case

For mode II crack tip field, the traction free conditions are the same as Eq.(44). The anti-symmetric conditions at  $\theta = 0$  are

$$\sigma_{rr}^{(0)}(0) = \sigma_{\theta\theta}^{(0)}(0) = m_{z\theta}^{(0)}(0) = 0 \quad m_{zr}^{(0)'}(0) = 0 \quad (50)$$

Only  $\sigma_{rr}^{(0)'}(0)$ ,  $\sigma_{r\theta}^{(0)}(0)$  and  $m_{zr}^{(0)}(0)$  are unknown. The shooting method is also used to solve the ordinary differential equations (30)~(32) and (35), (36). Without losing generality, the overall effective stress at  $\theta = 0$  can be assumed to be 1

$$\Sigma_e^{(0)}(0) = \left[ 3(\sigma_{r\theta}^{(0)})^2 + \frac{3}{2l^2}(m_{zr}^{(0)})^2 \right]^{1/2} \Big|_{\theta=0} = 1 \quad (51)$$

then at  $\theta = 0$ , we can write

$$\sigma_{r\theta}^{(0)}(0) = -\frac{\cos \varphi}{\sqrt{3}} \quad m_{zr}^{(0)}(0) = \frac{\sqrt{2}l \sin \varphi}{\sqrt{3}} \quad 0 \leq \varphi \leq 2\pi \quad (52)$$

The shooting method is used to search  $\varphi$  and  $\sigma_{rr}^{(0)'}(0)$  in order to satisfy the traction free condition (44) on the crack face. Only one solution is obtained for mode II plane stress case. The solution corresponds to  $\varphi = 0$  and gives a stress-dominated near tip field (couple stresses vanishing), i.e.

$$\sigma_{ij} = A\sigma_{ij}^{(0)}(\theta)r^{-1/(n+1)} \quad m_{z\alpha} = 0(r^{-1/(n+1)}) \quad (\alpha = r, \theta) \quad (53)$$

This agrees with the plane-strain analysis in [19] that couple stresses in mode II are less singular than stresses. Therefore, the combined measure of effective stress  $\Sigma_e$  degenerates to the Von Mises  $\sigma_e$  and the stress asymptotic field near the crack tip is the same as HRR solution. The corresponding angular distributions of stress components for mode II plane stress are shown in Fig.4 with the hardening exponent  $n = 10$ . Neither the normal stress nor the shear stress increases, which indicates that the rotational strain gradients hardly have any effect on the plane stress crack tip fields. The crack tip field is irrotational, which is the same conclusion as obtained by [19].

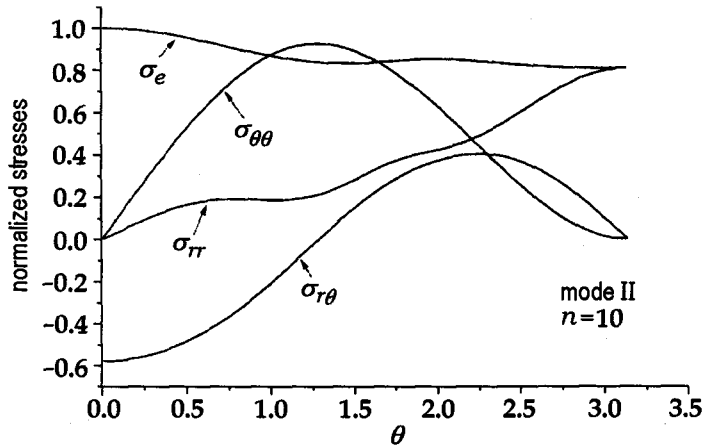


Fig.4 The angular distributions of the normalized stresses for mode II plane stress case

## 5.2 Plane Strain Case

For the plane strain case of mode II crack tip asymptotic field, the traction free conditions are the same as Eq.(44) and the initial values at  $\theta = 0$  are the same as Eq.(50) and the others are as follows

$$m_{zr}^{(0)'}(0) = 0 \quad \sigma_{\theta\theta}^{(0)'}(0) = -(p+2)\sigma_{rr}^{(0)}(0) \quad m_{z\theta}^{(0)}(0) = -(p+1)m_{zr}^{(0)}(0) \quad (54)$$

$$\Sigma_e^{(0)'}(0) = 0 \quad (55)$$

The shooting method is also used to solve the ordinary differential equations (30)~(32) and (40), (41). Without losing generality, at  $\theta = 0$ , we assume  $\Sigma_e^{(0)}(0) = \sigma_0 l^{1/(n+1)}$ , then

$$\frac{\sigma_{r\theta}^{(0)}|_{\theta=0}}{\sigma_0 l^{1/(n+1)}} = \frac{\cos \varphi}{\sqrt{3}} \quad \frac{l^{-1}m_{zr}^{(0)}|_{\theta=0}}{\sigma_0 l^{1/(n+1)}} = \frac{\sqrt{2}}{\sqrt{3}} \sin \varphi \quad 0 \leq \varphi \leq 2\pi \quad (56)$$

Now we can adjust the value of  $\varphi$  and  $\sigma_{rr}'(0)$  to meet the boundary conditions at  $\theta = \pi$ . Also only one solution is obtained for mode II plane strain case, corresponding to  $\varphi = 0$  and

this solution gives a stress-dominated near tip field (couple stresses vanishing). Furthermore the stress asymptotic field is the same as the classical mode II plane strain HRR solution.

The corresponding angular distributions of stress components for plane strain mode II asymptotic field are shown in Fig.5 with the hardening exponent  $n = 10$ .

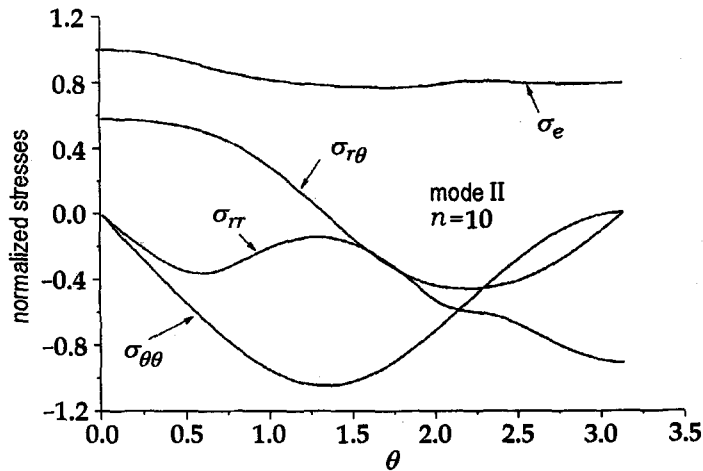


Fig.5 The angular distributions of the normalized stresses for mode II plane strain case

## 6 DISCUSSIONS

The near tip field for a power law hardening material with strain gradient effects consists of a stress field and a couple stress field. For mode I crack tip asymptotic field, the stress asymptotic field and the couple stress asymptotic field can not exist simultaneously. While the stress asymptotic field is dominated, it is consistent with the conventional HRR field. While the couple stress asymptotic field is dominated, it is consistent with the counterpart obtained in [19]. For both plane stress and plane strain mode II crack tip asymptotic field, the stress asymptotic field is dominated and consistent with the conventional plane stress and plane strain mode II HRR field, respectively. The couple stress asymptotic field is less singular than the stress asymptotic field.

The near tip stress asymptotic field of mode I and mode II obtained by means of the new strain gradient deformation theory is different from that obtained in [19] since the stress dominated asymptotic field (the couple asymptotic stress is less singular and hardly has contribution to the strain energy) obtained in [19] is different from the conventional HRR stress field.

Due to stress singularity, large strain gradients exist near the tip of a crack. We want to find the increase of stress components near the crack tip through analysis. Now, the increase, however, is not observed. There are two reasons: (1) the strain gradient plasticity theory used in the present study is based only on the rotational gradient of the deformation and the stretching gradient is not included in the present deformation theory, but the stretching gradient is dominated near the crack tip. (2) The stress asymptotic field can not exist simultaneously with the couple stress asymptotic field and the near-tip field is irrotational, thus the rotational strain gradient has hardly any influence on the stress asymptotic field.

Further work will be done using the new strain gradient deformation theory, in which the stretching gradient effect is considered.

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