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CRACK PROBLEMS OF PIEZOELECTRIC MATERIALS*

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ABSTRACT This paper presents an analysis of crack problems in homogeneous piezoelectrics or on the interfaces between two dissimilar piezoelectric materials based on the continuity of normal electric displacement and electric potential across the crack faces. The explicit analytic solutions are obtained for a single crack in piezoelectrics or on the interfaces of piezoelectric bimaterials. A class of boundary problems involving many cracks is also solved. For homogeneous materials it is found that the normal electric displacement D_2 induced by the crack is constant along the crack faces which depends only on the applied remote stress field. Within the crack slit, the electric fields induced by the crack are also constant and not affected by the applied electric field. For the bimaterials with real $\boldsymbol{H}$, the normal electric displacement D_2 is constant along the crack faces and electric field E_2 has the singularity ahead of the crack tip and a jump across the interface.

KEY WORDS piezoelectric materials, crack, interface

I. INTRODUCTION

Piezoelectric materials have been extensively used in smart devices as sensors and actuators. The combined mechanical and electrical loads give rise to sufficiently high stresses in these devices which reuslt in catastrophic failure. The fracture mechanics of piezoelectric materials have attracted many theoretical workers (Parton[1], Pohanka and Smith[2], Deeg[3], Pak and Herrmann[4], McMeeking[5], Pak[6], Sosa[7], Suo et al.[8], Suo[9], Zhang and Hack[10], Yang and Suo[11], Dunn[12], Wang[13], Wang and Huang[14], Zhang and Tong[15], Yu and Qin[16], Gao, Zhang and Tong[17] among others). There are two different aspects for the electric boundary condition along the crack faces. Parton[1] pointed out that the medium within the crack slit is electrically permeable, hence the electric potential and the normal electric displacement should be continuous across the crack slit:

$$D_n^+ = D_n^-, \qquad \varphi^+ = \varphi^- \tag{1}$$

This aspect has been supported by Zhang and Hack[10], Dunn[12] and Hao and Shen[18]. Pak[6], Sosa[7], Suo et al.[9] proposed another set of electric boundary condition on the crack faces:

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$$D_n^+ = D_n^- = 0 \tag{2}$$

This aspect ignores the electric field within the crack slit. The boundary condition (2) has been extensively used in literature. McMeeking[5] pointed out that the condition (2) may not be appropriate for a crack slit. The crack can be considered as an elliptic flaw with low permittivity. The parameter $\varepsilon_f a/\varepsilon_m b$ proposed by McMeeking[5] is an important quantity, which controls the electric boundary condition on the flaw surface. The boundary condition (2) is suitable for the case in which this parameter is small, otherwise one should use the boundary condition (1).

This paper presents an analysis for the crack problems in piezoelectric materials, or on interfaces between two dissimilar piezoelectric materials based on the boundary condition (1). The paper is organized as follows: Section 2 introduces the basic formulae of elastic piezoelectric materials. Key results for single crack in homogeneous piezoelectrics or on the interfaces of piezoelectric bimaterials are presented in section 3. A class of boundary problems including a set of cracks on the interface is also solved.

Ⅱ. BASIC FORMULAE

The constitutive equations for piezoelectric materials are

$$\begin{cases}\sigma_{ij} = c_{ijkl}\gamma_{kl} - e_{kij}E_k \\ D_i = e_{ikl}\gamma_{kl} + \varepsilon_{ik}E_k\end{cases} \tag{3}$$

where σ_{ij}, γ_{ij} are the stress tensor and strain tensor respectively. D_i, E_i are the electric displacement and electric field. $c_{ijkl}, e_{ikl}, \varepsilon_{ij}$ are the elastic, piezoelectric and dielectric constants. The mechanical and electrical equilibrium equations take the form

$$\begin{cases}\sigma_{ij,i} = 0 \\ D_{i,i} = 0\end{cases} \tag{4}$$

Strain γ_{ij} and electric field E_i can be expressed as

$$\begin{cases}\gamma_{ij} = \dfrac{1}{2}(u_{i,j} + u_{j,i}) \\ E_i = -\varphi_{,i}\end{cases} \tag{5}$$

where u_i is displacement, φ is the electric potential.

Substituting Eqs. (3), (5) into Eq. (4), one obtains

$$\begin{cases}(c_{ijkl}u_k + e_{lij}\varphi)_{,li} = 0 \\ (e_{ikl}u_k - \varepsilon_{il}\varphi)_{,li} = 0\end{cases} \tag{6}$$

Consider a two-dimensional problem, the general solution to which can be expressed by complex potential[8]

$$\{u_i, \varphi\} = \boldsymbol{a}f(\zeta_1 x + \zeta_2 y) \tag{7}$$

where $\boldsymbol{a}$ is a four-component column. $\zeta_1 = 1$, $\zeta_2 = p$. Substituting Eq. (5) into Eq. (6), it follows that

$$\begin{cases}(c_{\alpha jk\beta}a_k + e_{\alpha\beta j}a_4)\zeta_\alpha\zeta_\beta = 0 \\ (e_{\alpha\beta k}a_k - \varepsilon_{\alpha\beta}a_4)\zeta_\alpha\zeta_\beta = 0\end{cases} \tag{8}$$

where α, β take on the values 1 and 2. j, k have the values 1, 2 and 3. This is an eigen-problem for column $\boldsymbol{a}$. Suo et al. [8] have shown that the eigenequation of this problem has eight complex roots in forming four conjugate pairs. Let p_1, p_2, p_3, p_4 be the four roots each with a positive imaginary part. We have

$$\{u_i, \varphi\} = 2\mathrm{Re}\sum_{k=1}^{4} \boldsymbol{a}_k f_k(z_k) \tag{9}$$

where $z_k = x + p_k y$.

For the stress and electric displacement components, one obtains

$$\begin{cases} \{\sigma_{2j}, D_2\} = 2\mathrm{Re}\sum_{k=1}^{4} \boldsymbol{b}_k f_k(z_k) \\ \{\sigma_{1j}, D_1\} = -2\mathrm{Re}\sum_{k=1}^{4} \boldsymbol{b}_k p_k f_k(z_k) \end{cases} \tag{10}$$

Column $\boldsymbol{b}$ has the components

$$\begin{cases} b_j = (c_{2jk1}a_k + e_{12j}a_4) + (c_{2jk2}a_k + e_{22j}a_4)p \\ b_4 = (c_{21k}a_k - \varepsilon_{21}a_4) + (e_{22k}a_k - \varepsilon_{22}a_4)p \end{cases} \tag{11}$$

Introduce 4×4 matrices $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4], \qquad \boldsymbol{B} = [\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3, \boldsymbol{b}_4] \tag{12}$$

Define a function vector $\boldsymbol{f}(z)$ of a single variable

$$\boldsymbol{f}(z) = \{f_1(z),\ f_2(z),\ f_3(z),\ f_4(z)\} \tag{13}$$

Then the generalized displacement and the traction on the real axis can be expressed as

$$\boldsymbol{U}(x) = \{u_j, \varphi\} = \boldsymbol{A}\boldsymbol{f}(x) + \overline{\boldsymbol{A}}\,\overline{\boldsymbol{f}(x)} \tag{14}$$

$$\boldsymbol{t}(x) = \{\sigma_{2j}, D_2\} = \boldsymbol{B}\boldsymbol{f}'(x) + \overline{\boldsymbol{B}}\,\overline{\boldsymbol{f}'(x)} \tag{15}$$

Ⅲ. CRACK PROBLEMS

3.1 Mechanics Analysis

A finite crack of length $2a$ lies on the interface between two half spaces, the material of the upper half space is denoted by 1 and that of the lower by 2 as shown in Fig. 1. Both the materials 1 and 2 are piezoelectric materials. The crack segment is denoted by L. The continuity of generalized traction $\boldsymbol{t}(x)$ across the x-axis requires that

$$\boldsymbol{t}^+(x) = \boldsymbol{t}^-(x), \quad -\infty < x < +\infty \tag{16}$$

The above equation can be rewritten as

$$\boldsymbol{B}_1\boldsymbol{f}_1'^+(x) + \overline{\boldsymbol{B}}_1\overline{\boldsymbol{f}_1'}^-(x) = \boldsymbol{B}_2\boldsymbol{f}_2'^+(x) + \overline{\boldsymbol{B}}_2\overline{\boldsymbol{f}_2'}^+(x), \quad -\infty < x < +\infty \tag{17}$$

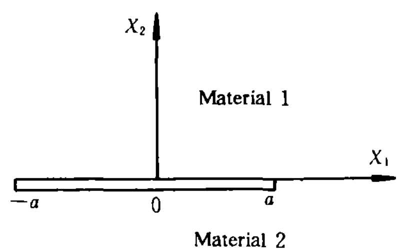

Fig. 1 A finite crack of length $2a$ on interface of piezoelectric bimaterials.

Hereafter the subscripts 1 and 2 attached to matrices and vectors indicate these quantities belong to material 1 and material 2, respectively.

The infinite plate is subjected to remote stresses $\sigma_{11}^{\infty}, \sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}$ and electric displacements D_1^{∞}, D_2^{∞} loadings. The solution is composed of two solutions: one is the homogeneous solution produced by the applied remote loading and the other is the inhomogeneous solution induced by the cracks. Our attentions will be focused on the inhomogeneous solution.

According to the work by Suo et al.[8], from Eq. (17) one obtains

$$\begin{cases} \boldsymbol{B}_1 \boldsymbol{f}_1'^{+}(z) = \overline{\boldsymbol{B}}_2 \overline{\boldsymbol{f}_2'}^{-}(z), & y > 0 \\ \boldsymbol{B}_2 \boldsymbol{f}_2'^{+}(z) = \overline{\boldsymbol{B}}_1 \overline{\boldsymbol{f}_1'}^{-}(z), & y < 0 \end{cases} \tag{18}$$

Furthermore one can obtain

$$i\boldsymbol{\delta}'(x) = \boldsymbol{H}\boldsymbol{B}_1 \boldsymbol{f}_1'^{+}(x) - \overline{\boldsymbol{H}}\boldsymbol{B}_2 \boldsymbol{f}_2'^{-}(x) \tag{19}$$

where $\boldsymbol{\delta}(x)$ is the generalized displacement, $\boldsymbol{\delta}(x) = \{u_i^+ - u_i^-, \varphi^+ - \varphi^-\}$.

A bimaterial matrix is defined as

$$\boldsymbol{H} = \boldsymbol{Y}_1 + \overline{\boldsymbol{Y}}_2 \tag{20}$$

where

$$\boldsymbol{Y}_1 = i\boldsymbol{A}_1 \boldsymbol{B}_1^{-1}, \qquad \boldsymbol{Y}_2 = i\boldsymbol{A}_2 \boldsymbol{B}_2^{-1} \tag{21}$$

When the crack is in a homogeneous material, $\boldsymbol{H}$ is a real matrix,

$$\boldsymbol{H} = 2\mathrm{Re}\boldsymbol{Y} \tag{22}$$

When the crack lies on the interface of bimaterials, $\boldsymbol{H}$ is generally spoken of as a complex matrix. If the material pair has certain symmetry, $\boldsymbol{H}$ can be real. This paper discusses only the case of real $\boldsymbol{H}$.

From the continuity of the generalized displacement across the bonded interface, one can conclude that the following function $\boldsymbol{h}(z)$ is analytic in the entire plane beside the cut L.

$$\boldsymbol{h}(z) = \begin{cases} \boldsymbol{B}_1 \boldsymbol{f}_1'(z), & y > 0 \\ \boldsymbol{B}_2 \boldsymbol{f}_2'(z), & y < 0 \end{cases} \tag{23}$$

On the crack faces, the generalized tractions $\boldsymbol{t}_1(x)$, $\boldsymbol{t}_2(x)$ satisfy the following boundary condition

$$\boldsymbol{t}_1(x) = \boldsymbol{t}_2(x) = -\boldsymbol{T}, \quad x \in L \tag{24}$$

where

$$\boldsymbol{T} = \{\sigma_{21}^{\infty}, \sigma_{22}^{\infty}, \sigma_{23}^{\infty}, d\} \tag{25}$$

where d is an unknown parameter. Substituting (17), (23) into (24), one obtains

$$\boldsymbol{h}^{+}(x) + \boldsymbol{h}^{-}(x) = -\boldsymbol{T}, \qquad x \in L \tag{26}$$

This is a typical Hilbert problem, the solution of which is not unique. The auxiliary conditions are needed in order to obtain a unique solution. They are: $\boldsymbol{h}^{*}(z) \to 0$ as $z \to \infty$; $\boldsymbol{h}(z)$ has square root singularity at the crack tip and the net Burgers vector for the finite crack vanishes. The last auxiliary condition can be represented as

$$\int_{-a}^{a} [\boldsymbol{h}^{+}(x) - \boldsymbol{h}^{-}(x)] \mathrm{d}x = \boldsymbol{0} \tag{27}$$

The solution, which satisfies the auxiliary conditions, is

$$\boldsymbol{h}(z) = \boldsymbol{T} f_0(z), \; f_0(z) = \frac{1}{2}\left[\frac{z}{\sqrt{z^2 - a^2}} - 1\right] \tag{28}$$

From Eq. (28) it follows that

$$\boldsymbol{t}(x) = 2\boldsymbol{T} f_0(x), \qquad |x| > a \tag{29}$$

$$\boldsymbol{\delta}(x) = \sqrt{a^2 - x^2}\boldsymbol{HT}, \qquad |x| < a \tag{30}$$

The continuity of the electric potential across the crack implies that

$$\delta_4(x) = \varphi^+ - \varphi^- = 0, \ x \in L \tag{31}$$

From Eqs. (30) and (31) it follows that

$$H_{41}\sigma_{21}^{\infty} + H_{42}\sigma_{22}^{\infty} + H_{43}\sigma_{23}^{\infty} + H_{44}d = 0 \tag{32}$$

Thus we have

$$d = -(H_{41}\sigma_{21}^{\infty} + H_{42}\sigma_{22}^{\infty} + H_{43}\sigma_{23}^{\infty})/H_{44} \tag{33}$$

Equation (33) shows that the parameter d depends only on the applied stress field. This implies that the inhomogeneous fields induced by the crack depend only on the applied remote stress fields and the applied remote electric fields have no effect on the inhomogeneous fields.

After determing the parameter d from Eq. (33), the column $\boldsymbol{T}$ is completely determined. From Eqs. (27) and (23), one can obtain $\boldsymbol{h}(z)$, $f_1(z)$ and $f_2(z)$. Instead of z, using z_k for each component function in Eqs. (9) and (10), one can obtain all the solutions. The intensity factors can be obtained from Eq. (29)

$$\begin{aligned} K_{\mathrm{I}} &= \sqrt{\pi a}\sigma_{21}^{\infty}, \quad K_{\mathrm{I}} = \sqrt{\pi a}\sigma_{22}^{\infty} \\ K_{\mathrm{I}} &= \sqrt{\pi a}\sigma_{23}^{\infty}, \quad K_{\mathrm{N}} = \sqrt{\pi a}d \end{aligned} \tag{34}$$

It should be emphasized that the electric displacement D_2 still has the singularity at the crack tip, despite the continuity of the electric displacement D_2 and the electric potential across the crack.

3.2 Crack Problem of Homogeneous Material

We only discuss the inhomogeneous fields induced by the crack.

3.2.1 Crack front coincides with the poling axis

The constitutive equations have transverse symmetry,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\ c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & & c_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & (c_{11} - c_{12})/2 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{23} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{bmatrix} - \begin{bmatrix} 0 & 0 & e_{31} \\ 0 & 0 & e_{31} \\ 0 & 0 & e_{33} \\ 0 & e_{15} & 0 \\ e_{15} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{35}$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & e_{15} & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \\ e_{31} & e_{31} & e_{33} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{23} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{36}$$

The (x, y) is an isotropic plane. The stress and the displacement can be expressed by complex potentials as

$$\begin{aligned} \sigma_y - i\tau_{yx} &= \Phi(z) + \overline{\Omega(z)} + (z - \bar{z})\overline{\Phi'(z)} \\ 2\mu(u + iv)'_x &= \kappa\Phi(z) - \Omega(z) - (z - \bar{z})\Phi'(z) \end{aligned} \tag{37}$$

On the real axis, we have

$$
\begin{aligned}
\sigma_y - i\tau_{yx} &= \Phi(x) + \overline{\Omega(x)} \\
2\mu(u+iv)'_{,x} &= \kappa\Phi(x) - \Omega(x)
\end{aligned}
\tag{38}
$$

Equation (38) can be represented as

$$\sigma_y = \mathrm{Re}\{\Phi(x) + \Omega(x)\}, \qquad \tau_{yx} = \mathrm{Re}\{i\Phi(x) - i\Omega(x)\} \tag{39}$$

$$2\mu u_{,x} = \mathrm{Re}\{\kappa\Phi(x) - \Omega(x)\}, \qquad 2\mu v_{,x} = \mathrm{Re}\{-i\kappa\Phi(x) - i\Omega(x)\} \tag{40}$$

On the other hand, the complex representations for u_z and φ given by Pak[6] are

$$w = u_z = 2\mathrm{Re}[f_3(z)] \tag{41}$$

$$\varphi = 2\mathrm{Re}[f_4(z)] \tag{42}$$

The functions $\Phi(z)$, $\Omega(z)$ can be considered as $f_1(z)$, $f_2(z)$. The four eigenvalues are $p_1 = p_2 = p_3 = p_4 = i$. We have

$$
\boldsymbol{A} = \begin{bmatrix} \dfrac{\kappa}{4\mu} & -\dfrac{1}{4\mu} & 0 & 0 \\ -i\dfrac{\kappa}{4\mu} & -\dfrac{i}{4\mu} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad
\boldsymbol{B} = \begin{bmatrix} \dfrac{i}{2} & -\dfrac{i}{2} & 0 & 0 \\ \dfrac{1}{2} & \dfrac{1}{2} & 0 & 0 \\ 0 & 0 & ic_{44} & ie_{15} \\ 0 & 0 & ie_{15} & -i\varepsilon_{11} \end{bmatrix}
\tag{43,44}
$$

$$
\boldsymbol{Y} = \begin{bmatrix} \dfrac{1-\nu}{\mu} & \left(\dfrac{1-2\nu}{2\mu}\right) & 0 & 0 \\ -i\left(\dfrac{1-2\nu}{2\mu}\right) & \dfrac{1-\nu}{\mu} & 0 & 0 \\ 0 & 0 & \dfrac{\varepsilon_{11}}{k} & \dfrac{e_{15}}{k} \\ 0 & 0 & \dfrac{e_{15}}{k} & -\dfrac{c_{44}}{k} \end{bmatrix}
\tag{45}
$$

where $k = e_{15}^2 + c_{44}\varepsilon_{11}$.

Thus

$$H_{41} = H_{42} = 0, \quad H_{43} = \frac{2}{k}e_{15}, \quad H_{44} = -\frac{2}{k}c_{44} \tag{46}$$

$$d = \frac{e_1}{c_{44}}\sigma_{23}^{\infty} \tag{47}$$

$$K_D = K_N = K_{\mathrm{I}}e_{15}/c_{44} \tag{48}$$

3.2.2 Crack perpendicular to the poling axis

The constitutive equations are

$$
\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix}
=
\begin{bmatrix} c_{11} & c_{13} & c_{12} & 0 & 0 & 0 \\ c_{13} & c_{33} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{13} & c_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & (c_{11}-c_{12})/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{44} \end{bmatrix}
\begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{23} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{bmatrix}
-
\begin{bmatrix} 0 & e_{31} & 0 \\ 0 & e_{33} & 0 \\ 0 & e_{31} & 0 \\ 0 & 0 & e_{15} \\ 0 & 0 & 0 \\ e_{15} & 0 & 0 \end{bmatrix}
\begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}
\tag{49}
$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & e_{15} \\ e_{31} & e_{33} & e_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & e_{15} & 0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_{11} \\ \gamma_{22} \\ \gamma_{33} \\ 2\gamma_{23} \\ 2\gamma_{31} \\ 2\gamma_{12} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{33} & 0 \\ 0 & 0 & \varepsilon_{11} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} \tag{50}$$

The eigen problem (8) becomes

$$\begin{bmatrix} c_{11} + c_{44}p^2 & (c_{13} + c_{44})p & 0 & (e_{31} + e_{15})p \\ (c_{13} + c_{44})p & c_{44} + c_{33}p^2 & 0 & e_{15} + e_{33}p^2 \\ 0 & 0 & (c_{11} - c_{12})/2 + c_{44}p^2 & 0 \\ (e_{31} + e_{15})p & e_{15} + e_{33}p^2 & 0 & -\varepsilon_{11} - \varepsilon_{33}p^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = 0 \tag{51}$$

The eigen equation is

$$(c_{11} - c_{12} + 2c_{44}p^2)(c_0p^6 + c_1p^4 + c_2p^2 + c_3) = 0 \tag{52}$$

where

$$\begin{aligned} c_0 &= c_{44}e_{33}^2 + c_{33}c_{44}\varepsilon_{33} \\ c_1 &= c_{33}(e_{31} + e_{15})^2 - 2c_{13}e_{33}(e_{31} + e_{15}) + c_{11}e_{33}^2 - 2c_{44}e_{31}e_{33} \\ &\quad + c_{33}c_{44}\varepsilon_{11} + (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})\varepsilon_{33} \\ c_2 &= 2c_{11}e_{15}e_{33} - 2c_{13}e_{15}(e_{15} + e_{31}) + c_{44}e_{31}^2 + (c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44})\varepsilon_{11} + c_{11}c_{44}\varepsilon_{33} \\ c_3 &= c_{11}e_{15}^2 + c_{11}c_{44}\varepsilon_{11} \end{aligned} \tag{53}$$

Referring to the work by Sosa and Park[15] one can obtain $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{H}$. The matrix $\boldsymbol{H}$ has the following structure

$$\boldsymbol{H} = \begin{bmatrix} \frac{2}{C_L} & 0 & 0 & 0 \\ 0 & \frac{2}{C_T} & 0 & \frac{2}{e} \\ 0 & 0 & \frac{2}{C_A} & 0 \\ 0 & \frac{2}{e} & 0 & -\frac{2}{\varepsilon} \end{bmatrix} \tag{54}$$

The parameters $C_L, C_T, C_A, \varepsilon$ and e can be obtained by numerical calculation. From Eq. (54), it follows that

$$H_{41} = H_{43} = 0, \quad H_{42} = 2/e, \quad H_{44} = -2/\varepsilon \tag{55}$$

Substituting them into Eqs. (33) and (34), we have

$$d = \frac{\varepsilon}{e}\sigma_{22}^{\infty} \tag{56}$$

$$K_D = K_N = \frac{\varepsilon}{e}K_{\mathrm{I}} \tag{57}$$

3.3 Interface Crack Problem

3.3.1 Crack front coincides with the poling axis

For this case, we find

$$H_{41}=H_{42}=0,\quad H_{43}=\left(\frac{e_{15}}{k}\right)^{(1)}+\left(\frac{e_{15}}{k}\right)^{(2)},\quad -H_{44}=\left(\frac{c_{44}}{k}\right)^{(1)}+\left(\frac{c_{44}}{k}\right)^{(2)} \tag{58}$$

The superscripts (1) and (2) indicate the quantities belong to material 1 and 2 respectively and we have

$$d=\frac{e_{15}^{(1)}[e_{15}^2+c_{44}\varepsilon_{11}]^{(2)}+e_{15}^{(2)}[e_{15}^2+c_{44}\varepsilon_{11}]^{(1)}}{c_{44}^{(1)}k^{(2)}+c_{44}^{(2)}k^{(1)}}\sigma_{23}^{\infty} \tag{59}$$

$$K_D=K_{\mathrm{N}}=K_{\mathrm{III}}=\frac{e_{15}^{(1)}k^{(2)}+e_{15}^{(2)}k^{(1)}}{c_{44}^{(1)}k^{(2)}+c_{44}^{(2)}k^{(1)}} \tag{60}$$

Now we discuss the electric field ahead of the crack tip. From the constitutive equation one obtains

$$E_2=(c_{44}D_2-e_{15}\sigma_{23})/k \tag{61}$$

In front of the crack tip, we have

$$\begin{aligned}E_2^+(x)&=2(c_{44}^{(1)}d-e_{15}^{(1)}\sigma_{23}^{\infty})f_0(x)/k^{(1)}\\&=2\frac{c_{44}^{(1)}e_{15}^{(2)}-c_{44}^{(2)}e_{15}^{(1)}}{c_{44}^{(1)}k^{(2)}+c_{44}^{(2)}k^{(1)}}\sigma_{23}^{\infty}f_0(x),\qquad |x|>a\end{aligned} \tag{62}$$

Hence the electric field E_2 has the singularity at the crack tip. The electric intensity factor is given by

$$K_E^+=\rho_E K_{\mathrm{III}} \tag{63}$$

$$\rho_E=\frac{c_{44}^{(1)}e_{15}^{(2)}-c_{44}^{(2)}e_{15}^{(1)}}{c_{44}^{(1)}k^{(2)}+c_{44}^{(2)}k^{(1)}} \tag{64}$$

Similarly, in material 2, one can find that

$$E_2^-(x)=-2\rho_E\sigma_{23}^{\infty}f_0(x)=-E_2^+(x) \tag{65}$$

This implies that there is a jump of E_2 across the interface. On the other hand one can easily prove that the E_1 has no singularity ahead of the crack tip and continuously across the interface.

Now we look at the electric field near the crack faces. Equations (23) and (27) give

$$\begin{aligned}\boldsymbol{f}_1'(z)&=\boldsymbol{B}_1^{-1}\boldsymbol{h}(z)=\boldsymbol{\Lambda}_1\boldsymbol{T}f_0(z),\qquad y>0\\ \boldsymbol{f}_2'(z)&=\boldsymbol{B}_2^{-1}\boldsymbol{h}(z)=\boldsymbol{\Lambda}_2\boldsymbol{T}f_0(z),\qquad y<0\end{aligned} \tag{66}$$

where $\boldsymbol{\Lambda}=\boldsymbol{B}^{-1}$. Thus the component functions are

$$\begin{aligned}f_{1j}(z)&=c_j^{(1)}f_0(z),\qquad y>0\\ f_{2j}(z)&=c_j^{(2)}f_0(z),\qquad y<0\end{aligned} \tag{67}$$

where

$$c_j^{(\alpha)}=\sum_{k=1}^{4}\Lambda_{jk}^{(\alpha)}T_k,\qquad j=1,2,3;\quad \alpha=1,2 \tag{68}$$

Equation (14) leads to

$$\begin{aligned}\boldsymbol{U}'(x)&=\boldsymbol{A}\boldsymbol{f}'(x)+\overline{\boldsymbol{A}}\,\overline{\boldsymbol{f}'(x)}\\ \boldsymbol{U}'^+(x)&=\boldsymbol{A}_1\boldsymbol{B}_1^{-1}\boldsymbol{T}f_0^+(x)+\overline{\boldsymbol{A}}_1\overline{\boldsymbol{B}_1^{-1}\boldsymbol{T}f_0^+(x)}\end{aligned} \tag{69}$$

Along the upper face of crack

$$f_0^+(x)=-\frac{1}{2}\left[1+i\frac{x}{\sqrt{a^2-x^2}}\right] \tag{70}$$

$$\boldsymbol{U}'^+(x)=2\mathrm{Re}\left\{\frac{1}{i}\boldsymbol{Y}_1\boldsymbol{T}f_0^+(x)\right\}=-\frac{x}{\sqrt{a^2-x^2}}\mathrm{Re}\{\boldsymbol{Y}_1\}\boldsymbol{T}-\mathrm{Im}\{\boldsymbol{Y}_1\}\boldsymbol{T} \tag{71}$$

$$E_1^+ = -\frac{\partial\varphi^+}{\partial x} = \frac{x}{\sqrt{a^2 - x^2}}\{e_{15}^{(1)}\sigma_{23}^\infty - c_{44}^{(1)}d\}/k^{(1)} = -\rho_E\sigma_{23}^\infty \frac{x}{\sqrt{a^2 - x^2}} \tag{72}$$

Similarly, one can confirm that

$$E_1^- = -\rho_E\sigma_{23}^\infty \frac{x}{\sqrt{a^2 - x^2}} = E_1^+, \qquad |x| < a \tag{73}$$

$$E_2^+ = -E_2^- = (c_{44}^{(1)}d - e_{15}^{(1)}\sigma_{23}^\infty)/k^{(1)} = \rho_E\sigma_{23}^\infty, \quad |x| < a \tag{74}$$

It can be seen that the E_1 is continuously across the crack and has the singularity, but the E_2 has a jump without singularity. Within the crack, we have

$$E_1 = -\rho_E\sigma_{23}^\infty \frac{x}{\sqrt{a^2 - x^2}}, \qquad E_2 = d/\varepsilon_0 \tag{75}$$

3.3.2 Crack perpendicular to the poling axis

One can easily find that

$$H_{41} = H_{43} = 0, \quad H_{42} = \frac{1}{c_e^{(1)}} + \frac{1}{c_e^{(2)}},$$

$$H_{44} = -\left(\frac{1}{\varepsilon^{(1)}} + \frac{1}{\varepsilon^{(2)}}\right), \quad d = \frac{\varepsilon^{(1)}\varepsilon^{(2)}}{e^{(1)}e^{(2)}} \cdot \frac{e^{(1)} + e^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}}\sigma_{22}^\infty \tag{76}$$

The parameter d depends only on the σ_{22}^∞.

$$K_D = K_N = \rho_D K_{\mathrm{I}} \tag{77}$$

$$\rho_D = \frac{e^{(1)}\varepsilon^{(2)}}{e^{(1)}e^{(2)}} \cdot \frac{e^{(1)} + e^{(2)}}{\varepsilon^{(1)} + \varepsilon^{(2)}} \tag{78}$$

From Eq. (9), it follows that

$$\varphi = 2\mathrm{Re}\Big[\sum_{k=1}^{4} A_{4k}f_k(z_k)\Big] \tag{79}$$

$$E_1 = -2\mathrm{Re}\Big[\sum_{k=1}^{4} A_{4k}f_k(z_k)\Big] \tag{80}$$

$$E_2 = -2\mathrm{Re}\Big[\sum_{k=1}^{4} A_{4k}p_k f_k(z_k)\Big] \tag{81}$$

Substituting (67) into the above equations one finds that

$$E_1 = -2\mathrm{Re}\Big[\sum_{\substack{k=1\\j=1}}^{4} A_{4k}B_{kj}^{-1}T_j f_0(z_k)\Big] \tag{82}$$

$$E_2 = -2\mathrm{Re}\Big[\sum_{\substack{k=1\\j=1}}^{4} A_{4k}B_{kj}^{-1}T_j p_k f_0(z_k)\Big] \tag{83}$$

On the real axis,

$$E_1 = -2\mathrm{Im}\Big\{\sum_{j=1}^{4} Y_{4j}T_j f_0\Big\} \tag{84}$$

$$E_2 = -2\mathrm{Re}\Big\{\sum_{j=1}^{4} A_{4k}p_k c_k f_0\Big\} \tag{85}$$

One can easily prove that $E_1^+ = E_1^-$, and generally speak $E_2^+ \neq E_2^-$. The E_1, E_2 have singularity at the crack tip.

Along the crack faces, $E_1^+ = E_1^-$, and generally speak $E_2^+ \neq E_2^-$. The E_1^+, E_1^-, E_2^+, E_2^- have singularity.

The governing equations (26), (27) and (31) are also valid for a class of boundary probems including many cracks on the interface. Let L denote a set of cracks on the interface. Now Eq. (7) should be met for each crack. The governing Eqs. (26),(27) and (31) do not involve any material parameters. Hence the solution can be directly obtained, provided the corresponding solutions for a homogeneous isotropic elastic solid are known.

Ⅳ. CONCLUSION AND DISCUSSION

The interface crack problems of piezoelectric bimaterials are analysed based on the boundary condition of the continuity of normal electric displacement and electric potential across the crack faces. The explicit analytic solutions are obtained for a single crack in piezoelectric or on the interface of piezoelectric bimaterials. A class of boundary problems involving many cracks is also solved.

It is revealed that the inhomogeneous mechanical and electric fields induced by the crack only depend on the applied remote stress fields. The applied electric fields have no effect on the inhomogeneous fields.

For homogeneous materials it is found that the normal electric displacement D_2 induced by the crack is constant along the crack faces which depends only on the applied remote stress field. Within the crack slit, the total electric fields are also constant, which are much larger than the remote electric fields in the matrix.

For the bimaterials with real $\boldsymbol{H}$, the normal electric displacement D_2 is also constant along the crack faces and has the singularity ahead of the crack tip despite the continuity of normal electric displacement and electric potential across the crack faces. The electric field E_2 has the singularity ahead of the crack tip and a jump across the interface.

The theoretical study[20] shows that the applied electric field should inhibit crack propagation irrespective of its sign. Park and Sun[2] measured the failure stresses of cracks perpendicular to the poling axis in compact tension and three-point bending specimens made of PZT-4 ceramics and found that the failure stresses decreased with an increase of the positive applied electric field(in the same direction as the poling axis) and increased with an increase of the applied negative electric field. They argued that the fracture process of ceramic materials is a pure mechanical process and should be controlled only by the mechanical part of the energy release rate.

The energy release rate can be calculated by the following integral

$$G = \frac{1}{2l}\int_0^l \boldsymbol{t}^{\mathrm{T}}(l-r)\boldsymbol{\delta}(r)\mathrm{d}r \tag{86}$$

where l is an arbitrary small length.

For homogeneous materials or bimaterials with real $\boldsymbol{H}$, substituting Eqs. (29) and (30) into (86), we obtain

$$G = \frac{\pi a}{4}\boldsymbol{T}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{T} = \frac{1}{4}\boldsymbol{k}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{k} \tag{87}$$

From Eqs. (31) and (32), one can conclude that the energy release rate G depends only on the mechanical forces and the applied electric field has no contribution to the energy release rate, which seems to support the argument given by Park and Sun[2].

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