Shaohua Chen^{a,b}, Huajian Gao^a^aMax Planck Institute for Metals Research, Stuttgart, Germany^bOn leave from LNM, Institute of Mechanics, Chinese Academy of Sciences, Beijing, China

Generalized Maugis–Dugdale model of an elastic cylinder in non-slipping adhesive contact with a stretched substrate

Dedicated to Professor Dr. Fritz Aldinger on the occasion of his 65th birthday

We have recently developed a generalized JKR model for non-slipping adhesive contact between an elastic cylinder and a stretched substrate where both tangential and normal tractions are transmitted across the contact interface. Here we extend this model to a generalized Maugis–Dugdale model by adopting a Dugdale-type adhesive interaction law to eliminate the stress singularity near the edge of the contact zone. The non-slipping Maugis–Dugdale model is expected to have a broader range of validity in comparison with the non-slipping JKR model. The solution shares a number of common features with experimentally observed behaviors of cell reorientation on a cyclically stretched substrate.

Keywords: Generalized MD model; Generalized JKR model; Adhesive contact; Cell adhesion

1. Introduction

The classical contact theory of Hertz assumes no adhesive interactions between elastic solids. This aspect of contact mechanics was successfully modeled by the JKR theory of Johnson, Kendall, and Roberts [1] in which the stress field near the contact edge is governed by a crack-like singularity and the size of the contact region is determined according to the thermodynamic equilibrium between elastic energy and surface energy (i. e., the Griffith criterion). The JKR theory is especially applicable for contact between relatively large and soft elastic bodies. For contact between small and rigid particles, it is more appropriate to use the theory by Derjaguin, Muller, and Toporov (DMT) [2], which models the adhesive interactions outside the contact area predicted by the Hertz model, but assumes the same Hertz stress distribution inside the contact area. Maugis [3] proposed a unified theory linking JKR and DMT models by extending the Dugdale model [4] of a plastically yielded crack to adhesive contact mechanics. Numerous extensions of these adhesive contact theories have been developed between elastic and viscoelastic bodies in the past few decades [5–20].

Recently, the theories of adhesive contact mechanics have been used to understand biological adhesion mechanisms such as the hierarchical adhesion structures on the feet

of Geckos [21–24], cells on stretched substrates [25] and cell–cell adhesion [26]. The set of experiments [27–31] which have motivated the present work have shown that cells cultured on a cyclically stretched substrate tend to reorient away from the stretch direction. An important observation [28] is that cells do not respond to small stretch amplitudes, suggesting that there exists a lower threshold of stretch amplitude around 1–2 % for cell reorientation. Above this threshold, cells become increasingly responsive to substrate deformation. Beyond a second threshold of stretch amplitude, around 5–6 %, almost all cells reorient away from the stretch direction [30]. These experimental observations indicate that cells respond sensitively to substrate deformation. There is already evidence that contact mechanics theories may help explain at least some aspects of these phenomena. Chu et al. [26] showed that the JKR theory can help predicting the adhesion energy between two S180 cells. Wang [31] has shown that the final aligning angle of cells on cyclically stretched substrates can be calculated based on the principle of minimum strain energy. A generalized JKR model for non-slipping adhesive contact between an elastic cylinder and a stretched substrate [25] is found to share a number of common features with experimental observations of cell reorientation on stretched substrates. In this paper, we extend the non-slipping JKR model to a generalized Maugis–Dugdale model by adopting a Dugdale-type adhesive interaction law to eliminate the unphysical singular stress field near the edge of the contact zone. The resulting non-slipping Maugis–Dugdale model is expected to have a broader range of applicability than the non-slipping JKR model discussed in [25].

The present study builds upon a class of two-dimensional plane strain adhesive contact theories, which have been investigated for a number of interesting problems in the past [32–36]. Our model can be viewed as an extension of the previous work by Baney and Hui [10], who developed the Maugis–Dugdale model for adhesion between elastic cylinders when the shear traction in the contact area is negligible. Recently, Sari et al. [35] have also considered rolling and sliding motion of a cylinder on substrate subjected to combined normal and tangential forces. The focus of our study is on how substrate strain affects the contact area between an elastic cylinder and an elastic half-space.

2. Model

Consider the plane strain problem of an elastic cylinder in non-slipping adhesive contact with an elastic half-space, as shown in Fig. 1. The contact region is assumed to be perfectly bonded with both normal and shear tractions along the contact interface. A Cartesian coordinate system (x, y) is placed at the center of the contact region (Fig. 1). Of interest here is how the size of the contact zone changes when the substrate is stretched in the x direction. The problem has been studied recently with a generalized JKR model [25]. The purpose of the present paper is to extend the generalized JKR model to a generalized Maugis–Dugdale model by using appropriately selected interaction laws to eliminate stress singularities near the edge of the contact zone, which is assumed to be small compared to the radius of the cylinder such that the local deformation of the cylinder can be approximated by that of an elastic half-space.

The defined non-slipping adhesive contact problem is analyzed as a superposition of two sub-problems. The first sub-problem (Fig. 1a) has both normal and shear tractions, $\sigma_{yy}(x)$ and $\sigma_{xy}(x)$ singular over the contact region $|x| \leq a$. The second sub-problem (Fig. 1b) involves an external crack $|x| > a$ subjected to a constant shear traction $q(x) = -\tau_0$ over a region $a < |x| \leq b_s$ and a constant normal traction $p(x) = -\sigma_0$ over $a < |x| \leq b_l$.

We choose our reference frame such that the center point of the contact zone ($x = 0, y = 0$) is fixed in space. The displacement continuity conditions within the contact region are

$$u_1 - u_2 = \varepsilon x, \quad |x| \leq a \quad (1)$$

$$w_1 + w_2 = -\frac{x^2}{2R}, \quad |x| \leq a \quad (2)$$

where w and u are the normal and tangential displacements along the contact interface; subscripts “1, 2” denote the upper and lower materials, respectively. The y coordinate of each material is assumed to point into the material.

Outside the contact region, one can define the tangential and normal separations between the deformed surfaces as

$$\Delta_x = -\varepsilon x + u_1 - u_2 \quad (3)$$

$$\Delta_y = \frac{x^2}{2R} + w_1 + w_2 \quad (4)$$

The displacements will be evaluated by superposing the two sub-problems, denoted by subscripts “A” and “C”, as

$$u_1 - u_2 = (u_1 - u_2)_A + (u_1 - u_2)_C = \delta_{xA} + \delta_{xC} \quad (5)$$

$$w_1 + w_2 = (w_1 + w_2)_A + (w_1 + w_2)_C = \delta_{yA} + \delta_{yC} \quad (6)$$

Here, subscript “A” denotes the singular adhesive contact problem of Fig. 1a and “C” denotes the Dugdale interaction problem of Fig. 1b.

3. General solution

The two sub-problems, i.e. the singular adhesive contact problem depicted in Fig. 1a and the Dugdale interaction

problem depicted in Fig. 1b, will be separately analyzed in this section and then superposed to construct the general solutions to the non-slipping Maugis–Dugdale model.

3.1. Solution to the singular adhesive contact subproblem

The singular adhesive contact problem shown in Fig. 1a is quite similar to the classical JKR model except that the contact region is assumed to be perfectly bonded and the substrate is subjected to an external strain ε . The solution to this problem can be found in our previous paper [25] and are summarized below for the convenience of the reader.

The tangential and normal stresses in the contact area are

$$\begin{cases} \sigma_{xy}(x) = 2\text{Re}\{G(x)\} + \frac{E^*x\beta}{2R(1-\beta^2)}, & |x| < a \\ \sigma_{yy}(x) = -2\text{Im}\{G(x)\} + \frac{E^*\beta\varepsilon}{2(1-\beta^2)} \end{cases} \quad (7)$$

Fig. 1. (a) Singular adhesive contact problem of an elastic cylinder in non-slipping adhesive contact with a stretched substrate. The constants (E_1, ν_1) , (E_2, ν_2) denote Young's modulus and Poisson's ratios of the cylinder and substrate, respectively. A Cartesian coordinate system (x, y) is attached to the center of the contact interface. (b) The Dugdale interaction problem of an external interface crack subjected to shear and normal crack face tractions $q(x)$ and $p(x)$ in the regions $b_s \geq |x| \geq a$ and $b_l \geq |x| \geq a$, respectively.

where

$$G(x) = -\frac{E^*(a+x)^{-\bar{r}}(a-x)^{-r}}{4\pi(1-\beta^2)} \times \left[\int_{-a}^a \left(\varepsilon + \frac{it}{R} \right) \frac{(a+t)^{\bar{r}}(a-t)^r}{t-x} dt \right] \quad (8)$$

$$r = \frac{1}{2} + i\kappa, \quad \kappa = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta} \quad (9)$$

R is the radius of the cylinder as shown in Fig. 1a and β is Dundurs' parameter [37],

$$\beta = \frac{1}{2} \left\{ \frac{(1-2\nu_1)/\mu_1 - (1-2\nu_2)/\mu_2}{(1-\nu_1)/\mu_1 + (1-\nu_2)/\mu_2} \right\} \quad (10)$$

μ_i, ν_i ($i = 1, 2$) being the shear moduli and Poisson ratios of the upper and lower materials, respectively; E^* is the reduced modulus related to Young's moduli E_i ($i = 1, 2$) and Poisson's ratios ν_i ($i = 1, 2$) of the two solids as

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2} \quad (11)$$

Equation (7) leads to a complex stress intensity factor,

$$K_A = -\sqrt{2\pi} \lim_{x \rightarrow a} (a-x)^r [\sigma_{yy}(x) + i\sigma_{xy}(x)] \\ = \frac{-iE^*(2a)^{-\bar{r}}}{\sqrt{2\pi}(1-\beta^2)} \int_{-a}^a \frac{\left(\varepsilon + \frac{it}{R} \right) (a+t)^{\bar{r}}(a-t)^r}{a-t} dt \quad (12)$$

The relative tangential and normal displacements at $x = -b_s$ and $x = -b_t$ are found to be

$$\delta_{xA}(-b_s) = -\varepsilon a - \frac{2}{\pi E^*} \int_{-a}^a \left[2\text{Re}\{G(\xi)\} + \frac{E^*\xi\beta}{2R(1-\beta^2)} \right] \\ \times \ln \frac{\xi + b_s}{\xi + a} d\xi \quad (13)$$

$$\delta_{yA}(-b_t) = \frac{-a^2}{2R} - \frac{2}{\pi E^*} \int_{-a}^a \left[-2\text{Im}\{G(\xi)\} + \frac{E^*\beta\varepsilon}{2(1-\beta^2)} \right] \\ \times \ln \frac{\xi + b_t}{\xi + a} d\xi \quad (14)$$

3.2. Solution to the Dugdale interaction subproblem

The Dugdale interaction problem is described in Fig. 1b as an external interfacial crack $|x| > a$ subjected to a constant shear traction $q(x) = -\tau_0$ over a region $a < |x| \leq b_s$ and a constant normal traction $p(x) = -\sigma_0$ over $a < |x| \leq b_t$.

The relation between stresses and displacements along the crack line can be formulated as (see e. g., [25])

$$\frac{A}{\pi} \int_{-a}^a \frac{1}{\xi - x} \begin{pmatrix} \sigma_{xy}(\xi) \\ \sigma_{yy}(\xi) \end{pmatrix} d\xi + B \begin{pmatrix} \sigma_{xy}(x) \\ \sigma_{yy}(x) \end{pmatrix} = \frac{A}{\pi} \begin{pmatrix} f_x(x) \\ f_y(x) \end{pmatrix} \\ |x| < a \quad (15)$$

where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \quad B = \begin{bmatrix} 0 & -\beta \\ \beta & 0 \end{bmatrix} \quad (16)$$

and

$$f_x(x) = \int_{-b_s}^{-a} \frac{-q}{\xi - x} d\xi + \int_a^{b_s} \frac{q}{\xi - x} d\xi \\ f_y(x) = \int_{-b_t}^{-a} \frac{-p}{\xi - x} d\xi + \int_a^{b_t} \frac{-p}{\xi - x} d\xi \quad (17)$$

Solving Eq. (15) yields the tangential and normal stresses at the interface (see Appendix),

$$\sigma_{xy}(x) = \frac{-\beta f_y(x)}{\pi(1-\beta^2)} + 2\text{Re}\{I(x)\} \\ + \frac{4}{\sqrt{1-\beta^2}} \text{Im}\{(a+x)^{-\bar{r}}(a-x)^{-r} c_1\}, \quad |x| \leq a \quad (18)$$

$$\sigma_{yy}(x) = \frac{\beta f_x(x)}{\pi(1-\beta^2)} - 2\text{Im}\{I(x)\} \\ + \frac{4}{\sqrt{1-\beta^2}} \text{Re}\{(a+x)^{-\bar{r}}(a-x)^{-r} c_1\}, \quad |x| \leq a \quad (19)$$

where c_1 is given in Eq. (A.23) in the Appendix and

$$I(x) = \frac{-(a+x)^{-\bar{r}}(a-x)^{-r}}{2\pi^2(1-\beta^2)} \\ \times \int_{-a}^a \frac{(a+t)^{\bar{r}}(a-t)^r [f_x(t) - if_y(t)]}{t-x} dt \quad (20)$$

The associated complex-valued stress intensity factor

$$K_C = -\sqrt{2\pi} \lim_{x \rightarrow a} (a-x)^r [\sigma_{yy}(x) + i\sigma_{xy}(x)] \quad (21)$$

is found to be

$$K_C = \frac{i\sqrt{2\pi}(2a)^{-\bar{r}}}{\pi^2(1-\beta^2)} \int_{-a}^a \frac{(a+t)^{\bar{r}}(a-t)^r [f_x(t) - if_y(t)]}{t-a} dt \\ + \frac{2\sqrt{2\pi}(b_t-a)p(2a)^{-\bar{r}}}{\int_{-a}^a (a+\xi)^{-\bar{r}}(a-\xi)^{-r} d\xi} \quad (22)$$

The relative tangential and normal displacements along the crack surfaces are

$$\delta_{xC}(x) = \frac{2}{\pi E^*} \left[\int_a^{b_s} q \ln \frac{|\xi - x|}{|\xi + x|} d\xi \right. \\ \left. - \int_{-a}^a \sigma_{xy}(x) \ln |\xi - x| d\xi + \pi(-p\beta x + e_1) \right] \quad (23)$$

$$\delta_c(x) = \frac{2}{\pi E^*} \left[\int_a^{b_1} -p \ln|\xi^2 - x^2| d\xi - \int_{-a}^a \sigma_{yy}(x) \ln|\xi - x| d\xi + \pi(q\beta x + e_2) \right] \quad (16)$$

where e_1 and e_2 are two unknown constants to be determined by the following boundary conditions,

$$\delta_c(-a) = \delta_{yc}(-a) = 0 \quad (17)$$

Finally, the relative crack surface displacements at $x = -b_s$ and $x = -b_l$ are obtained as

$$\delta_c(-b_s) = \frac{2}{\pi E^*} \left[\pi\beta p(b_s - a) - q \int_a^{b_s} \ln \frac{b_s - \xi}{\xi - a} d\xi + q \int_a^{b_s} \ln \frac{\xi + b_s}{\xi + a} d\xi - \int_{-a}^a \sigma_{xy}(\xi) \ln \frac{\xi + b_s}{\xi + a} d\xi \right] \quad (18)$$

$$\delta_c(-b_l) = \frac{2}{\pi E^*} \left[\pi\beta q(a - b_l) - p \int_a^{b_l} \ln \frac{b_l - \xi}{\xi - a} d\xi + p \int_a^{b_l} \ln \frac{\xi + a}{\xi + b_l} d\xi - \int_{-a}^a \sigma_{yy}(\xi) \ln \frac{\xi + b_l}{\xi + a} d\xi \right] \quad (19)$$

3.3. Solution to the non-slipping Maugis–Dugdale model

The solutions to the two sub-problems can be superposed to eliminate the stress singularity in the contact zone, i.e.

$$K_A + K_C = 0 \quad (20)$$

Substituting Eqs. (12) and (22) into (28) yields the following normalized equation relating b_s and b_l to a ,

$$\int_{-1}^1 \frac{(1+t)^{\tilde{r}}(1-t)^r}{1-t} dt - \frac{a}{R} \int_{-1}^1 \frac{t(1+t)^{\tilde{r}}(1-t)^r}{1-t} dt = \frac{\tilde{\tau}_0 t}{2} \int_{-1}^1 \frac{(1+t)^{\tilde{r}}(1-t)^r}{1-t} \ln \frac{n^2 - t^2}{1 - t^2} dt - \frac{\tilde{\sigma}_0}{2} \int_{-1}^1 \frac{(1+t)^{\tilde{r}}(1-t)^r}{1-t} \ln \frac{(m-t)(1+t)}{(1-t)(m+t)} dt - \frac{\pi^2 \tilde{\sigma}_0 (1 - \beta^2)(m-1)}{\int_{-1}^1 (1+t)^{-\tilde{r}}(1-t)^{-r} dt} \quad (21)$$

where we have substituted $p = -\sigma_0$ and $q = -\tau_0$ into the Dugdale interaction problem and

$$\tilde{\sigma}_0 = \frac{4\sigma_0}{\pi E^*}, \quad \tilde{\tau}_0 = \frac{4\tau_0}{\pi E^*}, \quad n = \frac{b_s}{a}, \quad m = \frac{b_l}{a} \quad (22)$$

Inserting Eqs. (13, 14, 26, 27) into Eqs. (5, 6), then into Eqs. (3, 4) yields the relative surface displacements Δ_x and Δ_y at $x = -b_s$ and $x = -b_l$. Substituting Δ_x and Δ_y into the following energy balance criterion

$$\tau_0 \Delta_x(-b_s) + \sigma_0 \Delta_y(-b_l) = \Delta\gamma \quad (23)$$

leads to an implicit relationship between the substrate strain ε and the contact half-width a . Numerical calculations will be used to calculate this relationship in Section 6.

4. Non-oscillatory solution

The solution discussed in Section 3 exhibits an oscillatory field near the edge of the contact zone, with an oscillation index κ which is related to Dundurs' parameter β . We have previously [25] found that the parameters κ and β play insignificant roles in the non-slipping JKR model and can be assumed to be zero for most practical purposes (this fact was known for interfacial crack problems; see, e.g. [38]). Here we also confirm this for the non-slipping Maugis–Dugdale model. In this section, we consider the non-oscillatory solution of non-slipping adhesive contact between two dissimilar elastic bodies, assuming

$$\beta = 0, \quad \kappa = 0, \quad r = 1/2 \quad (24)$$

Note that the homogeneous case, i.e. when two elastic bodies are made of the same material, is a special case for which the oscillatory solution becomes exact. We will show that the non-oscillatory solution can serve as an approximate solution in general cases even when Eq. (32) is not satisfied.

4.1. The singular adhesive contact sub-problem

In the non-oscillatory case, the shear and normal stresses inside the contact region can be obtained from Eq. (7) as

$$\sigma_{xy}(x) = \frac{E^* \varepsilon x}{2\sqrt{a^2 - x^2}}, \quad |x| \leq a \quad (25)$$

$$\sigma_{yy}(x) = \frac{-E^* x^2 - a^2/2}{2R \sqrt{a^2 - x^2}}, \quad |x| \leq a \quad (26)$$

with corresponding stress intensity factors

$$K_{AI} = \frac{E^* \sqrt{\pi a^3/2}}{4R} \quad (27)$$

$$K_{AII} = \frac{-E^* \varepsilon \sqrt{\pi a}}{2} \quad (28)$$

The relative displacements at $x = -b_s$ and $x = -b_l$ can be expressed as

$$\delta_{xA}(-b_s) = \varepsilon \sqrt{b_s^2 - a^2} - \varepsilon b_s \quad (29)$$

$$\delta_{yA}(-b_l) = \frac{b_l}{2R} \sqrt{b_l^2 - a^2} - \frac{b_l^2}{2R} \quad (30)$$

4.2. The Dugdale interaction subproblem

The non-oscillatory solution to shear and normal stresses along the crack line in the Dugdale interaction subproblem can be obtained from Eqs. (18, 19) as

$$\sigma_{xy}(x) = \frac{-2q(a^2 - x^2)^{-\frac{1}{2}}x}{\pi} \ln \frac{a}{b_s + \sqrt{b_s^2 - a^2}} + \frac{q}{\pi} \left(\operatorname{arctg} \frac{a^2 + b_s x}{\sqrt{b_s^2 - a^2} \sqrt{a^2 - x^2}} - \operatorname{arctg} \frac{a^2 - b_s x}{\sqrt{b_s^2 - a^2} \sqrt{a^2 - x^2}} \right), \quad |x| \leq a \quad (39)$$

$$\sigma_{yy}(x) = \frac{-2p}{\pi} \sqrt{\frac{b_t^2 - a^2}{a^2 - x^2}} + \frac{2p}{\pi} \arctan \sqrt{\frac{b_t^2 - a^2}{a^2 - x^2}}, \quad |x| \leq a \quad (40)$$

where

$$p = -\sigma_0, \quad q = -\tau_0 \quad (41)$$

The stress intensity factors near the contact edge can be separated into a mode I and mode II components as

$$K_{CI} = \frac{-2\sigma_0 \sqrt{b_t^2 - a^2}}{\sqrt{\pi a}} \quad (42)$$

$$K_{CII} = \frac{2\tau_0 \sqrt{a}}{\sqrt{\pi}} \ln \left[\frac{b_s}{a} + \sqrt{\left(\frac{b_s}{a} \right)^2 - 1} \right] \quad (43)$$

The relative crack face displacements are

$$\delta_{xC}(-b_s) = \frac{-4\tau_0 b_s}{\pi E^*} \times \left[\sqrt{1 - \left(\frac{a}{b_s} \right)^2} \cosh^{-1} \frac{b_s}{a} - \cosh^{-1} \left(\frac{a^2 + b_s^2}{2ab_s} \right) \right] \quad (44)$$

$$\delta_{yC}(-b_t) = \frac{-4\sigma_0 b_t}{\pi E^*} \ln \frac{b_t}{a} \quad (45)$$

4.3. The non-oscillatory solution

The above solutions are superposed to eliminate the stress singularity at the edge of the contact area, i. e.

$$K_{AI} + K_{CI} = 0 \quad (46)$$

$$K_{AII} + K_{CII} = 0 \quad (47)$$

which yields the following explicit relationships among a , b_s and b_t ,

$$b_s = \frac{a(1 + \alpha)}{2\sqrt{\alpha}}, \quad \alpha = e^{\frac{E^* \pi \varepsilon \sqrt{\pi}}{2\tau_0}} \quad (48)$$

$$b_t = \sqrt{a^2 + \frac{\pi^2 E^{*2} a^4}{64 R^2 \sigma_0^2}} \quad (49)$$

Using Eqs. (3, 4, 37, 38, 44, 45) and the energy balance criterion of Eq. (31) leads to

$$\left[\frac{\tilde{\sigma}_0 m}{2} \sqrt{m^2 - 1} + \frac{R \tilde{\tau}_0 \varepsilon}{a} \sqrt{n^2 - 1} \right] \frac{a^2}{4\omega R^2} - \left\{ \tilde{\sigma}_0^2 m \ln m + \tilde{\tau}_0^2 n \left[\sqrt{1 - \frac{1}{n^2}} \cosh^{-1} n - \cosh^{-1} \left(\frac{1}{2n} + \frac{n}{2} \right) \right] \right\} \times \frac{a}{4\omega R} = 1 \quad (50)$$

where the following dimensionless parameters have been used,

$$m = \frac{b_t}{a} = \sqrt{1 + \frac{a^2}{4R^2 \tilde{\sigma}_0^2}}, \quad n = \frac{b_s}{a} = \frac{1}{2} e^{\frac{\sqrt{\pi} a}{\tilde{\tau}_0}} + \frac{1}{2} e^{-\frac{\sqrt{\pi} a}{\tilde{\tau}_0}} \quad (51)$$

$$\omega = \frac{\Delta \gamma}{\pi E^* R} \quad (52)$$

Equation (50) relates the contact half-width a to the substrate strain ε .

From the second equation in (51), we find $n = 1$, corresponding to $b_s = a$, when $\varepsilon = 0$, in which case there is no shear stress along the contact interface. In the case of $\varepsilon = 0$, Eq. (50) becomes

$$\frac{\tilde{\sigma}_0 a_0^2 m \sqrt{m^2 - 1}}{8\omega R^2} - \frac{\tilde{\sigma}_0^2 a_0 m \ln m}{4\omega R} = 1 \quad (53)$$

5. Comparison with the Baney–Hui model

The two dimensional Maugis–Dugdale problem of frictionless adhesive contact between two elastic cylinders was solved by Baney and Hui [10]. In the absence of substrate deformation and an applied force, the non-oscillatory solution of the present non-slipping contact model should reduce to that of Baney and Hui. In the non-oscillatory case, letting $\varepsilon = 0$ in the second equation of (51) shows

$$n = 1$$

in which case the shear stress vanishes inside the contact region.

Equations (38) and (45) yield the surface separation $\Delta_y(-b_t)$ for the non-oscillatory case,

$$\Delta_y(-b_t) = \frac{b_t}{2R} \sqrt{b_t^2 - a^2} - \frac{4\sigma_0 b_t}{\pi E^*} \ln \frac{b_t}{a} \quad (55)$$

where b_t is given in Eq. (49).

Substituting Eq. (55) into the energy balance criterion

$$\sigma_0 \Delta \gamma (-b_t) = \Delta \gamma \quad (56)$$

leads to

$$\frac{\sigma_0^2 a_{MD}^2 m \sqrt{m^2 - 1}}{8 \omega R^2} - \frac{\sigma_0^2 a_{MD} m \ln m}{4 \omega R} = 1 \quad (57)$$

where

$$m = \frac{b_t}{a_{MD}} \quad (58)$$

σ_0 and ω are given in Eqs. (30) and (52), respectively. The contact half-width a_{MD} can be obtained from Eq. (57).

Using the following definitions,

$$\frac{\sigma_0 a_{MD}^2}{4 \omega R^2} = \hat{\sigma}_0 \hat{a}^2, \quad \frac{\sigma_0^2 a_{MD}}{2 \omega R} = \hat{\sigma}_0^2 \hat{a} \quad (59)$$

where $\hat{a}$ and $\hat{\sigma}_0$ were defined in Eqs. (10) and (20) by Baney and Hui [10],

$$\hat{a} = \frac{a_{MD}}{2 \left(\frac{R^2 \Delta \gamma}{\pi E^*} \right)^{1/3}}, \quad \hat{\sigma}_0 = \frac{4 \sigma_0}{\left(\frac{\pi^2 E^{*2} \Delta \gamma}{R} \right)^{1/3}} \quad (60)$$

Substituting Eqs. (59) into (57) leads to

$$\frac{\hat{\sigma}_0 \hat{a}^2 m \sqrt{m^2 - 1}}{2} - \frac{\hat{\sigma}_0^2 \hat{a} m \ln m}{2} = 1 \quad (61)$$

Letting the applied force $\tilde{P}$ vanish in Eq. (22) of [10] results in

$$\hat{a}^2 = \hat{\sigma}_0 \hat{a} \sqrt{m^2 - 1} \quad (62)$$

which when substituted into Eq. (23) of [10] leads to an equation identical to Eq. (61).

6. Approximate solution for general bimaterial cases

For general bimaterial cases, the relations among b_s , b_t and a in Eq. (29) can not be explicitly expressed. Furthermore, the relation between ε and a in Eq. (31) includes singular integrals. Similar to the generalized JKR model for non-slip contact [25], we find that the Dundurs parameter β also plays a negligible role in the present model so that the non-oscillatory solution discussed in Section 4 can serve as an approximate solution for most practical applications. Comparing Eqs. (53) and (57) shows that

$$a_0 = a_{MD} \quad (63)$$

We thus propose the following approximate solution for non-slipping adhesive contact between an elastic cylinder and a substrate subjected to external strain ε

$$\left[\frac{\hat{\sigma}_0 m}{2} \sqrt{m^2 - 1} + \frac{\lambda \tilde{\tau}_0 \varepsilon}{\hat{a}} \sqrt{n^2 - 1} \right] \frac{\tilde{a}^2}{4 \omega \lambda^2} \left\{ \hat{\sigma}_0^2 m \ln m + \tilde{\tau}_0^2 n \left[\sqrt{1 - \frac{1}{n^2}} \cosh^{-1} n - \cosh^{-1} \left(\frac{1}{2n} + \frac{n}{2} \right) \right] \right\} \times \frac{\tilde{a}}{4 \omega \lambda} = 1 \quad (64)$$

where

$$\tilde{a} = \frac{a}{a_{MD}}, \quad \lambda = \frac{R}{a_0} \quad (65)$$

$$m = \sqrt{1 + \frac{\tilde{a}^2}{4 \lambda^2 \tilde{\sigma}_0^2}}, \quad n = \frac{1}{2} e^{\frac{\sqrt{n}}{\tilde{\tau}_0}} + \frac{1}{2} e^{-\frac{\sqrt{n}}{\tilde{\tau}_0}} \quad (66)$$

$$\tilde{\sigma}_0 = \frac{4 \sigma_0}{\pi E^*}, \quad \tilde{\tau}_0 = \frac{4 \tau_0}{\pi E^*}, \quad \omega = \frac{\Delta \gamma}{\pi E^* R} \quad (67)$$

$a_0 = a_{MD}$ being defined in Eqs. (53, 57).

Figure 2 plots the relationship between the normalized contact half-width a/a_0 and the substrate strain ε for a fixed λ and different values of $\tilde{\tau}_0$. The result indicates that the behavior of a/a_0 can be characterized by three distinct regimes with two threshold strain levels: (1) the contact width is hardly influenced by the applied loading when the substrate strain is below the first threshold; (2) as the substrate

Fig. 2. Variation of normalized contact half-width a/a_0 as a function of the substrate strain ε as predicted by Eq. (64) of the text for a fixed $\lambda = R/a_0$ and different values of $\tilde{\tau}_0$. Three different regimes of the contact solutions can be found. a_0 is the contact half-width in the absence of external loading ε .

Fig. 3. Variation of normalized contact half-width a/a_0 as a function of the substrate strain ε as predicted by Eq. (64) of the text for a fixed $\tilde{\tau}_0$ and different values of $\lambda = R/a_0$.

Fig. 4. The non-oscillatory solution of shear and normal tractions in the contact region. (a)–(d) Variation of normalized shear traction $\Sigma_{xy} = \sigma_{xy}/E^*$ vs. $A = x/a$ for a fixed substrate strain ε and different values of parameter $\bar{\tau}_0$. The corresponding shear traction from the generalized JKR model [25] is shown for comparison. (e)–(g) Variation of normalized normal traction $\Sigma_{yy} = \sigma_{yy}/E^*$ vs. $A = x/a$ for a fixed ratio a/R and different values of parameter $\bar{\sigma}_0$. The corresponding normal traction from the generalized JKR model [25] is shown for comparison.

strain increases to between the two threshold values, the contact width begins to decrease significantly in response to the applied loading; (3) the adhesion fails with almost no contact possible when the substrate strain exceeds the second threshold. Also note that, for a fixed λ , the larger the shear strength of the contact interface is, the larger the substrate strain is at final decohesion.

Figure 3 shows the relationship between the normalized contact half-width a/a_0 and the substrate stretch ε for a fixed $\tilde{\tau}_0$ and different values of λ . In this case, the shear strength of interface is fixed. One can see that the larger the value of λ is (i.e., the smaller the value of a_0), the smaller the substrate strain is at a given extent of decohesion.

Although the above results are derived for non-slipping adhesive contact between an elastic cylinder and a stretched substrate with Dugdale interaction zones outside the contact region, the behaviors of the contact area share a number of features with cells cultured on a cyclically stretched substrate where experiments show three characteristic regimes with two threshold stretch amplitudes [28, 30–31]. It was found that cells do not respond to stretch amplitudes smaller than 1–2 % [28]. Once this first threshold is reached, an increasing number of cells begin to respond to substrate stretch by re-orientating themselves away from the stretch direction. The cell reorientation leads to decreased contact width in the direction of stretch and increased contact width in the transverse direction. As the stretch amplitude increases beyond a second threshold level around 5–6 %, almost all cells reorient away from the stretch direction [30]. These features and the associated strain levels are not inconsistent with non-slipping adhesive contact models as discussed in [25] and this paper.

7. Shear and normal tractions inside the contact region

After the singularity near the contact edge is cancelled in the present generalized MD model, the non-oscillatory solutions for the shear and normal tractions in the contact region are

$$\sigma_{xyMD}(x) = \frac{-\tau_0}{\pi} \left(\operatorname{arccctg} \frac{a^2 + b_s x}{\sqrt{b_s^2 - a^2} \sqrt{a^2 - x^2}} - \operatorname{arccctg} \frac{a^2 - b_s x}{\sqrt{b_s^2 - a^2} \sqrt{a^2 - x^2}} \right), \quad |x| \leq a \quad (68)$$

$$\sigma_{yyMD}(x) = \frac{E^*}{2R} (a^2 - x^2)^{\frac{1}{2}} - \frac{2\sigma_0}{\pi} \arctan \sqrt{\frac{b_t^2 - a^2}{a^2 - x^2}}, \quad |x| \leq a \quad (69)$$

In the generalized JKR model, the corresponding non-oscillatory solutions are [25]

$$\sigma_{xyJKR}(x) = \frac{E^* \varepsilon x}{2\sqrt{a^2 - x^2}}, \quad |x| \leq a \quad (70)$$

$$\sigma_{yyJKR}(x) = \frac{-E^*}{2R} \frac{x^2 - a^2/2}{\sqrt{a^2 - x^2}}, \quad |x| \leq a \quad (71)$$

Using dimensionless parameters

$$A = \frac{x}{a}, \quad \Sigma_{xy} = \frac{\sigma_{xy}}{E^*}, \quad \Sigma_{yy} = \frac{\sigma_{yy}}{E^*} \quad (72)$$

Eqs. (68–71) can be rewritten as

$$\begin{cases} \Sigma_{xyMD}(A) = \frac{-\tilde{\tau}_0}{4} \times \left(\operatorname{arccctg} \frac{1 + nA}{\sqrt{n^2 - 1} \sqrt{1 - A^2}} - \operatorname{arccctg} \frac{1 - nA}{\sqrt{n^2 - 1} \sqrt{1 - A^2}} \right) & |A| < 1 \\ \Sigma_{xyMD}(A) = \frac{\pi \tilde{\tau}_0}{4} & 1 < |A| < n \end{cases} \quad (73)$$

$$\begin{cases} \Sigma_{yyMD}(A) = \frac{a}{2R} (1 - A^2)^{\frac{1}{2}} - \frac{\tilde{\sigma}_0}{2} \arctan \sqrt{\frac{m^2 - 1}{1 - A^2}} & |A| < 1 \\ \Sigma_{yyMD}(A) = -\frac{\pi \tilde{\sigma}_0}{4} & 1 < |A| < m \end{cases} \quad (74)$$

and

$$\Sigma_{xyJKR}(A) = \frac{\varepsilon A}{2\sqrt{1 - A^2}}, \quad |A| < 1 \quad (75)$$

$$\Sigma_{yyJKR}(A) = \frac{-a}{2R} \frac{A^2 - 1/2}{\sqrt{1 - A^2}}, \quad |A| < 1 \quad (76)$$

Equations (73) and (75) are used to plot Σ_{xy} vs. A for various values of ε and $\tilde{\tau}_0$, as in Figs. 4a–d, and Eqs. (74) and (76) are used to plot Σ_{yy} vs. A for various values of $\tilde{\sigma}_0$ and $\tilde{\tau}_0$, as in Figs. 4e–g. In general, the generalized JKR model gives a rather good qualitative description of both shear and normal tractions in comparison with the physically more realistic Maugis–Dugdale description.

Figures 4e–g show that the normal traction predicted by the generalized JKR model becomes a good approximation to the corresponding result from the generalized Maugis–Dugdale model as soon as $\tilde{\sigma}_0 > a/R$.

Support of this work has been provided by the Max Planck Society, the Chang Jiang Scholar Program through the Ministry of Education of China and NSFC (10202023).

Appendix: Solutions to equation (15)

It is convenient to rewrite Eq. (15) in a matrix form as

$$\frac{1}{\pi} \int_{-a}^a \frac{A}{\xi - x} t(\xi) d\xi + Bt(x) = C \quad (A.1)$$

where

$$t(\xi) = \begin{pmatrix} \sigma_{xy}(\xi) \\ \sigma_{yy}(\xi) \end{pmatrix} = \begin{pmatrix} t_1(\xi) \\ t_2(\xi) \end{pmatrix}, \quad C = \frac{1}{\pi} \begin{bmatrix} f_x(x) \\ f_y(x) \end{bmatrix} \quad (A.2)$$

To solve Eq. (A.1), we define

$$F_k(z) = \frac{1}{2\pi i} \int_{-a}^a \frac{t_k(\xi)}{\xi - z} d\xi, \quad k = 1, 2 \quad (A.3)$$

where $z = x + iy$ and $i = \sqrt{-1}$. Equation (A.3) leads to the following relations

$$t_k(x) = F_k^+(x) - F_k^-(x) \quad (\text{A.4})$$

$$F_k^+(x) + F_k^-(x) = \frac{1}{\pi i} \int_{-a}^a \frac{t_k(\xi)}{\xi - x} d\xi \quad (\text{A.5})$$

where superscripts “+” and “−” stand for the limits of $F_k(z)$ as $y \rightarrow 0^+$ and $y \rightarrow 0^-$, respectively. With these relations, Eq. (A.1) can be written as

$$UF^+(x) = VF^-(x) + C \quad (\text{A.6})$$

where

$$U = B + Ai, \quad V = B - Ai \quad (\text{A.7})$$

Before solving Eq. (A.6), we first consider the eigenvalue problem

$$[U - \lambda V] W = 0 \quad (\text{A.8})$$

which has two eigenvalues and two normalized eigenvectors

$$\lambda_1 = \frac{\beta - 1}{\beta + 1}, \quad \lambda_2 = \frac{\beta + 1}{\beta - 1} \quad (\text{A.9})$$

$$W = [W_1 \quad W_2] \quad (\text{A.10})$$

where

$$W_1 = [1 \quad i]^T, \quad W_2 = [1 \quad -i]^T \quad (\text{A.11})$$

Introducing the following auxiliary functions

$$W^{-1}F^+ = T^+, \quad W^{-1}F^- = T^- \quad (\text{A.12})$$

and multiplying Eq. (A.6) by W^{-1} leads to the following decoupled equation

$$U'T^+ - V'T^- = C' \quad (\text{A.13})$$

where

$$U' = W^{-1}UW, \quad V' = W^{-1}VW, \quad C' = W^{-1}C \quad (\text{A.14})$$

The decoupled matrix Eq. (A.13) consists of two inhomogeneous Hilbert equations

$$\begin{cases} (1 - \beta) iT_1^+ + (1 + \beta) iT_1^- = \frac{1}{2\pi} [f_x(x) - if_y(x)] \\ (1 + \beta) iT_2^+ + (1 - \beta) iT_2^- = \frac{1}{2\pi} [f_x(x) + if_y(x)] \end{cases} \quad (\text{A.15})$$

whose solution can be obtained following the standard procedure as [39]

$$\begin{aligned} T_1^+ = & -\frac{(a+x)^{-\bar{r}}(a-x)^{-r}}{4\pi^2(1-\beta)} \\ & \times \int_{-a}^a \frac{(a+t)^{\bar{r}}(a-t)^r [f_x(t) - if_y(t)]}{t-x} dt \\ & + \frac{f_x(x) - if_y(x)}{4\pi i(1-\beta)} + c_1(a+x)^{-\bar{r}}(a-x)^{-r} e^{-\pi ri} \end{aligned} \quad (\text{A.16})$$

$$\begin{aligned} T_1^- = & -\frac{(a+x)^{-\bar{r}}(a-x)^{-r} e^{2\pi ri}}{4\pi^2(1-\beta)} \\ & \times \int_{-a}^a \frac{(a+t)^{\bar{r}}(a-t)^r [f_x(t) - if_y(t)]}{t-x} dt \\ & + \frac{f_x(x) - if_y(x)}{4\pi i(1+\beta)} + c_1(a+x)^{-\bar{r}}(a-x)^{-r} e^{\pi ri} \end{aligned} \quad (\text{A.17})$$

$$\begin{aligned} T_2^+ = & -\frac{(a+x)^{-r}(a-x)^{-\bar{r}}}{4\pi^2(1+\beta)} \\ & \times \int_{-a}^a \frac{(a+t)^r(a-t)^{\bar{r}} [f_x(t) + if_y(t)]}{t-x} dt \\ & + \frac{f_x(x) + if_y(x)}{4\pi i(1+\beta)} + c_2(a+x)^{-r}(a-x)^{-\bar{r}} e^{-\pi ri} \end{aligned} \quad (\text{A.18})$$

$$\begin{aligned} T_2^- = & -\frac{(a+x)^{-r}(a-x)^{-\bar{r}} e^{2\pi ri}}{4\pi^2(1+\beta)} \\ & \times \int_{-a}^a \frac{(a+t)^r(a-t)^{\bar{r}} [f_x(t) + if_y(t)]}{t-x} dt \\ & + \frac{f_x(x) + if_y(x)}{4\pi i(1-\beta)} + c_2(a+x)^{-r}(a-x)^{-\bar{r}} e^{\pi ri} \end{aligned} \quad (\text{A.19})$$

where

$$r = \frac{1}{2} + i\kappa, \quad \kappa = \frac{1}{2\pi} \ln \frac{1+\beta}{1-\beta} \quad (\text{A.20})$$

and κ is an oscillation index; r is the stress singularity near the contact edge; c_1 and c_2 are two constants to be determined using the force equilibrium conditions.

The tangential and normal tractions inside the contact region ($|x| \leq a$) can be obtained from the following relations,

$$\begin{Bmatrix} \sigma_{xy}(x) \\ \sigma_{yy}(x) \end{Bmatrix} = \begin{Bmatrix} t_1(x) \\ t_2(x) \end{Bmatrix} = \begin{Bmatrix} T_1^+ - T_1^- + T_2^+ - T_2^- \\ i(T_1^+ - T_1^-) - i(T_2^+ - T_2^-) \end{Bmatrix} \quad (\text{A.21})$$

In order to determine the two constants, c_1 and c_2 , the equilibrium of forces in x and y directions should satisfy

$$\int_{-a}^a \sigma_{xy}(x) dx = 0, \quad \int_{-a}^a \sigma_{yy}(x) dx = -2p(b_l - a) \quad (\text{A.22})$$

We obtain (details are similar to [25]),

$$c_1 = \frac{-p(b_l - a)\sqrt{1-\beta^2}}{2 \int_{-a}^a (a+\xi)^{-\bar{r}}(a-\xi)^{-r} d\xi} + O(\beta^3), \quad c_2 = -\bar{c}_1 \quad (\text{A.23})$$

References

- [1] K.L. Johnson, K. Kendall, A.D. Roberts: *Proc. Roy. Soc. Lond. A* 324 (1971) 301.
- [2] B.V. Derjaguin, V.M. Muller, Y.P. Toporov: *J. Coll. Interface Sci.* 53 (1975) 314.
- [3] D. Maugis: *J. Coll. Interface Sci.* 150 (1992) 243.
- [4] D.S. Dugdale: *J. Mech. Phys. Solids* 8 (1960) 100.
- [5] D. Maugis, M. Barquins: *J. Phys. D11* (1978) 1989.
- [6] V.M. Muller, V.S. Yushenko, B.V. Derjaguin: *J. Colloid Interface Sci.* 77 (1980) 91.
- [7] R.W. Carpick, N. Agrait, D.F. Ogletree, M. Salmeron: *Langmuir* 12 (1996) 3334.
- [8] J.A. Greenwood: *Proc. Roy. Soc. Lond. A* 453 (1997) 1277.
- [9] K.L. Johnson, J.A. Greenwood: *J. Coll. Interface Sci.* 192 (1997) 326.
- [10] J.M. Baney, C.Y. Hui: *J. Adhesion Sci. Technol.* 11 (1997) 393.
- [11] C.Y. Hui, J.M. Baney, E.J. Kramer: *Langmuir* 14 (1998) 6570.
- [12] E. Barthel: *J. Coll. Interface Sci.* 200 (1998) 7.
- [13] F. Robbe-Valloire, M. Barquins: *Int. J. Adhesion and Adhesives* 18 (1998) 29.
- [14] J.A. Greenwood, K.L. Johnson: *J. Phys. D: Appl. Phys.* 31 (1998) 3279.
- [15] K.S. Kim, R.M. McMeeking, K.L. Johnson: *J. Mech. Phys. Solids* 46 (1998) 243.
- [16] Y.Y. Lin, C.Y. Hui, J.M. Baney: *J. Phys. D: Appl. Phys.* 32 (1999) 2250.
- [17] E. Barthel, G. Haiat: *Langmuir* 18 (2002) 9362.
- [18] C. Morrow, M. Lovell, X. Ning: *J. Phys. D: Appl. Phys.* 36 (2003) 534.
- [19] U.D. Schwarz: *J. Coll. Interface Sci.* 261 (2003) 99.
- [20] G. Haiat, M.C.P. Huy, E. Barthel: *J. Mech. Phys. Solids* 51 (2003) 69.
- [21] H. Gao, X. Wang, H. Yao, S. Gorb, E. Arzt: *Mech. Mater.* 37 (2005) 275.
- [22] H. Gao, H. Yao: *Proc. Nat. Acad. Sci. USA* 101 (2004) 7851.
- [23] N.J. Glassmaker, A. Jagota, C.Y. Hui, J. Kim: *J. R. Soc. Interface* 1 (2004) 23.
- [24] C.Y. Hui, N.J. Glassmaker, T. Tang, A. Jagota: *J. R. Soc. Interface* 1 (2004) 35.
- [25] S. Chen, H. Gao: *Proc. Roy. Soc. Lond. A* 462 (2006) 211.
- [26] Y.-S. Chu, S. Dufour, J.P. Thiery, E. Perez, F. Pincet: *Phys. Rev. Lett.* 94 (2005) 028102-1.
- [27] R.C. Buck: *Exp. Cell. Res.* 127 (1980) 470.
- [28] P.C. Dartsch, H. Hammerle: *Eur. J. Cell Biol.* 41 (1986) 339.
- [29] V.P. Shirinsky, A.S. Antonov, K. Birukov, A. Sobolevsky, Y. Romanov, N. Kabaeva, G. Antonova, V. Smirnov: *J. Cell. Biol.* 109 (1989) 331.
- [30] C. Neidlinger-Wilke, H.J. Wilke L. Claes: *J. Orthop. Res.* 12 (1994) 70.
- [31] J.H.C. Wang: *J. Theor. Biol.* 202 (2000) 33.
- [32] A.D. Roberts, A.G. Thomas: *Wear* 33 (1975) 45.
- [33] M. Barquins: *J. Adhesion* 26 (1988) 1.
- [34] M.K. Chaudhury, T. Weaver, C.Y. Hui, E.J. Kramer: *J. Appl. Phys.* 80 (1996) 30.
- [35] O.T. Sari, G.G. Adams, S. Muftu: *J. Appl. Mech.* 72 (2005) 633.
- [36] K.L. Johnson: *Contact Mechanics*. Cambridge University Press (1985).
- [37] J. Durdurs: *Appl. Mech.* 36 (1969) 650.
- [38] Z. Suo: *Proc. Roy. Soc. Lond. A* 427 (1990) 331.
- [39] G.F. Carrier, M. Krook, C.E. Pearson: *Functions of a complex variable*. Hod brooks, Ithaca, New York (1983).

(Received November 23, 2005; accepted January 15, 2006)

Correspondence address

Professor Huajian Gao
 Max Planck Institute for Metals Research
 Heisenbergstraße 3, D-70569 Stuttgart, Germany
 Tel.: +49 711 689 3510
 Fax: +49 711 689 3512
 E-mail: hjgao@mf.mpg.de

You will find the article and additional material by entering the document number MK101275 on our website at www.ijmr.de