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An incremental micromechanical scheme for nonlinear particulate composites

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Abstract

A general incremental micromechanical scheme for the nonlinear behavior of particulate composites is presented in this paper. The advantage of this scheme is that it can reflect partly the effects of the third invariant of the stress on the overall mechanical behavior of nonlinear composites. The difficulty involved is the determination of the effective compliance tensors of the anisotropic multiphase composites. This is completed by making use of the generalized self-consistent Mori–Tanaka method which was recently developed by Dai et al. (*Polymer Composites* 19 (1998) 506–513; *Acta Mechanica Solida* 18 (1998) 199–208). Comparison with existing theoretical and numerical results demonstrates that the present incremental scheme is quite satisfactory. Based on this incremental scheme, the overall mechanical behavior of a hard-particle reinforced metal matrix composite with progressive particle debonding damage is investigated. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Incremental method; Composites; Micromechanics; Nonlinear behavior

1. Introduction

It is well known that most widely used metal matrix composites and polymer matrix composites behave in the nonlinear range under mechanical loadings. Therefore, theoretical prediction of the overall mechanical behavior of this class of nonlinear composites is important. However, due to the difficulty in solving nonlinear boundary value problems, exact solutions by analytical methods

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cannot be obtained except for some special cases, such as composites with regular microstructure [1]. Instead of finding the exact solutions, several powerful approximate or linearized approaches have been developed to estimate or bound the overall properties of nonlinear composite materials during the past two decades. Among the most typical approaches are the secant modulus method, the nonlinear variational approach, and the simplified incremental method. Based on the idea of using the linear effective properties of a comparison composite to estimate its nonlinear properties, Tandon and Weng [2] proposed the so-called classical secant modulus method (CSMM) to analyze the plastic behavior of particulate composites. Their idea evolved out of the earlier work on polycrystal plasticity [3,4]. Since the von Mises equivalent stress of the matrix was calculated directly from its mean deviatoric stress, CSMM cannot predict the plastic response of porous materials subjected to a pure hydrostatic loading. To eliminate this drawback, Qiu and Weng [5,6] and Hu [7] developed the modified secant modulus method (MSMM) which allows one to evaluate the equivalent stress of the matrix by making use of the second moment of the matrix deviatoric stress. Ju and Tseng [8] proposed a new approach called the matrix stress norm to evaluate the equivalent stress of the nonlinear matrix. Another powerful tool to estimate or bound the overall behavior of nonlinear composites is the nonlinear variational method [9,10]. Recently, Hu [7] demonstrated that the use of the exact second moment of the equivalent stress in the context of MSMM leads to the same results as the nonlinear variational method. Recently, a powerful incremental damage theory was developed by Tohgo et al. [11–13] by making use of Eshelby's equivalent inclusion concept [14] and the Mori–Tanaka mean-field theory [15].

It is noted that, although these homogenization methods are concise in concept and convenient in operation, they still suffer the limitation that they cannot reflect the effect of the third stress invariant on the overall behavior of composites. However, recent research work by Lee and Mear [16] has demonstrated that the effect of the third stress invariant on the overall response cannot be ignored for some power-law materials reinforced by rigid particles. Therefore, some new approaches which can remove, at least partly, this limitation need to be developed for predicting accurately the overall behavior of nonlinear composites.

According to the aforementioned observations, a general incremental scheme is presented in this paper. This scheme involves the instantaneous anisotropic tangent modulus tensors of nonlinear phases thus allowing one to consider partly the effects of the third stress invariant on the overall behavior of nonlinear composites. The validity of the present incremental scheme is checked by comparing it with the available micromechanical methods.

2. Effective compliance of anisotropic composites

Since the incremental scheme involves the anisotropic tangent compliance tensor of the phases and the composite, we present a short description of their determination by extending the generalized self-consistent Mori–Tanaka method (GSCMTM) developed initially for isotropic composites by Dai et al. [17,18] to an anisotropic case in this section.

Consider a two-phase anisotropic composite material with inclusions distributed uniformly in a matrix. Inspired by the idea of the generalized self-consistent method (GSCM) [19], the composite can be modeled by a three-phase geometric configuration: inclusion/matrix/effective composite medium in order to investigate the effective moduli or compliance of the composite. The

elastic modulus tensors of the matrix, the inclusion, and the composite are denoted, respectively, by $\mathbf{L}^{(0)}$, $\mathbf{L}^{(1)}$, $\bar{\mathbf{L}}$, and the corresponding compliance tensors can be obtained by the relation $\mathbf{M} = \mathbf{L}^{-1}$. When the homogeneous displacement boundary condition $u|_{\partial V} = \mathbf{E} \cdot \mathbf{x}$ is imposed at infinity, the strain at an arbitrary point in the composite can be determined by the following Lippman–Schinger–Dyson-type integral equation [20]:

$$\boldsymbol{\varepsilon}(\mathbf{x}) = \mathbf{E} - \int_V \boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}') : \delta \mathbf{L} : \boldsymbol{\varepsilon}(\mathbf{x}') d\mathbf{x} \quad (1)$$

where $\delta \mathbf{L}$ is the fluctuating part of the elastic moduli tensor of the composite and defined by: $\delta \mathbf{L}(\mathbf{x}) = \mathbf{L}(\mathbf{x}) - \bar{\mathbf{L}}$, where $\mathbf{L}(\mathbf{x})$ is the local elastic moduli tensor of the composite. $\boldsymbol{\Gamma}(\mathbf{x} - \mathbf{x}')$ is the modified Green function associated with the homogeneous comparison material. By taking the effective composite medium as the comparison material and making use of the strain equivalent condition of GSCM and Hill's interfacial operation theory [21], one can derive the effective moduli of anisotropic composites as follows [22]:

$$\bar{\mathbf{L}} = \mathbf{L}^{(0)} + c_1(\mathbf{L}^{(1)} - \mathbf{L}^{(0)}) : [c_1 \mathbf{I}^{(4)} + (1 - c_1)\mathbf{H}]^{-1}. \quad (2)$$

Similarly, the effective compliance of the composite can be easily obtained:

$$\bar{\mathbf{M}} = \mathbf{M}^{(0)} + c_1(\mathbf{M}^{(1)} - \mathbf{M}^{(0)}) : [c_0 \mathbf{M}^{(0)-1} : \mathbf{H} : \mathbf{M}^{(1)} + c_1 \mathbf{I}]^{-1}, \quad (3)$$

where c_0 , c_1 are the volume fraction of matrix and inclusion, respectively, $\mathbf{I}^{(4)}$ is the fourth rank unit tensor. Likewise, \mathbf{H} is defined as

$$\mathbf{H} = \mathbf{I}^{(4)} + \mathbf{P} : (\mathbf{M}^{(1)-1} - \mathbf{M}^{(0)-1}), \quad (4)$$

where \mathbf{P} is related to the Eshelby tensor of inclusion \mathbf{S} by $\mathbf{P} = \mathbf{S} : \mathbf{M}^{(0)}$ and \mathbf{S} is defined by Eq. (12). It is interesting to find that the derived result of Eq. (2) or Eq. (3) is identical to that of the Mori–Tanaka method (MTM) [23]. Since the method starts from GSCM and leads to the same result of MTM, we call our present method as the generalized self-consistent Mori–Tanaka method (GSCMTM). The advantages of GSCMTM are that it can be suitable for arbitrary shape inclusion and anisotropic phases cases. Furthermore, in comparison with MTM, GSCMTM can easily be extended to the coated inclusion-based composites [18].

3. Incremental scheme for two-phase nonlinear composites

Consider inclusion/matrix two-phase nonlinear composites. The local constitutive behavior of individual phases can be characterized by a strain potential $U(\boldsymbol{\sigma})$

$$\boldsymbol{\varepsilon} = \frac{\partial U(\boldsymbol{\sigma})}{\partial \boldsymbol{\sigma}}. \quad (5)$$

This constitutive model is commonly used to represent a number of nonlinear mechanical phenomena, including time-independent plastic deformation (i.e. deformation theory of plasticity) and time-dependent viscous deformation (e.g. high-temperature creep) of metals. In the first case, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are the infinitesimal stress and strain tensors. In the second case, $\boldsymbol{\sigma}$ and $\dot{\boldsymbol{\varepsilon}}$ should be identified with the Cauchy stress and Eulerian strain rate, respectively. For simplicity, attention is focused on a class of isotropic materials for which $U(\boldsymbol{\sigma})$ reads as

$$U^{(r)}(\boldsymbol{\sigma}) = \frac{1}{2k^{(r)}}\sigma_m^2 + \varphi^{(r)}(\sigma_e) \quad (r = 0,1), \quad (6)$$

where $\sigma_m = \text{tr}(\boldsymbol{\sigma})/3$ and $\sigma_e = (3\mathbf{s}:\mathbf{s}/2)^{1/2}$ are the first two invariants of $\boldsymbol{\sigma}$, while \mathbf{s} is the deviatoric stress. By differentiation with respect to time, the constitutive relation (5) can be expressed in the incremental form

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{M}_t(\boldsymbol{\sigma}) : \dot{\boldsymbol{\sigma}}, \quad (7)$$

\mathbf{M}_t is the instantaneous tangent compliance tensor and can be written as

$$\mathbf{M}_t(\boldsymbol{\sigma}) = \frac{1}{3k}\mathbf{J} + \frac{1}{2\mu_t}\mathbf{K} + \frac{1}{\lambda}\mathbf{F}, \quad (8)$$

where

$$\mathbf{J} = \mathbf{I} \otimes \mathbf{I} / 3, \quad \mathbf{K} = \mathbf{I} - \mathbf{I} \otimes \mathbf{I} / 3,$$

$$\mathbf{F} = \hat{\mathbf{s}} \otimes \hat{\mathbf{s}}, \quad \hat{\mathbf{s}} \equiv \mathbf{s}/\sigma_e \quad (9)$$

and

$$\begin{aligned} k &= \sigma_m / [3\partial U(\boldsymbol{\sigma}) / \partial \text{tr}(\boldsymbol{\sigma})], \quad \mu_t = \sigma_e / 3\varphi'(\sigma_e), \\ \lambda &= \left(\frac{4}{9}\right) [\sigma_e / (\sigma_e \varphi''(\sigma_e) - \varphi'(\sigma_e))], \\ \varphi'(\sigma_e) &= \partial \varphi(\sigma_e) / \partial \sigma_e. \end{aligned} \quad (10)$$

In a general case, the local tangent compliance of the phases is anisotropic. The tangent compliance varying from one point to another, an approximation is made and constant tangent tensors are considered in each phase. It is noted that the tangent compliance defined by Eq. (8) depends not only on the first two stress invariants (σ_m, σ_e) but also on $\hat{\mathbf{s}} \otimes \hat{\mathbf{s}}$. While $\hat{\mathbf{s}} \otimes \hat{\mathbf{s}}$ cannot be completely determined by the first two stress invariants, it also depends on the third stress invariant. Hence, the presently defined tangent compliance can, at least partly reflect the effect of the third invariant of stress. It is this point that differs from the existing methods, such as the secant modulus method [2,4–7] and the incremental method where the isotropic tangent modulus tensors are adopted [11–13].

The overall behavior of nonlinear composites can be characterized by the following incremental constitutive relation:

$$\dot{\mathbf{E}} = \bar{\mathbf{M}}_t(\boldsymbol{\Sigma}) : \dot{\boldsymbol{\Sigma}}, \quad (11)$$

where \mathbf{E} , $\mathbf{\Sigma}$ are the macroscopic strain and stress tensors, respectively. Usually, determining the anisotropic effective compliance $\bar{\mathbf{M}}_t$ is very difficult due to the anisotropic matrix inclusion problem which has to be solved. In our present approach, $\bar{\mathbf{M}}_t$ will be determined by GSCMTM. The involved anisotropic Eshelby tensor \mathbf{S} was given by Mura [24]:

$$S_{ijkl} = (1/8\pi)M_{mnkl}^{(0)}(\bar{G}_{imjn} + \bar{G}_{j\min}), \tag{12}$$

where

$$\bar{G}_{ijkl} = 2\pi \int_{-1}^1 R[G_{ijkl}/Z] d\bar{\xi}_3, \tag{13}$$

in which $R[G_{ijkl}/Z]$ is the sum of the residues of the function G_{ijkl}/Z existing within the unit circle $|Z| = 1$ under fixed value of $\bar{\xi}_3$, while G_{ijkl} is defined by

$$G_{ijkl} = \bar{\xi}_k \bar{\xi}_l N_{ij}(\bar{\xi})/D(\bar{\xi}), \tag{14}$$

where $N_{ij}(\bar{\xi})$, $D(\bar{\xi})$ are the cofactors and the determinant of Christoffel matrix. The closed form of solutions of \bar{G}_{ijkl} can only be obtained for orthotropic and transversely isotropic materials. According to this fact, we only consider uniaxial tension and axisymmetric tension loading cases which ensure that the tangent compliance is transversely isotropic.

The overall behavior of nonlinear composites can be modeled by a series of incremental steps. In each incremental step, the nonlinear composite will be characterized approximately by a linear one. In this paper we only consider the case where the matrix phase is nonlinear while the spherical inclusion phase remains elastic during the entire deformation procedure. Because the tangent compliance depends on the instantaneous stress state, the average stresses of the matrix in each step should be determined. If the average stresses in the last step are known, then the average stresses of the matrix in the present step can be obtained by the approach to linear composites [22]:

$$d\bar{\sigma}_m^{(0)} = \frac{1}{(1 - c_1) + c_1(k^{(1)}/k^{(0)})h_k} d\Sigma_m, \tag{15}$$

$$d\bar{\sigma}_e^{(0)} = \frac{1}{(1 - c_1) + c_1(\mu^{(1)}/\mu_t^{(0)})h_\mu} d\Sigma_e, \tag{16}$$

where

$$h_k = \frac{3k^{(1)} + 4\mu_t^{(0)}}{3k^{(0)} + 4\mu_t^{(0)}}, \tag{17}$$

$$h_\mu = 1 + \frac{6(\mu^{(1)} - \mu_t^{(0)})(k^{(0)} + 2\mu_t^{(0)})}{\mu_t^{(0)}(3k^{(0)} + 4\mu_t^{(0)})}, \tag{18}$$

where k and μ are bulk and shear moduli, respectively, and Σ_m and Σ_e are the first two stress invariants of the imposed stress tensor $\mathbf{\Sigma}$. Once the local tangent compliance of the phases is obtained, the overall tangent compliance of the composite can be determined by Eq. (3) for each incremental step. Therefore, the overall stress–strain curve can be obtained point by point by making use of the incremental constitutive relation of Eq. (7).

Consider uniaxial tension: $\Sigma_{11} = \Sigma_{22} = 0, \Sigma_{33} \neq 0$. For this case, the tangent compliance tensor and the other related tensor are transversely isotropic tensors which can be expressed and calculated by a set of rules given by Walpole [25]. The overall stress–strain relation of nonlinear composites can be written as

$$\dot{E}_{33} = \bar{M}_{3333}^t \dot{\Sigma}_{3333} \quad (19)$$

\bar{M}_{3333}^t is the overall longitudinal tangent compliance tensor of the composite and is given by

$$\bar{M}_{3333}^t = M_{3333}^{(0)t} + c_1 \frac{y_1}{y_1 y_4 - 2y_2 y_3}, \quad (20)$$

where

$$\begin{aligned} y_1 &= [2(k^{(1)} + \mu^{(1)}/3)/w_1 - M_{3333}^{(0)t}]/w_2 + c_0 q_1, \\ y_2 &= [-(k^{(1)} - 2\mu^{(1)}/3)/w_1 - M_{1133}^{(0)t}]/w_2 + c_0 q_2, \\ y_3 &= [-(k^{(1)} - 2\mu^{(1)}/3)/w_1 - M_{3311}^{(0)t}]/w_2 + c_0 q_3, \\ y_4 &= [(k^{(1)} + 4\mu^{(4)}/3)/w_1 - (M_{1111}^{(0)t} + M_{1122}^{(0)t})]/w_2 + c_0 q_4, \end{aligned} \quad (21)$$

where w_j, q_j ($j = 1, 2, 3, 4$) are given in the appendix. The local tangent compliance tensors of the matrix $M_{ijkl}^{(0)t}$ can be obtained from Eq. (8):

$$\begin{aligned} M_{1111}^{(0)} &= M_{2222}^{(0)} = \frac{1}{9k^{(0)}} + \frac{1}{3\mu_t^{(0)}} + \frac{1}{9}\xi, \\ M_{3333}^{(0)} &= \frac{1}{9k^{(0)}} + \frac{1}{3\mu_t^{(0)}} + \frac{4}{9}\xi, \\ M_{1122}^{(0)} &= \frac{1}{9k^{(0)}} - \frac{1}{6\mu_t^{(0)}} + \frac{1}{9}\xi, \\ M_{1133}^{(0)} &= \frac{1}{9k^{(0)}} - \frac{1}{6\mu_t^{(0)}} - \frac{2}{9}\xi, \\ M_{3131}^{(0)} &= \frac{1}{4\mu_t^{(0)}}, \\ M_{1212}^{(0)} &= \frac{1}{4\mu_t^{(0)}}, \end{aligned} \quad (22)$$

where $\xi = 1/\lambda$.

To demonstrate the utility of the present incremental scheme, we illustrate two examples. In the first example, we consider a porous material. The material properties of the matrix are: $E^{(0)}/\sigma^{(0)} = 200, \nu^{(0)} = 0.30, n = 0.10$, and the volume fraction of voids is 6.5%. The constitutive

response of the isotropic power-law hardening elastoplastic matrix can be characterized by the following strain potential:

$$U^{(0)}(\boldsymbol{\sigma}) = \frac{1}{2k^{(0)}}\sigma_m^2 + \frac{\sigma_0^2}{3\mu^{(0)}} \left[\frac{n}{n+1} \left(\frac{\sigma_e}{\sigma_0} \right)^{n+1/n} - \frac{1}{2} \left(\frac{\sigma_e}{\sigma_0} \right)^2 \right]. \tag{23}$$

In the initial calculation, we have used several sizes of the stress increment, i.e., $\Delta\Sigma = 0.1, 0.2, 0.5, 0.8$ MPa. We find that the results are insensitive to the selected stress increment size. To speed up the calculation and obtain enough data of the calculated points, we finally select $\Delta\Sigma = 0.5$ MPa in our formal calculations and this incremental size was also used in the following examples. Fig. 1 presents a comparison of the overall uniaxial stress–strain curve of the composite predicted by the present incremental method with that by the finite element method made by Hom and McMeeking [26]. The comparison shows that the result from the incremental method is slightly stiffer than that from the finite element method.

In the second example, a SiC_p/2124Al metal matrix composite material is investigated. The material properties of the phases are: $E^{(1)} = 450$ GPa, $\nu^{(1)} = 0.17$; $E^{(0)} = 73$ GPa, $\nu^{(0)} = 0.33$, $\sigma_0 = 280$ MPa, $n = 7.66$ [27]. The volume fraction of the inclusion is taken to be 0.30. The local strain potential of the isotropic power-law hardening elastoplastic metal matrix is given by

$$U^{(0)}(\boldsymbol{\sigma}) = \frac{1}{2k^{(0)}}\sigma_m^2 + \frac{1}{6\mu^{(0)}}\sigma_e^2 + \frac{n-1}{n+1} \frac{\sigma_0^2}{6\mu^{(0)}} \left[1 + \frac{2}{n+1} \left(\frac{\sigma_e}{\sigma_0} \right)^{n+1} \right] H(\sigma_e - \sigma_0), \tag{24}$$

where $H(x)$ denotes Heaviside step function. The overall uniaxial stress–strain curve is calculated by the present incremental method and compared with those calculated by the classical secant

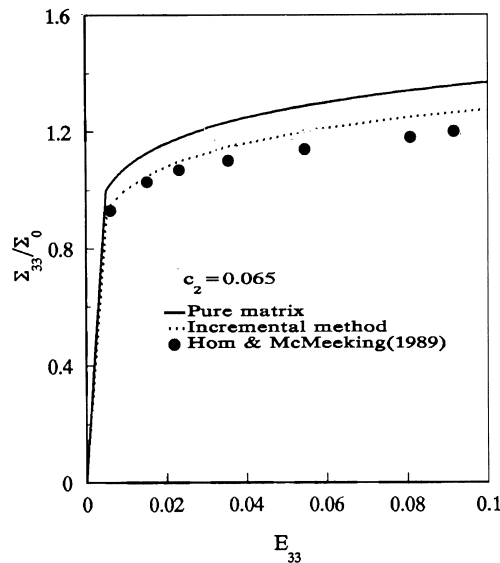


Fig. 1. Comparison of the stress–strain behavior of the incremental method with the finite element analysis for uniaxial tension ($\Sigma_{11} = \Sigma_{22} = 0, \Sigma_{33} = \Sigma$).

modulus method (CSMM) and the modified secant modulus method (MSMM) (Fig. 2). It is seen from Fig. 2 that the solutions from the incremental method are much softer than those from CSMM and close to those from MSMM. For an axisymmetric tension case: $\Sigma_{11} = \Sigma_{22} = -0.5\Sigma$, $\Sigma_{33} = \Sigma$, the tangent compliance tensors are also transversely isotropic. So, the overall stress–strain curves of this composite material can also be obtained (Fig. 3). From Fig. 3 we find

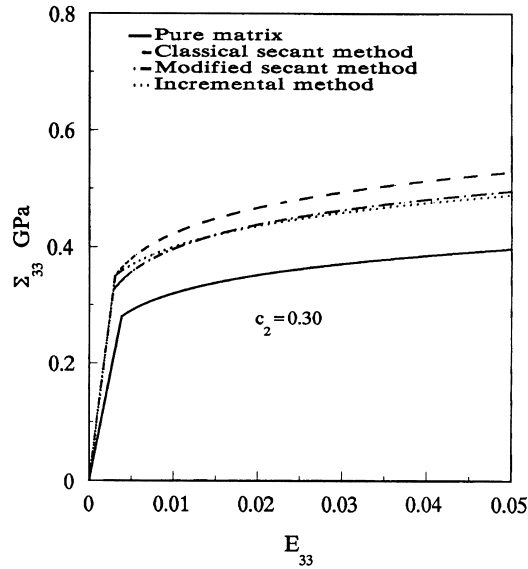


Fig. 2. Comparison of the stress–strain behavior of the incremental method with the classical secant modulus method and the modified secant modulus method for uniaxial tension ($\Sigma_{11} = \Sigma_{22} = 0, \Sigma_{33} = \Sigma$).

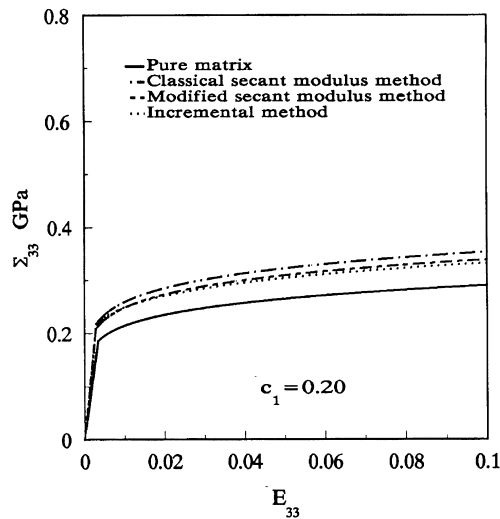


Fig. 3. Comparison of the stress–strain behavior of the incremental method with the classical secant modulus method and the modified secant modulus method for axisymmetric tension ($\Sigma_{11} = \Sigma_{22} = -0.5\Sigma, \Sigma_{33} = \Sigma$).

that the result by the present incremental method is slightly softer than that by MSMM. Because the estimation of the effective properties of the associated linearized composites are all followed by the generalized self-consistent Mori–Tanaka model, the difference shown in Figs. 2 and 3 may be considered to result from the effect of the third stress invariant. Since the stress triaxiality in the present two examples is not high, the effect of the third stress invariant is not significant. These three examples demonstrate the utility of the present incremental scheme to characterize the overall behavior of nonlinear composites.

4. Application to composites with debonding damage

Under the influence of external mechanical load, stress concentrations develop at particle interfaces in particulate composites. This changes the initial two-phase composite, where all particles are bonded to the matrix into a three-phase composite where some particles become debonded, thereby creating voids. As theorized by many investigators, there are two prevailing criteria for void nucleation from the interfaces, namely, an energy condition and a stress condition [28]. Usually, the size of reinforcing particles in metal matrix composites (such as SiCp/Al composites) is of the order of 10 μm . According to most of the previous studies, the energy criterion for void nucleation on this size is automatically satisfied [29]. That is, the stress criterion solely governs occurrence of the particles debonding (void nucleation). Apparently, the hard particles have not only a reinforcing effect but also a weakening effect due to particle debonding damage. Therefore, the overall mechanical behavior of the composites depends on the competitive result of these two effects. In this section, we will incorporate these two effects into the incremental formation presented in the previous section for predicting the overall behavior of particulate composites with debonding damage.

In order to describe the debonding process, as was done by Tohgo [11–13] in their research, the following assumptions are made: (1) the debonding of particles is controlled by the average hydrostatic stress of the particles and the statistical behavior of the particle–matrix interfacial strength. Here, the particle stress is used because the interfacial stress is given as a function of it [30]. Recent microscopic observations on the local deformation fields in SiCp/Al composite made by Derrien et al. [31] have demonstrated that the matrix region adjacent to broken particles has high hydrostatic tension. This means that the hydrostatic stress is an important factor controlling particle debonding in particulate metal matrix composites. (2) Once the particles debond from the interfaces, these particles become voids; therefore the volume fraction of the debonded particles turns into a void volume fraction. Usually, the bonding strength of the interfaces is not uniform. Therefore, the debonding process can be described by probability theory. According to the experimental observations of Brechet et al. [32] and following the strategy adopted by Tohgo et al. [11–13], the cumulative probability of the debonding can be characterized by the Weibull distribution

$$P_v(\bar{\sigma}_m^{(1)}) = 1 - \exp\left[-\left(\frac{\bar{\sigma}_m^{(1)}}{\beta}\right)^\alpha\right], \quad (25)$$

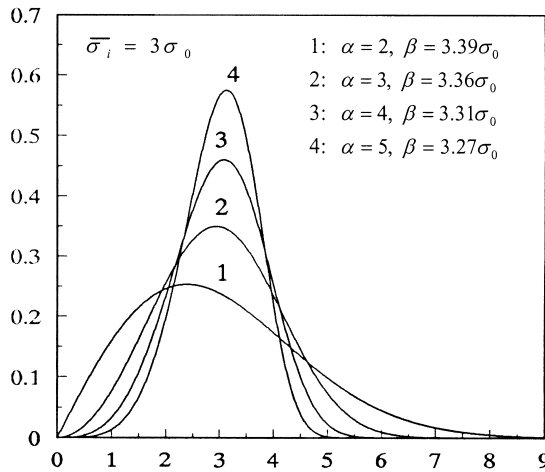


Fig. 4. Variations of the cumulative probability density of particle debonding with the distribution parameters α, β .

where $\bar{\sigma}_m^{(1)} = \bar{\sigma}_{kk}^{(1)}/3$ is the hydrostatic part of the stress of particles, and α and β are the scale parameter and the shape parameter, respectively. This probability controls the volume fraction of the debonded particles or voids. With this function, the average interfacial strength of the interface, denoted by $\bar{\sigma}_i$, is related to the Gamma function $\Gamma(\cdot)$ as

$$\bar{\sigma}_i = \beta \Gamma\left(1 + \frac{1}{\alpha}\right) \tag{26}$$

and the volume fraction of the debonded particles or voids c_2 is given by

$$c_2 = c_p P_v(\bar{\sigma}_m^{(1)}) = c_p \left\{ 1 - \exp\left[-\left(\frac{\bar{\sigma}_m^{(1)}}{\beta}\right)^\alpha\right] \right\}, \tag{27}$$

where c_p is the initial volume fraction of particles. c_p is related to the volume fraction of the particles at the current state c_1 and the volume fraction of the debonded particles or voids c_2 as

$$c_1 + c_2 = c_p. \tag{28}$$

Fig. 4 presents the variations of the probability density of the debonding with the distribution parameters α and β .

Once particles debond from the interfaces, an initial particle/matrix two-phase composite turns into a particle/void/matrix three-phase hybrid composite. In this case, the overall tangent compliance tensor of the hybrid composite can be determined by the GSCMTM and written as

$$\bar{\mathbf{M}}_t = [c_0 \mathbf{I} + c_1 \mathbf{H}^{(1)-1} + c_2 \mathbf{H}^{(2)-1}] : M_t^{(0)} : [c_0 \mathbf{I} + c_1 \mathbf{T}^{(1)-1}]^{-1}, \tag{29}$$

where

$$\mathbf{T}^{(1)} = \mathbf{I} + \mathbf{Q} : (\mathbf{M}^{(1)} - \mathbf{M}^{(0)}) \quad (30)$$

and

$$\mathbf{Q} = \mathbf{M}^{(0)-1} - \mathbf{M}^{(0)-1} : \mathbf{P} : \mathbf{M}^{(0)-1} \quad (31)$$

in which $\mathbf{P} = \mathbf{S} : \mathbf{M}^{(0)}$, and \mathbf{S} is the inclusion Eshelby tensor.

Since there is only one independent variable among Σ , c_1 and c_2 , the overall incremental constitutive relation of the three-phase hybrid composite can still be written in the form of Eq. (11). The effect of the damage on the overall behavior of the composite is included in the scheme by recalculating the values of c_1 and c_2 at each incremental step according to the assumed damage evolution law given by Eq. (27). Following a procedure similar to that presented in the previous section, the overall stress–strain curves of nonlinear composite with progressive debonding damage can be obtained. Here, we only consider the uniaxial tension case. An initially perfectly bonded hard elastic particle-reinforced soft elastic–plastic metal matrix composite is investigated. The elastic constants of the phases are given respectively as $E^{(1)} = 5000\sigma_0$, $\nu(1) = 0.17$; $E^{(0)} = 500\sigma_0$, $\nu^{(0)} = 0.30$. The constitutive behavior of the isotropic power-law hardening elastoplastic matrix is characterized by the following strain potential:

$$U(\boldsymbol{\sigma}) = \frac{1}{2k^{(0)}}\sigma_m^2 + \frac{\sigma_0^2}{E^{(0)}} \left[\frac{1}{n+1} \left(\frac{\sigma_e}{\sigma_0} \right)^{n+1} + \frac{2(1+\nu^{(0)})}{3} \left(\frac{\sigma_e}{\sigma_0} \right)^2 - \left(\frac{\sigma_e}{\sigma_0} \right) \right], \quad (32)$$

where σ_0 is the yield stress of the matrix, and material constant $n = 10$. The cumulative probability of particles debonding is given by Eq. (25) and the distribution parameters are, respectively, taken as $\beta = 3.27\sigma_0$, $\alpha = 5$. According to Eq. (26), this set of parameters gives the average interfacial strength $\bar{\sigma}_i = 3\sigma_0$. Figs. 5a–d present the overall uniaxial stress–strain curves for the initial particle volume fraction $c_1 = 0.05, 0.10, 0.20, 0.30$, respectively. From Fig. 5 we find that particle debonding has a significant weakening effect on the overall behavior of the composite. For the case of $c_1 = 0.20$, when the overall strain approaches 0.1, the hard particle reinforcing effect is almost counteracted by the particle debonding. Therefore, the particle debonding weakening effect should be considered when this class of hard-particle reinforced composites is applied in engineering fields.

5. Conclusions

Based on the generalized self-consistent Mori–Tanaka method, a general incremental micro-mechanical scheme is developed in this paper. The advantage of the scheme is that it can reflect, at least partly, the effects of the third invariant of the stress. Comparison with the other existing theoretical and numerical results demonstrates that the present incremental scheme is very satisfactory. By making use of this scheme, the effects of the progressive particle debonding damage on the overall behavior of nonlinear composites are investigated. The results show that the weakening effects of particle debonding should be considered when hard-particle reinforced composites are used in engineering fields.

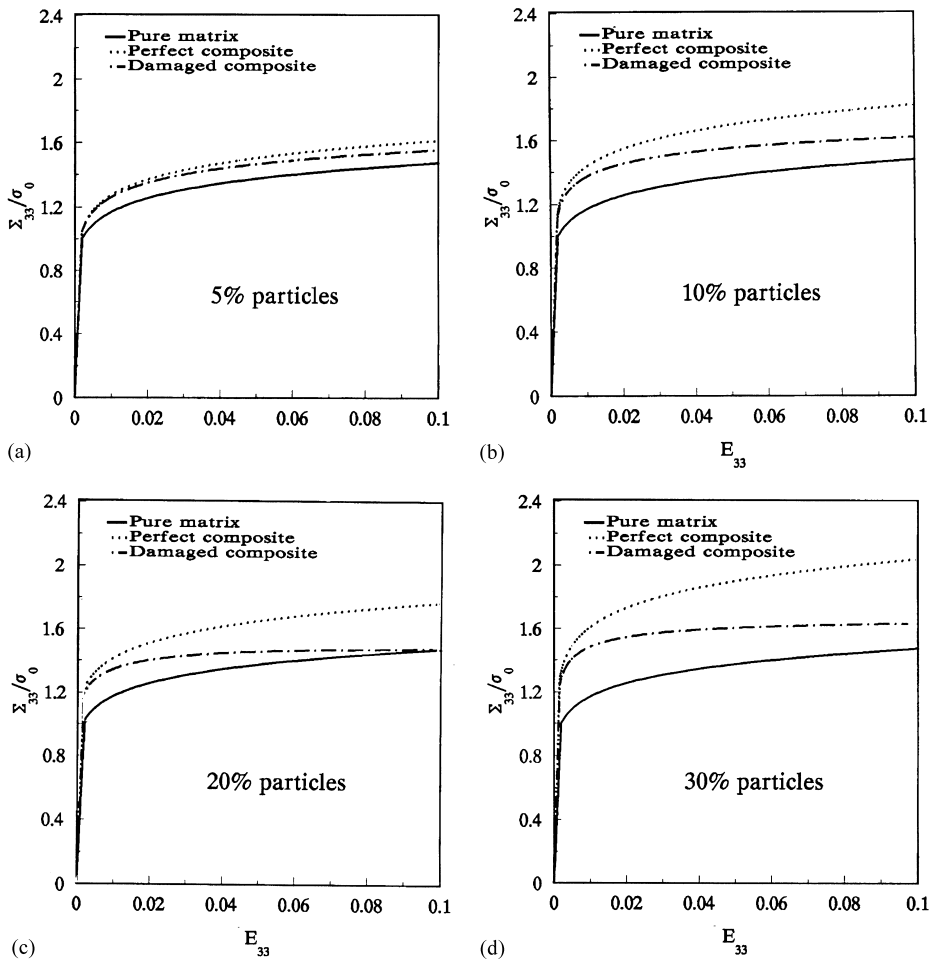


Fig. 5. Stress–strain behavior of pure matrix material, perfect composite, and damage composite under uniaxial tension ($\Sigma_{11} = \Sigma_{22} = 0, \Sigma_{33} = \Sigma$): (a) $c_p = 5\%$; (b) $c_p = 10\%$; (c) $c_p = 20\%$; (d) $c_p = 30\%$.

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Appendix. The related parameters in Eq. (20)

The related parameters used in Eq. (20) are given, respectively, as follows:

$$y_1 = [2(k^{(1)} + \mu^{(1)}/3)/w_1 - M_{3333}^{(0)}]/w_2 + c_0 q_1,$$

$$\begin{aligned}
y_2 &= [-(k^{(1)} - 2\mu^{(1)}/3)/w_1 - M_{1133}^{(0)}]/w_2 + c_0 q_2, \\
y_3 &= [-(k^{(1)} - 2\mu^{(1)}/3)/w_1 - M_{3311}^{(0)}]/w_2 + c_0 q_3, \\
y_4 &= [(k^{(1)} + 4\mu^{(4)}/3)/w_1 - (M_{1111}^{(0)} + M_{1122}^{(0)})]/w_2 + c_0 q_4,
\end{aligned} \tag{A.1}$$

where

$$\begin{aligned}
w_1 &= 2(k^{(1)} + 4\mu^{(1)}/3)(k^{(1)} + \mu^{(1)}/3) - 2(k^{(1)} - 2\mu^{(1)}/3)^2, \\
w_2 &= [2(k^{(1)} + \mu^{(1)}/3)/w_1 - M_{3333}^{(0)}][(k^{(1)} + 4\mu^{(4)}/3)/w_1 - (M_{1111}^{(0)} + M_{1122}^{(0)})] \\
&\quad - 2[-(k^{(1)} - 2\mu^{(1)}/3)/w_1 - M_{1133}^{(0)}][-(k^{(1)} - 2\mu^{(1)}/3)/w_1 - M_{3311}^{(0)}]
\end{aligned} \tag{A.2}$$

and

$$\begin{aligned}
q_1 &= r_1 - a_1, \quad q_2 = r_2 - a_2, \\
q_3 &= r_3 - a_3, \quad q_4 = r_4 - a_4,
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}
r_1 &= M_{3333}^{(0)}/[(M_{1111}^{(0)} + M_{1122}^{(0)})M_{3333}^{(0)} - 2M_{1133}^{(0)}M_{3311}^{(0)}], \\
r_2 &= -M_{1133}^{(0)}/[(M_{1111}^{(0)} + M_{1122}^{(0)})M_{3333}^{(0)} - 2M_{1133}^{(0)}M_{3311}^{(0)}], \\
r_3 &= -M_{3311}^{(0)}/[(M_{1111}^{(0)} + M_{1122}^{(0)})M_{3333}^{(0)} - 2M_{1133}^{(0)}M_{3311}^{(0)}], \\
r_4 &= (M_{1111}^{(0)} + M_{1122}^{(0)})/[(M_{1111}^{(0)} + M_{1122}^{(0)})M_{3333}^{(0)} - 2M_{1133}^{(0)}M_{3311}^{(0)}]
\end{aligned} \tag{A.4}$$

and

$$\begin{aligned}
a_1 &= r_1(p_1 r_1 + 2p_3 r_2) + 2r_3(p_2 r_1 + p_4 r_2), \\
a_2 &= r_2(p_1 r_1 + 2p_3 r_2) + r_4(p_2 r_1 + p_4 r_2), \\
a_3 &= r_3(p_4 r_4 + 2p_2 r_3) + r_1(p_3 r_4 + p_1 r_3), \\
a_4 &= r_4(p_4 r_4 + 2p_2 r_3) + 2r_2(p_3 r_4 + p_1 r_3),
\end{aligned} \tag{A.5}$$

where

$$\begin{aligned}
p_1 &= \frac{1}{4\pi}(\bar{G}_{1111} + \bar{G}_{1212}), \\
p_2 &= \frac{1}{4\pi}\bar{G}_{3131}, \\
p_3 &= \frac{1}{4\pi}\bar{G}_{1313}, \\
p_4 &= \frac{1}{4\pi}\bar{G}_{3333}.
\end{aligned} \tag{A.6}$$

Here \bar{G}_{ijkl} are written as

$$\begin{aligned}\bar{G}_{1111} &= \bar{G}_{2222} = \frac{\pi}{2} \int_0^1 \omega(1-x^2) \{ [f(1-x^2) + h\rho^2x^2][(3e+d)(1-x^2) + 4f\rho^2x^2] \\ &\quad - g^2\rho^2x^2(1-x^2) \} dx, \\ \bar{G}_{3333} &= 4\pi \int_0^1 \omega\rho^2x^2 [d(1-x^2) + f\rho^2x^2] [e(1-x^2) + f\rho^2x^2] dx, \\ \bar{G}_{1122} &= \frac{\pi}{2} \int_0^1 \omega(1-x^2) \{ [f(1-x^2) + h\rho^2x^2][(e+3d)(1-x^2) + 4f\rho^2x^2] \\ &\quad - 3g^2\rho^2x^2(1-x^2) \} dx, \\ \bar{G}_{1133} &= \bar{G}_{2233} = 2\pi \int_0^1 \omega\rho^2x^2 \{ [(d+e)(1-x^2) + 2f\rho^2x^2][f(1-x^2) + h\rho^2x^2] \\ &\quad - g^2\rho^2x^2(1-x^2) \} dx, \\ \bar{G}_{3311} &= \bar{G}_{3322} = 2\pi \int_0^1 \omega(1-x^2) [d(1-x^2) + f\rho^2x^2] [e(1-x^2) + f\rho^2x^2] dx, \\ \bar{G}_{1212} &= \frac{\pi}{2} \int_0^1 \omega(1-x^2)^2 \{ g^2\rho^2x^2 - (d-e)[f(1-x^2) + h\rho^2x^2] \} dx, \\ \bar{G}_{11313} &= \bar{G}_{2323} = -(2\pi) \int_0^1 \omega g\rho^2x^2(1-x^2) [e(1-x^2) + f\rho^2x^2] dx, \tag{A.7}\end{aligned}$$

where

$$\omega^{-1} = [e(1-x^2) + f\rho^2x^2] \{ [d(1-x^2) + f\rho^2x^2][f(1-x^2) + h\rho^2x^2] - g^2\rho^2x^2(1-x^2) \},$$

in which ρ is the aspect ratio of the inclusion.

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