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Asymptotic analysis based on K–S constitutive law for a rubber wedge compressed by a line load at its tip

S.H. Chen [a,*], Y.C. Gao [b]

[a] *LNM, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, China*
[b] *Institute of Mechanics, Northern Jiaotong University, Beijing 100044, China*

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Abstract

In the present paper, a rubber wedge compressed by a line load at its tip is asymptotically analyzed using a special constitutive law proposed by Knowles and Sternberg (K–S elastic law) [J. Elasticity 3 (1973) 67]. The method of dividing sectors proposed by Gao [Theoret. Appl. Fract. Mech. 14 (1990) 219] is used. Domain near the wedge tip can be divided into one expanding sector and two narrowing sectors. Asymptotic equations of the strain–stress field near the wedge tip are derived and solved numerically. The deformation pattern near a wedge tip is completely revealed. A special case, i.e. a half space compressed by a line load is solved while the wedge angle is π.

Keywords: Rubber wedge; K–S elastic law; Large strain asymptotic analysis

1. Introduction

It is well known that in the framework of linear elastic theory, the typical problem of a wedge acted by a line force was solved by Michell (1902) and a typical solution was given. According to the linear solution (Michell, 1902), when the wedge tip is approached, the stress and strain will tend to infinity, i.e. the field possesses singularity and the singular field contains a large deformation area. For ordinary engineering materials one can only consider the small deformation domain. Rubber materials can sustain large strain, therefore the stress–strain behavior of singular field must be analyzed by nonlinear theory. There are two obstacles for solving the nonlinear singular field: firstly, the inherently nonlinear geometry is difficult to describe; secondly, when the singular point is approached, because of the tremendously large strain, the ordinary constitutive relation may become physically invalid. In 1973, Knowles and Sternberg gave an elastic constitutive relation, which contained three parameters. Their constitutive relation was used to analyze the singular field near a plane strain crack tip (Knowles and Sternberg, 1973, 1974). In 1994, a rubber half space was analyzed for tension case by Simmonds and Warne (1994) based on the K–S elastic law. Problems of a rubber wedge tensioned by a line load (Chen and Gao, 2001a), a rubber cone tensioned by a concentrated force at its apex (Gao and Chen, 2000), a rigid wedge contacting with a rubber notch (Chen and Gao, 2001b) and a rubber wedge contacting with a rigid notch (Chen and Li, 2000) were also

* Corresponding author.
E-mail address: chenshaohua72@hotmail.com (S.H. Chen).

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solved with K–S elastic law. As for the compression case of a rubber wedge or a rubber half space with K–S elastic law, there are still no results published. In 1990, Gao gave a constitutive relation for rubber-like materials in which the stress tensor was decomposed into a spherical part and a deviatoric part. This constitutive relation kept reasonable when strain tended to infinity, and it was used to analyze the singular field of a plane strain crack (Gao, 1990). For some typical problems such as a rubber wedge (Gao and Gao, 1994a,b) tensioned or compressed by a line load, a rubber cone (Gao and Liu, 1995) tensioned by a concentrated force were also solved using the constitutive relation proposed by Gao (1990). In 1997, Gao gave another elastic law, which reflects two other kinds of material responses, i.e. tension and compression by tension stress and by compression stress (cf. Gao, 1997). With this kind of elastic law, the problem of a rubber wedge under tension by a line force was solved by Chen et al. (2000). The problem of a cone under tension by a concentrated force was solved by Gao and Chen (2001). An important problem is whether the mechanical behaviors of the large strain field, revealed by Gao et al. are due to this particular elastic law or not. Typical problems should be investigated to answer it. In the present paper, a rubber wedge compressed by a line load at its tip is analyzed using the K–S constitutive relation but with the same method as that used in Gao and Gao (1994b) and Gao (1998). Though only the elastic law changed, the analysis must be totally redone.

Basic definitions and K–S elastic law for nonlinear theory are given in Section 2. In Section 3, domain near a wedge tip is divided into one expanding sector and two narrowing sectors and the corresponding mapping functions for the two kinds of sectors are given, respectively. Solutions to the expanding sector are given in Section 4 and in Section 5 the narrowing sector is solved. In Section 6 the continuity conditions between the expanding sector and the narrowing sector are given.

2. Basic definitions and K–S elastic law

A three-dimensional domain of material is considered. Let x^i $(i = 1, 2, 3)$ denote the Lagrangian coordinates of a point. $\boldsymbol{P}$ and $\boldsymbol{Q}$ denote the position vectors of a point before and after deformation, respectively. $\boldsymbol{u}$ is the displacement of the point, then we have

$$\boldsymbol{Q} = \boldsymbol{P} + \boldsymbol{u}. \tag{1}$$

Two sets of local triads are defined

$$\boldsymbol{P}_i = \frac{\partial \boldsymbol{P}}{\partial x^i}, \quad \boldsymbol{Q}_i = \frac{\partial \boldsymbol{Q}}{\partial x^i}. \tag{2}$$

The displacement gradient tensor is

$$\boldsymbol{F} = \boldsymbol{Q}_i \otimes \boldsymbol{P}^i \tag{3}$$

in which $\otimes$ is the dyadic symbol and the summation rule is implied, $\boldsymbol{P}^i$ is the conjugate of $\boldsymbol{P}_i$, i.e.

$$\boldsymbol{P}^i \cdot \boldsymbol{P}_j = \delta^i_j. \tag{4}$$

The right and left Cauchy–Green strain tensors are

$$\boldsymbol{D} = \boldsymbol{F}^{\mathrm{T}} \cdot \boldsymbol{F} = (\boldsymbol{Q}_i \cdot \boldsymbol{Q}_j)\boldsymbol{P}^i \otimes \boldsymbol{P}^j, \tag{5}$$

$$\boldsymbol{d} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathrm{T}} = (\boldsymbol{P}^i \cdot \boldsymbol{P}^j)\boldsymbol{Q}_i \otimes \boldsymbol{Q}_j, \tag{6}$$

where the superscript T indicates transposition. $\boldsymbol{D}$, $\boldsymbol{d}$ possess the same invariant, for example

$$I = \boldsymbol{D} : \boldsymbol{E} = \boldsymbol{d} : \boldsymbol{E}, \tag{7}$$

where $\boldsymbol{E}$ denotes the unit tensor, and : denotes dual multiplication. Besides, a commonly used invariant is the volume inflation ratio J:

$$J = \det \mid \boldsymbol{Q}_i \cdot \boldsymbol{P}^j \mid = \frac{(\boldsymbol{Q}_1, \boldsymbol{Q}_2, \boldsymbol{Q}_3)}{(\boldsymbol{P}_1, \boldsymbol{P}_2, \boldsymbol{P}_3)}, \tag{8}$$

where $(*_1, *_2, *_3)$ denotes the mixed product of $*_1, *_2, *_3$.

The elastic law can be introduced from the strain energy per unit undeformed volume given in Knowles and Sternberg (1973)

$$U = (AI + BJ + CIJ^{-2})^n \tag{9}$$

in which A, B, C, n are the material constants. Some detailed discussions on the material constants were given in Knowles and Sternberg (1973), and the restriction on the constants can be written as

$$A > 0, \quad C > 0, \quad |B| < 2A, \quad 1/2 < n < \infty. \tag{10}$$

The Kirchhoff stress is

$$\boldsymbol{\sigma} = 2\frac{\partial U}{\partial \boldsymbol{D}} = 2n(AI + BJ + CIJ^{-2})^{n-1}\left[A\boldsymbol{E} + \tfrac{1}{2}BJ\boldsymbol{D}^{-1} + CJ^{-2}(\boldsymbol{E} - I\boldsymbol{D}^{-1})\right]. \tag{11}$$

The corresponding Cauchy stress is

$$\boldsymbol{\tau} = J^{-1}\boldsymbol{F}\cdot\boldsymbol{\sigma}\cdot\boldsymbol{F}^{\mathrm{T}} = 2nJ^{-1}(AI + BJ + CIJ^{-2})^{n-1}\left[A\boldsymbol{d} + \tfrac{1}{2}BJ\boldsymbol{E} + CJ^{-2}(\boldsymbol{d} - I\boldsymbol{E})\right]. \tag{12}$$

Under consideration is the plane strain case, so there is the following relation:

$$\boldsymbol{d} - I\boldsymbol{E} = -J^2\boldsymbol{d}^{-1}. \tag{13}$$

Substituting Eq. (13) into Eq. (12), we obtain

$$\boldsymbol{\tau} = 2nJ^{-1}(AI + BJ + CIJ^{-2})^{n-1}\left[A\boldsymbol{d} + \tfrac{1}{2}BJ\boldsymbol{E} - C\boldsymbol{d}^{-1}\right]. \tag{14}$$

The equilibrium equation is expressed as follows:

$$\boldsymbol{Q}^i\cdot\frac{\partial\boldsymbol{\tau}}{\partial x^i} = 0. \tag{15}$$

3. Dividing of sectors

We consider the deformation pattern of a wedge under compression. The cross-section of a wedge before loading is shown in Fig. 1(a), whereas Fig. 1(b) shows the same cross-section after loading. For simplicity, only the symmetric loading case is considered here. In order to describe the deformation, the wedge tip field is divided into three sectors, one is called expanding sector E, and another two are called narrowing sectors N and N'. Before deformation N and N' occupy almost the whole domain surrounding the wedge tip, while E is very narrow. But after deformation E becomes very wide and occupies almost the whole domain surrounding the wedge tip, while N and N' shrivel to be two narrowing sectors as shown in Figs. 1(a) and (b). The features of deformation in domain N (or N') and E are quite different so that they must be described individually. Two sets of coordinate systems are introduced. One set is (R,Θ) that denotes polar coordinate before deformation and another set is (r,θ) that denotes polar coordinate after deformation as shown in Figs. 1(a) and (b). Θ_0 denotes a half of the wedge angle. The mapping functions for a point before deformation and after deformation in expanding sector E are

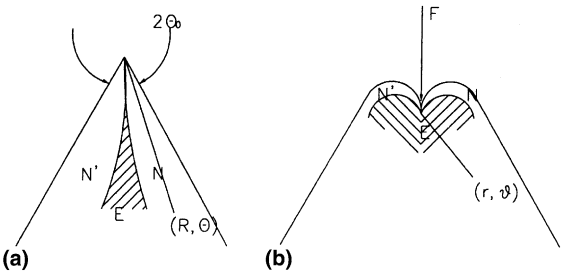

Fig. 1. Plots of polar coordinate systems and the dividing sectors in the present paper, E and N (N') denote the expanding sector and narrowing sectors before deformation, respectively. Θ_0 denotes a half of the wedge angle. (R,Θ) is the polar coordinate system before deformation; (r,θ) is the polar coordinate system after deformation.

$$R = r^{1-\delta}f(\theta),\qquad \Theta = r^{l}m(\theta),\quad |\theta| < \pi \tag{16}$$

in which δ, l are the positive exponents to be determined. $f(\theta)$ and $m(\theta)$ are the unknown functions. The mapping functions in narrowing sector N are

$$R = r^{1+\beta}h(\xi),\qquad \Theta = g(\xi),\quad \xi = r^{-\alpha}(\pi-\theta),\ \ \theta\to\pi \tag{17}$$

in which $0 < \xi < \infty$, α, β are the positive exponents to be determined. $h(\xi)$ and $g(\xi)$ are the unknown functions.

The mapping functions in narrowing sector N' can be similarly given but omitted here, since only the symmetric problem is considered in the present paper.

Actually, there is no strict boundary between different sectors. Mapping functions (16) and (17) should be convertible to each other at boundaries.

4. Expanding sector, E

4.1. Asymptotic equations

We define the following two sets of unit vectors:

$$\boldsymbol{e}_R = \frac{\partial \boldsymbol{P}}{\partial R} = \boldsymbol{P}_R, \quad \boldsymbol{e}_\Theta = \frac{1}{R}\frac{\partial \boldsymbol{P}}{\partial \Theta} = \frac{1}{R}\boldsymbol{P}_\Theta, \tag{18}$$

$$\boldsymbol{e}_r = \frac{\partial \boldsymbol{Q}}{\partial r} = \boldsymbol{Q}_r, \quad \boldsymbol{e}_\theta = \frac{1}{r}\frac{\partial \boldsymbol{Q}}{\partial \theta} = \frac{1}{r}\boldsymbol{Q}_\theta. \tag{19}$$

Using Eqs. (16), (18), (19) we obtain

$$\begin{aligned} \boldsymbol{P}_r &= r^{-\delta} f[(1-\delta)\boldsymbol{e}_R + l r^l m \boldsymbol{e}_\Theta], \\ \boldsymbol{P}_\theta &= r^{1-\delta}[f'\boldsymbol{e}_R + r^l m' f \boldsymbol{e}_\Theta], \end{aligned} \tag{20}$$

then

$$\begin{aligned} \boldsymbol{P}^r &= r^{\delta-l} s^{-1}(r^l f m' \boldsymbol{e}_R - f' \boldsymbol{e}_\Theta), \\ \boldsymbol{P}^\theta &= r^{\delta-l-1} s^{-1} f[-l r^l m \boldsymbol{e}_R + (1-\delta)\boldsymbol{e}_\Theta] \end{aligned} \tag{21}$$

in which

$$s = f[(1-\delta) f m' - l f' m]. \tag{22}$$

Using Eqs. (3)–(6) and (18)–(21) we have

$$\begin{aligned} \boldsymbol{d} = r^{2\delta-2l} s^{-2}[&f'^2 \boldsymbol{e}_r \otimes \boldsymbol{e}_r + f^2(1-\delta)^2 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta \\ &- (1-\delta) f f'(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r)], \end{aligned} \tag{23}$$

$$\begin{aligned} \boldsymbol{d}^{-1} = r^{-2\delta}[&(1-\delta)^2 f^2 \boldsymbol{e}_r \otimes \boldsymbol{e}_r + f'^2 \boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta \\ &+ (1-\delta) f f'(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r)] \end{aligned} \tag{24}$$

then

$$I_1 = r^{2\delta-2l} T s^{-2}, \quad J = r^{2\delta-l} s^{-1}, \tag{25}$$

where $T = (1-\delta)^2 f^2 + f'^2$.

Substituting Eqs. (23)–(25) into Eq. (14) and assuming $\boldsymbol{d}$ and $\boldsymbol{d}^{-1}$ are the same order, we have

$$l = 2\delta. \tag{26}$$

Using the equilibrium condition of external and internal loading, i.e. $\boldsymbol{\tau} \sim r^{-1}$, then

$$\delta = 1/2n. \tag{27}$$

Eq. (27) in conjunction with Eq. (16) implies that $n > 1/2$ in order that the displacements at the wedge tip do not become infinite. This is consistent with Eq. (10), which is arrived in Knowles and Sternberg (1973).

The components of stress can be written as

$$\begin{aligned} \tau_{rr} &= 2nsT^{n-1}P^{n-1}[As^{-2}f'^2 - C(1-\delta)^2 f^2] r^{-1}, \\ \tau_{\theta\theta} &= 2nsT^{n-1}P^{n-1}[As^{-2}f^2(1-\delta)^2 - Cf'^2] r^{-1}, \\ \tau_{r\theta} &= -2ns(1-\delta)T^{n-1}P^n f f' r^{-1}, \end{aligned} \tag{28}$$

where

$$P = As^{-2} + C. \tag{29}$$

For convenience, we define

$$\tau_{ij} = T_{ij} r^{-1} \tag{30}$$

then

$$\begin{aligned} T_{rr} &= 2nsT^{n-1}P^{n-1}[As^{-2}f'^2 - C(1-\delta)^2 f^2], \\ T_{\theta\theta} &= 2nsT^{n-1}P^{n-1}[As^{-2}f^2(1-\delta)^2 - Cf'^2], \\ T_{r\theta} &= -2ns(1-\delta)T^{n-1}P^n f f'. \end{aligned} \tag{31}$$

Substituting Eqs. (28)–(30) into equilibrium equation (15), we have

$$\frac{\partial T_{r\theta}}{\partial \theta} - T_{\theta\theta} = 0, \quad \frac{\partial T_{\theta\theta}}{\partial \theta} + T_{r\theta} = 0. \tag{32}$$

Combining Eqs. (31) and (32), we obtain the asymptotic equations for the expanding sector

$$\begin{aligned} a_{11} f'' + a_{12} m'' + a_{13} &= 0, \\ a_{21} f'' + a_{22} m'' + a_{23} &= 0, \end{aligned} \tag{33}$$

where

$$\begin{aligned} a_{11} &= (1-\delta)[-lms^2 P f^2 f' + 2(n-1)s^3 T^{-1} P f f'^2 \\ &\quad + 2nAlmf^2 f' + s^3 P f], \\ a_{12} &= (1-\delta)^2 f^3 f'(s^2 P - 2nA), \\ a_{13} &= (1-\delta)\{f f'^2[(1-\delta) f m' - l f' m](s^2 P - 2nA) \\ &\quad + f^2 f'[(1-\delta) f' m' - l f' m'](s^2 P - 2nA) \\ &\quad + 2(n-1)(1-\delta)^2 s^3 T^{-1} P f^2 f'^2 + s^3 P f'^2\} \\ &\quad + s[Af'^2(1-\delta)^2 - Cs^2 f'^2], \end{aligned} \tag{34}$$

$$
\begin{aligned}
a_{21} &= \{2(n-1)s^3T^{-1}f' - [s^2 - 2A(n-1)P^{-1}]flm\} \\
&\quad \times [As^{-2}f^2(1-\delta)^2 - Cf'^2] \\
&\quad + 2Al(1-\delta)^2mf^3 - 2Cs^3f', \\
a_{22} &= (1-\delta)f^2[s^2 - 2A(n-1)P^{-1}] \\
&\quad \times [As^{-2}f^2(1-\delta)^2 - Cf'^2] - 2A(1-\delta)^3f^4, \\
a_{23} &= \{f'[s^2 - 2A(n-1)P^{-1}][(1-\delta)fm' - lf'm] \\
&\quad + [s^2 - 2A(n-1)P^{-1}]f[(1-\delta)f'm' - lf'm'] \\
&\quad + 2(n-1)T^{-1}s^3(1-\delta)^2ff'\} \\
&\quad \times [As^{-2}f^2(1-\delta)^2 - Cf'^2] + 2A(1-\delta)^2 \\
&\quad \times \{sff' - f^2f'[(1-\delta)fm' - lf'm] \\
&\quad - f^3[(1-\delta)f'm' - lf'm']\} - (1-\delta)Ps^3ff'.
\end{aligned} \tag{35}
$$

4.2. Solutions to sector E

Since only the symmetric case is considered in the present paper, at the bisector, i.e. the line of $\theta = 0$, the shear stress vanishes, that is

$$\tau_{r\theta}(0) = 0, \quad \text{i.e. } f'(0) = 0. \tag{36}$$

At the bisector $\theta = 0$, which is corresponding to $\Theta = 0$ before deformation, from the second equation of Eq. (16), we know that

$$m(0) = 0. \tag{37}$$

Defining the initial values of functions $f(\theta)$ and $m'(\theta)$ as

$$f(0) = f_0, \quad m'(0) = m_1, \tag{38}$$

where f_0, m_1 are the parameters to be determined.

In order to connect the different sectors, when $\theta \to \pi$, the natural boundary conditions are required

$$f(\pi) = 0, \quad m(\pi) = \infty. \tag{39}$$

Eq. (33) under conditions (36)–(39) can be solved numerically. From the numerical results, we find surprisingly whatever the values of the existing parameters are, when $\theta \to \pi$, the following results are always true:

$$s = \left[\frac{A}{C}(2n-1)\right]^{1/2}, \quad s' = 0. \tag{40}$$

If we let

$$s \equiv \left[\frac{A}{C}(2n-1)\right]^{1/2}$$

for the domain $\theta \in [0, \pi)$ and substitute it into Eq. (33), we find the two equations of (33) become the same one, which means that

$$s \equiv \left[\frac{A}{C}(2n-1)\right]^{1/2}$$

is the true solution to Eq. (33). The reduced equation is

$$
\begin{aligned}
&(1-\delta)^3f^3 + (1-\delta)(4n^2 - 4n^2\delta - 4n \\
&\quad + 4n\delta + 1)ff'^2 + 2n(2n-1)f'^2f'' \\
&\quad + 2n(1-\delta)^2f^2f'' = 0.
\end{aligned} \tag{41}
$$

Using Eq. (40), we rewrite Eq. (22) as follows:

$$m' = \frac{1}{(1-\delta)f^2}\left\{\left[\frac{A}{C}(2n-1)\right]^{1/2} + lff'm\right\}. \tag{42}$$

Eq. (42) shows that $m'(0) = m_1$ is no longer a free parameter but related with the value of parameter f_0, that is

$$m_1 = \frac{1}{(1-\delta)f_0^2}\left[\frac{A}{C}(2n-1)\right]^{1/2}. \tag{43}$$

When $\theta \to \pi$, from numerical calculation we find that the following result is always true:

$$f \approx C_f(\pi - \theta), \quad m \approx C_m(\pi - \theta)^{-1}. \tag{44}$$

Combining Eqs. (22), (40) and (44), we get the following relation:

$$C_f^2C_m = (1+\delta)^{-1}(2n-1)^{1/2}\left(\frac{A}{C}\right)^{1/2}. \tag{45}$$

The curves of $\theta - f$, $\theta - m$ from numerical solution to Eqs. (41) and (42) are shown in Fig. 2 for $A = 2.0$, $B = 3.0$, $C = 2.0$ and various values of n.

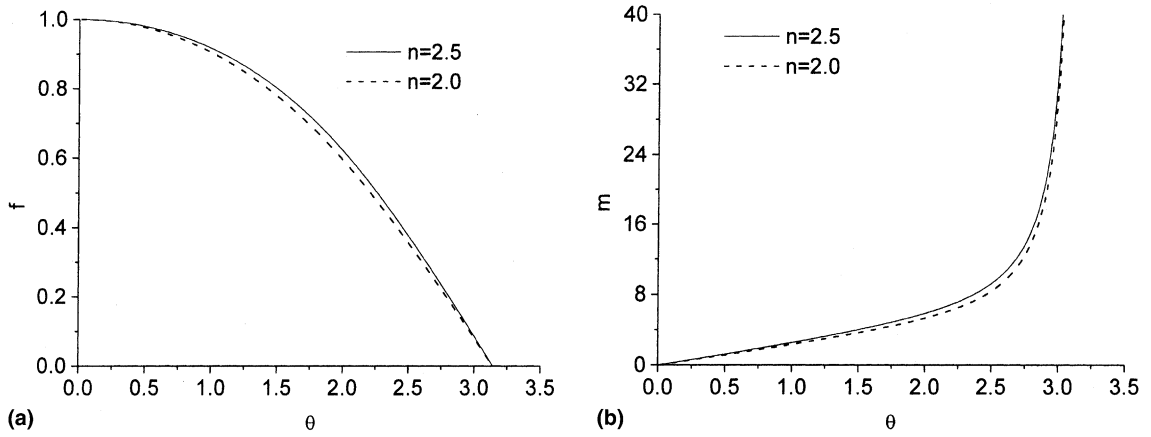

Fig. 2. Curves of $\theta - f$ (a) and $\theta - m$ (b) with $A = 2.0$, $B = 3.0$, $C = 2.0$ and various values of n for the expanding sector.

5. Narrowing sector, N

5.1. Asymptotic equations

Using Eqs. (17)–(19) we obtain

$$\begin{aligned}\boldsymbol{P}_r &= r^{\beta}\{[(1+\beta)h - \alpha\xi h']\boldsymbol{e}_R - \alpha\xi h g'\boldsymbol{e}_\Theta\},\\ \boldsymbol{P}_\theta &= r^{1+\beta-\alpha}[-h'\boldsymbol{e}_R - hg'\boldsymbol{e}_\Theta]\end{aligned} \tag{46}$$

then

$$\begin{aligned}\boldsymbol{P}^r &= r^{-\beta}v^{-1}(-hg'\boldsymbol{e}_R + h'\boldsymbol{e}_\Theta),\\ \boldsymbol{P}^\theta &= r^{\alpha-\beta-1}v^{-1}\{\alpha\xi hg'\boldsymbol{e}_R + [(1+\beta)h - \alpha\xi h']\boldsymbol{e}_\Theta\},\end{aligned} \tag{47}$$

where

$$v = -(1+\beta)h^2 g'. \tag{48}$$

Combining Eqs. (3)–(6) and (46)–(48) we have

$$\begin{aligned}\boldsymbol{d} = {} & r^{-2\beta}v^{-2}(h^2g'^2 + h'^2)\boldsymbol{e}_r \otimes \boldsymbol{e}_r + r^{2\alpha-2\beta}v^{-2}\\ & \times \{\alpha^2\xi^2h^2g'^2 + [(1+\beta)h - \alpha\xi h']^2\}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta\\ & + r^{\alpha-2\beta}v^{-2}\{-\alpha\xi h^2g'^2 + h'[(1+\beta)h - \alpha\xi h']\}\\ & \times (\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r),\end{aligned} \tag{49}$$

$$\begin{aligned}\boldsymbol{d}^{-1} = {} & r^{2\beta}\{[(1+\beta)h - \alpha\xi h']^2 + \alpha^2\xi^2h^2g'^2\}\boldsymbol{e}_r \otimes \boldsymbol{e}_r\\ & + r^{2\beta-2\alpha}(h'^2 + h^2g'^2)\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + r^{2\beta-\alpha}\{\alpha\xi h^2g'^2\\ & - h'[(1+\beta)h - \alpha\xi h']\}(\boldsymbol{e}_r \otimes \boldsymbol{e}_\theta + \boldsymbol{e}_\theta \otimes \boldsymbol{e}_r)\end{aligned} \tag{50}$$

then

$$I_1 = r^{-2\beta}v^{-2}(h^2g'^2 + h'^2), \quad J = r^{\alpha-2\beta}v^{-1}. \tag{51}$$

In view that the volume inflation ratio has no singularity, we obtain

$$\alpha = 2\beta. \tag{52}$$

Using the equilibrium condition of external and internal loading, i.e. $\tau \sim r^{-1}$, then

$$\beta = 1/2n. \tag{53}$$

Then the components of stress in narrowing sector can be written as

$$\begin{aligned}\tau_{rr} &= 2nAu^nv^{-1}(Av^{-2} + C)^{n-1}r^{-1},\\ \tau_{\theta\theta} &= -2nCu^nv(Av^{-2} + C)^{n-1}r^{-1},\\ \tau_{r\theta} &= 2nvu^{n-1}(Av^{-2} + C)^n\{-\alpha\xi h^2g'^2 + h'[(1+\beta)h\\ &\quad - \alpha\xi h']\}r^{-1+\alpha},\end{aligned} \tag{54}$$

where

$$u = h'^2 + h^2g'^2. \tag{55}$$

In order to simplify Eq. (54), we introduce the following coordinate transformation:

$$\begin{aligned}\eta &= r\left(1 + \frac{\alpha}{2}\theta^{*2} + \frac{\alpha^2}{8}\theta^{*4} + \frac{\alpha^3}{48}\theta^{*6} + \cdots\right),\\ \xi &= r^{-\alpha}\theta^*, \quad \theta^* = \pi - \theta.\end{aligned} \tag{56}$$

The series in Eq. (56) are calculated according to the orthogonal condition of ξ, η coordinates. The coordinate lines of (η, ξ) are shown in Fig. 3. Us-

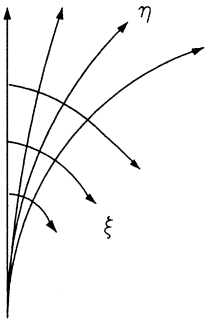

Fig. 3. Plot of (n,ξ) coordinate system introduced in the narrowing sectors. Vectors η and ξ are orthogonal.

ing Eq. (56) and only taking the dominant terms, we obtain

$$\boldsymbol{Q}_\eta = \frac{\partial \boldsymbol{Q}}{\partial \eta} = \boldsymbol{e}_r - \alpha\xi\eta^\alpha \boldsymbol{e}_\theta, \tag{57}$$

$$\boldsymbol{Q}_\xi = \frac{\partial \boldsymbol{Q}}{\partial \xi} = -\eta^{1+\alpha}(\alpha\xi\eta^\alpha \boldsymbol{e}_r + \boldsymbol{e}_\theta) \tag{58}$$

then

$$\boldsymbol{Q}^\eta = \boldsymbol{Q}_\eta = \boldsymbol{e}_r - \alpha\xi\eta^\alpha \boldsymbol{e}_\theta, \tag{59}$$

$$\boldsymbol{Q}^\xi = \eta^{-2-2\alpha}\boldsymbol{Q}_\xi = -\eta^{-1-\alpha}(\alpha\xi\eta^\alpha \boldsymbol{e}_r + \boldsymbol{e}_\theta). \tag{60}$$

We define that

$$\boldsymbol{Q}_\eta = \boldsymbol{e}_\eta, \quad \boldsymbol{Q}_\xi = \eta^{1+\alpha}\boldsymbol{e}_\xi \tag{61}$$

then the relations of unit vectors can be written as

$$\boldsymbol{e}_r = \boldsymbol{e}_\eta - \alpha\xi\eta^\alpha \boldsymbol{e}_\xi, \quad \boldsymbol{e}_\theta = -\boldsymbol{e}_\xi - \alpha\xi\eta^\alpha \boldsymbol{e}_\eta. \tag{62}$$

Combining Eqs. (62) and (54), the components of stress in the (η,ξ) coordinate system are expressed as

$$\begin{aligned}
\tau_{\eta\eta} &= 2nAu^n v^{-1}(Av^{-2}+C)^{n-1}\eta^{-1},\\
\tau_{\xi\xi} &= -2nCu^n v(Av^{-2}+C)^{n-1}\eta^{-1},\\
\tau_{\xi\eta} &= -2nvu^{n-1}(Av^{-2}+C)^n(1+\beta)hh'\eta^{-1+\alpha}.
\end{aligned} \tag{63}$$

Let $\tau_{\xi\xi} = T_{\xi\xi}\eta^{-1}$, $\tau_{\eta\eta} = T_{\eta\eta}\eta^{-1}$, $\tau_{\xi\eta} = T_{\xi\eta}\eta^{-1+\alpha}$, then

$$\begin{aligned}
T_{\eta\eta} &= 2nAu^n v^{-1}(Av^{-2}+C)^{n-1},\\
T_{\xi\xi} &= -2nCu^n v(Av^{-2}+C)^{n-1},\\
T_{\xi\eta} &= -2nvu^{n-1}(Av^{-2}+C)^n(1+\beta)hh'.
\end{aligned} \tag{64}$$

Considering the singularity of the stress components, the equilibrium equations in (η,ξ) coordinate system can be simplified as

$$\begin{aligned}
&\frac{\partial T_{\xi\xi}}{\partial \xi} = 0,\\
&(1+\alpha)(T_{\eta\eta} - T_{\xi\xi}) + \frac{\partial T_{\xi\eta}}{\partial \xi} - 2n\beta T_{\eta\eta} = 0.
\end{aligned} \tag{65}$$

From the first equation of Eq. (65), we know that

$$T_{\xi\xi} = \text{constant}. \tag{66}$$

Assuming the constant to be D^*, then combining the second equation of Eq. (64), we have

$$u^n v(Av^{-2}+C)^{n-1} = D, \tag{67}$$

where

$$D = \frac{-D^*}{2nC}. \tag{68}$$

Substituting Eq. (64) into Eq. (65), we obtain the finally asymptotic equations of narrowing sector N

$$\begin{aligned}
&a_{11}h'' + a_{12}g'' + a_{13} = 0,\\
&a_{21}h'' + a_{22}g'' + a_{23} = 0
\end{aligned} \tag{69}$$

in which

$$\begin{aligned}
a_{11} &= -(1+\beta)u^{-1}(Av^{-2}+C)h(1-2u^{-1}h'^2),\\
a_{12} &= -2(1+\beta)h^3h'[A(1+\beta)u^{-1}v^{-3}\\
&\quad - u^{-2}(Av^{-2}+C)g'],\\
a_{13} &= (1+\alpha)(Av^{-2}+C) - (1+\beta)\\
&\quad \times [-2u^{-2}(Av^{-2}+C)h^2h'^2g'^2\\
&\quad + 4A(1+\beta)v^{-3}u^{-1}h^2h'^2g'\\
&\quad + u^{-1}(Av^{-2}+C)h'^2] - 2nA\beta v^{-2},
\end{aligned} \tag{70}$$

$$\begin{aligned}
a_{21} &= 2nvh'(Av^{-2}+C),\\
a_{22} &= 2nvh^2g'(Av^{-2}+C) - (1+\beta)h^2u(Av^{-2}+C)\\
&\quad + 2(n-1)Auv^{-2}(1+\beta)h^2,\\
a_{23} &= 2nv(Av^{-2}+C)hh'g'^2 - 2(1+\beta)u(Av^{-2}+C)\\
&\quad \times hh'g' + 4(n-1)A(1+\beta)uv^{-2}hh'g'.
\end{aligned} \tag{71}$$

5.2. Solution to sector N

From Eqs. (66) and (67), we know that at $\theta = \pi$ (or $\xi = 0$) there exists the following equation:

$$\tau_{\xi\xi}(0) = -2nCD\eta^{-1}, \tag{72}$$

where D can be considered as a parameter that is related to the amplitude of the load. In order to give the other boundary conditions for Eq. (69), attention must be paid to the different deformation features that in the expanding sector E, $R > r$; while in the narrowing sector N, $R < r$. Furthermore, the continuum condition must be met on the boundary between the narrowing and expanding sectors. While $\xi \to \infty$, we have the following condition:

$$h(\infty) = \infty. \tag{73}$$

Considering the feature before and after deformation, we know that the line of $\xi \to \infty$ in narrowing sector is corresponding to that of $\Theta = 0$ before deformation, so we have

$$g(\infty) = 0. \tag{74}$$

The wedge tip under the compression of a line load will form a locally closed notch and the contact boundary is assumed to be frictionless, i.e. at $\theta = \pi$ (or $\xi = 0$), $\tau_{\xi\eta}(0) = 0$, so the corresponding boundary condition is

$$h'(0) = 0. \tag{75}$$

The line of $\xi = 0$ is corresponding to that of $\Theta = \Theta_0$ before deformation, thus we know

$$g(0) = \Theta_0. \tag{76}$$

On the other hand, we define that

$$h(0) = h_0, \quad g'(0) = g_1. \tag{77}$$

Using the second equation of Eq. (63), we can write Eq. (72) as

$$-(1+\beta)h_0^{2(n+1)}g_{(0)}'^{2n+1}[A(1+\beta)^{-2}h_0^{-4}g_{(0)}'^{-2} + C]^{n-1} = D. \tag{78}$$

For a set of given parameters of Θ_0, n, A, B, C, we can adjust the values of h_0 and $g'(0)$ to meet Eq. (74). The corresponding external load can be obtained through D, which denotes the amplitude of the load. While the values of h_0 and $g'(0)$ are known, from Eq. (78), we can obtain the value D. On the other hand, the value of D can also be firstly given, h_0 and $g'(0)$ are not independent through Eq. (78), we adjust one of them to meet Eq. (74).

Numerical solution to Eq. (69) shows that when $\xi \to \infty$:

$$v(\infty) = \left[\frac{A}{C}(2n-1)\right]^{1/2} \tag{79}$$

and the following special forms of $h(\xi)$ and $g(\xi)$ satisfy all the boundary conditions and equilibrium equations at $\xi \to \infty$:

$$h(\xi) = C_h\xi, \quad g(\xi) = C_g\xi^{-1}. \tag{80}$$

It can be easily proved and details are omitted here.

From Eq. (48) and Eqs. (79) and (80), it follows that:

$$C_h^2 C_g = (1+\beta)^{-1}\left(\frac{A}{C}\right)^{1/2}(2n-1)^{1/2}, \tag{81}$$

$$C_h = \left[\frac{A}{C}(2n-1)\right]^{-1/4n} C^{(1-n)/2n}\left(\frac{2n}{2n-1}\right)^{(1-n)/2n} D^{1/2n},$$
$$C_g = A^{(n+1)/n}\left(\frac{C}{2n-1}\right)^{(n-3)/2n}(2n)^{(n-1)/n}(1+\beta)^{-1}D^{-1/n}. \tag{82}$$

Evidently, C_h and C_g are not influenced by Θ_0.

Numerical results for the narrowing sector are given in the present paper. During the process of numerical calculation, the value of ξ tends to be ∞. In order to adjust initial values precisely, we introduce a transformation that is $\gamma = \operatorname{arctg}\xi$, then

$$\frac{\mathrm{d}}{\mathrm{d}\xi} = \cos^2\gamma\frac{\mathrm{d}}{\mathrm{d}\gamma}, \quad \frac{\mathrm{d}^2}{\mathrm{d}\xi^2} = \cos^4\gamma\left(\frac{\mathrm{d}^2}{\mathrm{d}\gamma^2} - 2tg\gamma\frac{\mathrm{d}}{\mathrm{d}\gamma}\right). \tag{83}$$

The boundary conditions become

$$h|_{\gamma=\pi/2} = \infty, \quad g|_{\gamma=\pi/2} = 0. \tag{84}$$

Other boundary conditions keep unchanged. In the calculation, we take $h(0) = 1$, $A = 2.0$, $B = 3.0$, $C = 2.0$, $n = 2.5$ and adjust the value of $g'(0)$ to meet the second equation of Eq. (84) for various Θ_0. The curves of $\gamma - h(\gamma)$, $\gamma - g(\gamma)$ are shown in Fig. 4. Table 1 shows the values of Θ_0 and the corresponding values of $g'(0)$. Fig. 5 shows the curves of $\gamma - h(\gamma)$, $\gamma - g(\gamma)$ for another set of parameters: $h(0) = 1$, $A = 2.0$, $B = 3.0$, $C = 2.0$,

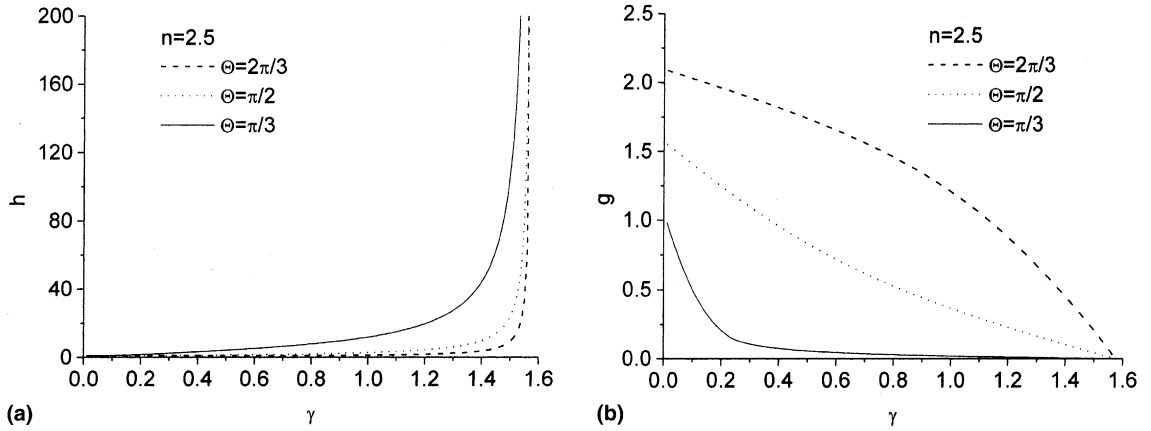

Fig. 4. Curves of γ–h (a) and γ–g (b) with $h(0) = 1$, $A = 2.0$, $B = 3.0$, $C = 2.0$, $n = 2.5$ and various values of Θ_0 for the narrowing sector.

Table 1
($h(0) = 1$, $A = 2.0$, $C = 2.0$, $n = 2.5$)

Θ_0	$\pi/3$	$\pi/2$	$2\pi/3$
$g'(0)$	−6.243	−1.671	−0.6685

$\Theta_0 = \pi/3$. Table 2 shows the values of n and the corresponding values of $g'(0)$.

5.3. Special case

A special case is that a half space is compressed by a line force while $\Theta_0 = \pi/2$. In this case, from numerical calculation we find that

$$v = \left[\frac{A}{C}(2n-1)\right]^{-1/2} \tag{85}$$

then the equilibrium equations become

$$\begin{aligned} &nu^{n-1}u'v(Av^{-2}+C)^{n-1} = 0,\\ &(1+\alpha)(Av^{-2}+C) - (1+\beta)\{-u^{-2}u'(Av^{-2}+C)hh'\\ &\quad + u^{-1}(Av^{-2}+C)h^{2} + u^{-1}(Av^{-2}+C)hh''\}\\ &\quad - 2nA\beta v^{-2} = 0. \end{aligned} \tag{86}$$

From the first equation of Eq. (86), we know that

$$u' = 0. \tag{87}$$

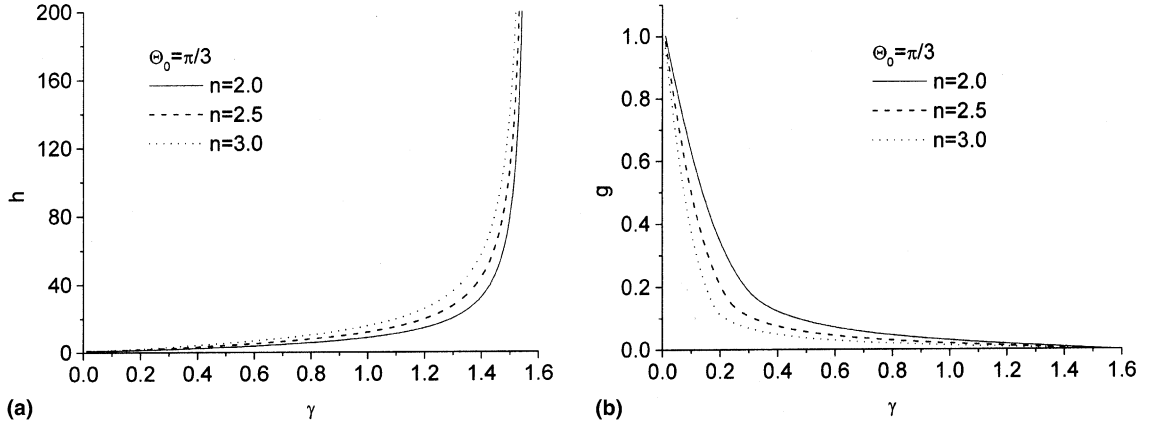

Fig. 5. Curves of γ–h (a) and γ–g (b) with $h(0) = 1$, $A = 2.0$, $B = 3.0$, $C = 2.0$, $\Theta_0 = \pi/3$ and various values of n for the narrowing sector.

Table 2
($h(0) = 1$, $A = 2.0$, $B = 3.0$, $C = 2.0$, $\Theta_0 = \pi/3$)

n	2.0	2.5	3.0
$g'(0)$	−4.4827	−6.2430	−8.2844

That is u is a constant and we assume the constant to be t, i.e. $u = t$. Substituting Eqs. (85) and (87) into the second equation of Eq. (86), we get a reduced equation

$$hh'' + h'^2 = t. \tag{88}$$

The solution to Eq. (88) under the boundary conditions of Eqs. (73)–(76) is

$$h = h_0(1 + K^2\xi^2)^{1/2} \tag{89}$$

then

$$g = -\frac{[A(2n-1)/C]^{1/2}}{Kh_0^2(1+\beta)}\operatorname{arctg}(K\xi) + \frac{\pi}{2}, \tag{90}$$

where

$$K = \frac{\sqrt{t}}{2h_0}. \tag{91}$$

6. Matching of sectors N and E

Since f, m and h, g represent the same quantities, they must be consistent with each other on the boundary of $\theta \to \pi$ and $\xi \to \infty$. Now we consider the continuity condition. According to Eqs. (16) and (44) we have

$$R = C_f r^{1-\delta}(\pi - \theta),$$
$$\Theta = C_m r^l (\pi - \theta)^{-1} \quad \text{on } \theta \to \pi. \tag{92}$$

From Eqs. (17) and (80) we have

$$R = C_h r^{1-\beta}(\pi - \theta),$$
$$\Theta = C_g r^\alpha (\pi - \theta)^{-1} \quad \text{on } \xi \to \infty. \tag{93}$$

Comparing Eq. (92) with Eq. (93), the continuity conditions require that

$$C_f = C_h, \quad C_m = C_g, \tag{94}$$
$$\beta = \delta, \quad \alpha = l. \tag{95}$$

Eqs. (26), (52) and (95) are consistent. Eqs. (45), (81) and (82) give the expression of C_f, C_h, C_m, C_g. f_0 and D are not independent so there is only one parameter that indicates the amplitude of the field.

7. Concluding remarks

- The asymptotic solutions in the present paper disclose the behaviors of the stress and strain field near the singular point, which is the wedge tip point compressed by a line load.
- For rubber like materials, a wedge tip or a notch corner under the compression of a line load will form a locally closed notch.
- The deformation field contains a singular point and the stresses possess the order of r^{-1}. The deformation field can be divided into one expanding sector and two narrowing sectors and the solutions for these sectors are matched completely.
- Specially, from the analysis we can know the feature of the field of a half rubber-like space compressed by a line load.
- While the angle of the wedge tip is larger than π, the solution in the present paper is also adopted, that is the solution is also adapted to the problem of a rubber notch compressed by a line load.
- Typical solution to this kind of typical problem is obtained and it is different from the classical solution to the linear elastic wedge compressed by a line at its tip.
- Comparing the results of this paper with those obtained by Gao and Gao (1994b) and Gao (1998), we find that the basic deformation pattern and the stress singularity do not depend on the concrete form of an individual elastic law.

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