Time correlations of pressure in isotropic turbulence

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The time correlations of pressure modes in stationary isotropic turbulence are investigated under the Kraichnan and Tennekes "random sweeping" hypothesis. A simple model is obtained which predicts a universal form for the time correlations. It implies that the decorrelation process of pressure fluctuations in time is mainly dominated by the sweeping velocity, and the pressure correlations have the same decorrelation time scales as the velocity correlations. These results are verified using direct numerical simulations of isotropic turbulence at two moderate Reynolds numbers; the mode correlations collapse to the universal form when the time separations are scaled by wavenumber times the sweeping velocity, and the ratios of the correlation coefficients of pressure modes to those of velocity modes are approximately unity for the entire range of time separation. *© 2008 American Institute of Physics*. [DOI: 10.1063/1.2870111]

I. INTRODUCTION

In an incompressible flow, pressure fluctuations are related to velocity fluctuations through the Poisson equation

$$\nabla^2 p = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}. \tag{1}$$

All variables used in this article are dimensionless, with the velocity normalized by a reference velocity U_{ref}, pressure normalized by ρU_{ref}^2, spatial coordinates by a reference length L_{ref}, and time by $L_{\mathrm{ref}}/U_{\mathrm{ref}}$, where ρ is density. Since velocity derivatives only appear as a source on the right-hand side of Eq. (1), the small scale pressure fluctuations are mainly determined by the small scale velocity fluctuations. However, in contrast to the $-5/3$ scaling law for the velocity energy spectra $E_u(k)$ in the inertial range, the inertial scaling law for the pressure spectra is controversial.[1] Kolmogorov's theory predicts that pressure spectra should scale as $E_p(k) \sim k^{-7/3}$ (Refs. 2 and 3) in the inertial range. This result has been supported by the experiments[4,5] but challenged by several direct numerical simulations (DNS) of isotropic turbulence.[6–8] The DNS results claim that the pressure spectra should have the same scaling as kinetic energy spectra, that is, $E_p(k) \sim k^{-3/5}$. The recent DNS with 1024^3 grid points by Gotoh and Fukayama[9,10] points out that the pressure spectra scale approximately with $k^{-7/3}$ at lower wavenumbers, followed by a bump of nearly $k^{-5/3}$ scaling at higher wavenumbers. In spite of disagreements on the slope of the spectra, all previous experiments and DNS support Kolmogorov's prediction on the universality of pressure spectra. Note that the pressure spectra mentioned above are the wavenumber spectra. A related problem is the frequency spectra, or more generally the two-time, two-point correlations of pressure fluctuations in physical space. In this paper, we will address the universal form of pressure time correlations.

The spatio-temporal statistics of hydrodynamic pressure fluctuations are relevant to flow-generated sound and flow-structure interaction.[11–13] Kraichnan and Tennekes propose the random sweeping hypothesis for isotropic turbulence,[14–16] which yields a universal form of velocity correlations at two times. This universality is demonstrated by He *et al.*[13] in decaying isotropic turbulence using DNS and LES. He and Wang further suggest an elliptic model of velocity correlations for turbulent shear flows,[17] and the model is verified in turbulent channel flows,[18] in the sense that the time correlations for several spatial separations collapse when the time separations are scaled using this elliptic model. Therefore, the time correlations for velocity have a universal form in isotropic turbulence as well as turbulent shear flows. Those results motivated us to explore the universal form of pressure correlations at two times.

The paper is organized as follows: In Sec. II, a random sweeping model is developed for pressure time correlations in isotropic turbulence. This model is verified using DNS of isotropic turbulence at two moderate Reynolds numbers in Sec. III. The conclusions are presented in Sec. IV.

II. A RANDOM SWEEPING MODEL FOR PRESSURE TIME CORRELATIONS

For brevity, we henceforth refer to the time correlation for a pressure mode simply as the pressure time correlation, and the one for a velocity mode as the velocity time correlation. The pressure time correlation is defined as

$$C_p(k,\tau) = \langle p(\mathbf{k},t)p(-\mathbf{k},t+\tau)\rangle, \tag{2}$$

and its normalized form is defined as

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TABLE I. Relevant parameters and statistical quantities in DNS.[a]

Case	Nodes	$\triangle t$	u_{rms}	L	λ	ξ	CFL	Re_λ
Case 1	128^3	0.002	0.843	1.597	0.599	1.445	0.194	78.75
Case 2	256^3	0.001	0.864	1.471	0.341	1.207	0.267	148.92

[a]Here L is the integral length scale, λ is the Taylor microscale, $\xi = k_{\max}\eta$ indicates the spatial resolution, CFL is the CFL number, and Re_λ is the Taylor-scale Reynolds number.

$$c_p(k,\tau) = \frac{\langle p(\mathbf{k},t)p(-\mathbf{k},t+\tau)\rangle}{\langle p(\mathbf{k},t)p(-\mathbf{k},t)\rangle}. \tag{3}$$

The latter is also called as the correlation coefficient. Similarly, we will define the velocity time correlation and its correlation coefficient, respectively, as

$$C_u(k,\tau) = \langle u_i(\mathbf{k},t)u_i(-\mathbf{k},t+\tau)\rangle \tag{4}$$

and

$$c_u(k,\tau) = \frac{\langle u_i(\mathbf{k},t)u_i(-\mathbf{k},t+\tau)\rangle}{\langle u_i(\mathbf{k},t)u_i(-\mathbf{k},t)\rangle}. \tag{5}$$

The random sweeping hypothesis suggests that the decorrelation process of fluctuating velocities in time is dominated by the sweeping of small eddies by energy-containing eddies. The process can be described mathematically as follows:[15] Consider a Fourier mode $\mathbf{u}(\mathbf{k},t)$ of fluctuating velocity convected by a large scale velocity field $\mathbf{v}$, where $\mathbf{v}$ is a uniform Gaussian field and independent of the fluctuating velocity $\mathbf{u}(\mathbf{k},t)$. Following the assumption in the random sweeping hypothesis that the viscous effect and nonlinear terms in $\mathbf{u}$ may be neglected in the Navier-Stokes equations, we obtain

$$\frac{\partial \mathbf{u}(\mathbf{k},t)}{\partial t} + i(\mathbf{k}\cdot\mathbf{v})\mathbf{u}(\mathbf{k},t) = 0. \tag{6}$$

The solution of Eq. (6) is

$$\mathbf{u}(\mathbf{k},t) = \mathbf{u}(\mathbf{k},0)\exp[-i(\mathbf{k}\cdot\mathbf{v})t]. \tag{7}$$

Then, the velocity time correlation can be expressed as[13,15]

$$C_u(k,\tau) = \langle u_i(\mathbf{k},t)u_i(-\mathbf{k},t)\rangle \exp\left(-\frac{1}{2}v^2k^2\tau^2\right), \tag{8}$$

where the sweeping characteristic velocity $v=|\mathbf{v}|$ is the rms of velocity fluctuations. This is the random sweeping model for velocity time correlations in isotropic turbulence.

Now we solve the Poisson equation for the fluctuating pressure in Fourier space

$$p(k,t) = -\frac{k_ik_j}{k^2}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}[u_i(\mathbf{p},t)u_j(\mathbf{q},t)\delta(\mathbf{p}+\mathbf{q}-\mathbf{k})]. \tag{9}$$

Introduction of Eq. (7) into Eq. (9) at $t+\tau$ gives

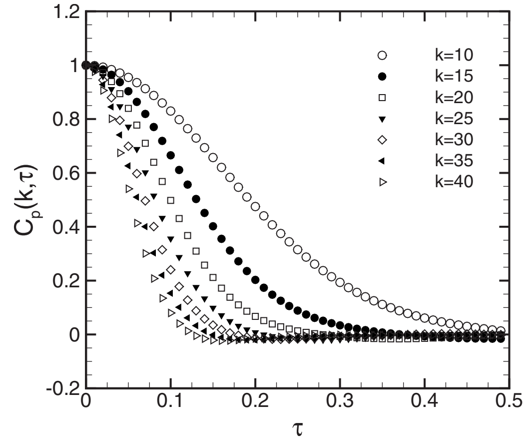

FIG. 1. Normalized time correlations of pressure modes vs time separations for wavenumbers $k=10,15,20,25,30,35,40$, at $\mathrm{Re}_\lambda=78.75$.

$$\begin{aligned} p(k,t+\tau) &= -\frac{k_ik_j}{k^2}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}[u_i(\mathbf{p},t+\tau)u_j(\mathbf{q},t+\tau) \\ &\quad \times \delta(\mathbf{p}+\mathbf{q}-\mathbf{k})] \\ &= -\frac{k_ik_j}{k^2}\sum_{p=-\infty}^{\infty}\sum_{q=-\infty}^{\infty}[u_i(\mathbf{p},t)u_j(\mathbf{q},t) \\ &\quad \times \delta(\mathbf{p}+\mathbf{q}-\mathbf{k})]\exp[-i(\mathbf{p}+\mathbf{q})\cdot\mathbf{v}\tau]. \end{aligned} \tag{10}$$

Substitution of Eqs. (9) and (10) into the pressure time correlations (2) yields

$$\begin{aligned} C_p(k,\tau) &= \frac{k_ik_jk_mk_n}{k^4}\sum_{p_1=-\infty}^{\infty}\sum_{p_2=-\infty}^{\infty}\sum_{q_1=-\infty}^{\infty}\sum_{q_2=-\infty}^{\infty} \\ &\quad \times \langle u_i(\mathbf{p}_1,t)u_j(\mathbf{q}_1,t)u_m(\mathbf{p}_2,t)u_n(\mathbf{q}_2,t)\rangle \\ &\quad \times \delta(\mathbf{p}_1+\mathbf{q}_1-\mathbf{k})\delta(\mathbf{p}_2+\mathbf{q}_2+\mathbf{k}) \\ &\quad \times \langle \exp[-i(\mathbf{p}_1+\mathbf{q}_1)\cdot\mathbf{v}\tau]\rangle \\ &= \langle p(\mathbf{k},t)p(-\mathbf{k},t)\rangle \exp\left(-\tfrac{1}{2}k^2v^2\tau^2\right), \end{aligned} \tag{11}$$

where the first step invokes the independence of $\mathbf{v}$ from $\mathbf{u}$, and the second one uses the assumption that $\mathbf{v}$ is uniform Gaussian. The latter leads to

$$\langle \exp(-i\mathbf{k}\cdot\mathbf{v}\tau)\rangle = \exp\left(-\tfrac{1}{2}k^2v^2\tau^2\right). \tag{12}$$

However, the derivation does not invoke the quasinormal approximation, in contrast to earlier work in which pressure spectra are often deduced using the quasinormal assumption.[1]

The model given by Eq. (11) predicts a universal form for time correlations of pressure modes in isotropic turbulence. It suggests that the fluctuating pressure has the same decorrelation time scales "kv" as the fluctuating velocity. Therefore, the decorrelation process of fluctuating pressure is also dominated by the sweeping velocity.

III. COMPARISON WITH NUMERICAL RESULTS

We now use the data from DNS of isotropic turbulence to verify the random sweeping model for pressure time cor-

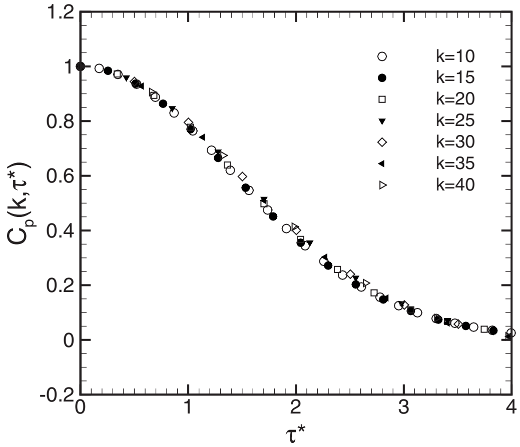

FIG. 2. Normalized time correlations of pressure modes vs time separation scaled by the sweeping time scale $(vk)^{-1}$, $\tau^*=vk\tau$, for wavenumbers $k=10,15,20,25,30,35,40$, at $\mathrm{Re}_\lambda=78.75$.

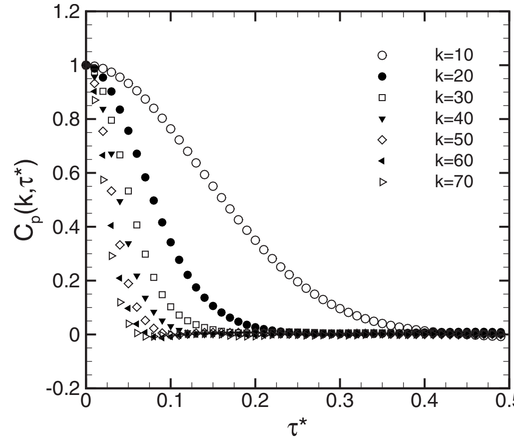

FIG. 4. Normalized time correlations of pressure modes vs time separations for wavenumbers $k=10,20,30,40,50,60,70$, at $\mathrm{Re}_\lambda=148.92$.

relations. The DNS for isotropic turbulence was performed using a pseudospectral method. The computational domain is a box of length 2π on each side, where the periodic boundary conditions are applied. To keep the turbulence stationary, an external force $f(k)$ is imposed on the first two shells of wavenumbers $k=1,2$. Aliasing errors are removed through the two-thirds truncation rule. The Adams-Bashforth scheme is used for time advance. Two cases are run in the present study: Case 1 with $\mathrm{Re}_\lambda=78.75$ on a 128^3 grid and case 2 with $\mathrm{Re}_\lambda=148.92$ on a 256^3 grid, respectively. The relevant parameters in the DNS are listed in Table I.

The time correlation $C_p(k,\tau)$ for each wave number k is computed as a function of time lag τ. The ensemble average is performed over the wave number shell $k=|\mathbf{k}|$ and the different start times t as defined in Eq. (2). In the present case, t is chosen from 0.0 to 2.0 eddy turnover times with an increment of 0.25.

Figures 1 and 2 plot the normalized time correlations of pressure modes for wavenumbers $k=10,15,20,25,30,35,40$. All curves show a decrease in correlation, as separation increases, with vanishing slopes at zero separation. The time separation in Fig. 2 is scaled using the scale-dependent similarity variable vk, while the time separation in Fig. 1 is not scaled. It is observed from Fig. 2 that with the vk scaling all points collapse onto a single curve. The result is in agreement with the prediction of the random sweeping hypothesis on pressure time correlations. Therefore, the random sweeping model given by Eq. (11) for pressures indeed predicts a universal form of the time correlations of pressure modes.

To investigate the model's prediction of the decorrelation time scales, we examine the ratio of the correlation coefficient for the kth pressure mode to that for the velocity modes

$$r(k,\tau)=\frac{c_p(k,\tau)}{c_v(k,\tau)}=\frac{c_p(k,0)}{c_v(k,0)}. \tag{13}$$

Based on the models given by Eqs. (8) and (11), the ratio should be unity for each wavenumber. Figure 3 plots the ratios for wavenumbers $k=10,15,\ldots,40$. It is observed that the ratios for all wavenumbers are almost unity, although some very small deviations are observed at large time separations. These deviations are shown to decrease with increasing statistical sample size and thus do not affect the interpretation of our results. Therefore, the DNS result confirms the

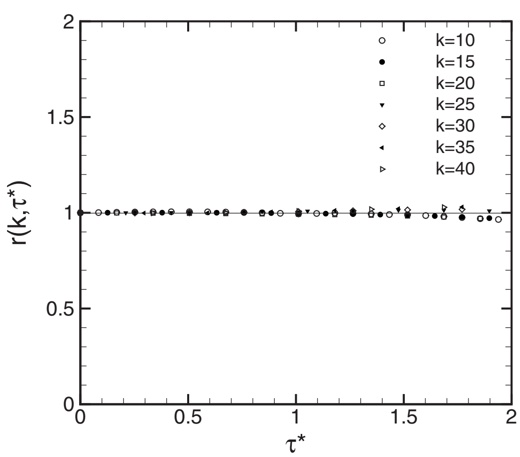

FIG. 3. The ratios of pressure correlation coefficients to velocity correlation coefficients as a function of scaled time separations $\tau^*=vk\tau$ at different wavenumbers, for $\mathrm{Re}_\lambda=78.75$. The solid line represents $r=1.0$.

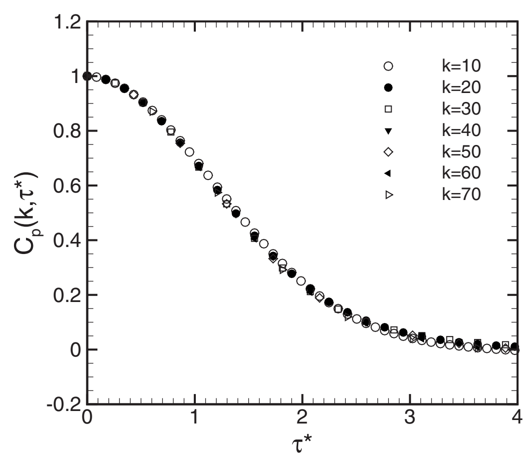

FIG. 5. Normalized time correlations of pressure modes vs time separation scaled by the sweeping time scale $(vk)^{-1}$, $\tau^*=vk\tau$, for wavenumbers $k=10,20,30,40,50,60,70$, at $\mathrm{Re}_\lambda=148.92$.

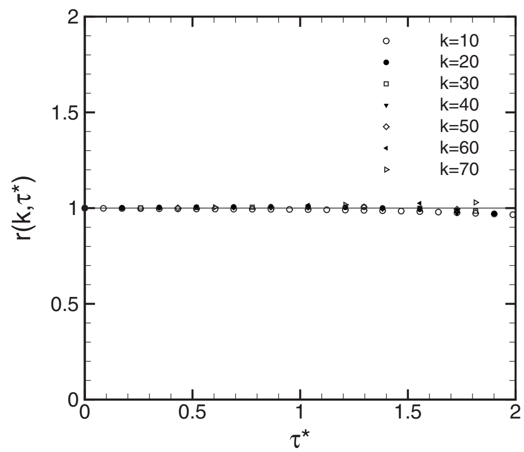

FIG. 6. The ratios of pressure correlation coefficients to velocity correlation coefficients as a function of scaled time separations $\tau^* = v k \tau$ at different wavenumbers, for $Re_\lambda = 148.92$. The solid line represents $r = 1.0$.

theoretical prediction that pressure time correlations have the same decorrelation time scales as the velocity time correlations.

To investigate the effect of Reynolds number on the random sweeping model for pressure time correlations, we use the data from the higher Reynolds number case (case 2) to plot the mode correlations against the un-scaled and scaled time separations in Figs. 4 and 5, respectively, for wavenumbers $k = 10, 20, 30, 40, 50, 60, 70$. The ratios of pressure correlation coefficients to velocity correlation coefficients are plotted in Fig. 6. As in case 1, Fig. 5 exhibits a very good collapse of correlation data based on the sweeping hypothesis and Fig. 6 shows that the correlation ratios are well approximated by the straight line $r = 1.0$. The random sweeping model for pressure time correlations is as accurate for $Re = 148.92$ (case 2) as for $Re = 78.75$ (case 1).

IV. CONCLUSIONS

In summary, a simple model for the time correlations of pressure in Fourier space is developed for isotropic turbulence. The model is based on the Kraichnan and Tennekes "random sweeping" hypothesis. It predicts that, as with velocity correlations, the random sweeping effect dominates the decorrelation of pressure fluctuations, and that the decorrelation time scales for pressure fluctuations are the same as the ones for velocity fluctuations. These results are verified using DNS of stationary isotropic turbulence at two moderate Reynolds numbers. The model exhibits a universal form in the sense that the correlation coefficients collapse when plotted against time separations normalized by the sweeping time scale. An interesting question to be asked is whether pressure correlations share the same decorrelation time scales with velocity correlations in turbulent shear flows, where random sweeping is not the dominant effect.[18]

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[1] M. Nelkin, "Universality and scaling in fully developed turbulence," Adv. Phys. **43**, 143 (1994).
[2] A. S. Monin and A. M. Yaglom, *Statistical Fluid Mechanics* (MIT, Cambridge, 1975), Vol. II.
[3] G. K. Batchelor, "Pressure fluctuations in isotropic turbulence," Proc. Cambridge Philos. Soc. **47**, 359 (1951).
[4] R. J. Hill and J. M. Wilczak, "Pressure structure functions and spectra for locally isotropic turbulence," J. Fluid Mech. **296**, 247 (1995).
[5] W. K. George, P. D. Beuther, and R. E. A. Arndt, "Pressure spectra in turbulent free shear flows," J. Fluid Mech. **148**, 155 (1984).
[6] D. I. Pullin and R. S. Rogallo, "Pressure and higher-order spectra for homogeneous isotropic turbulence," in *Proceedings of the Summer Program*, Center for Turbulence Research, Stanford University and NASA Ames Research Center (1994).
[7] N. Cao, S. Chen, and G. D. Doolen, "Statistics and structures of pressure in isotropic turbulence," Phys. Fluids **11**, 2235 (1999).
[8] P. Vedula and P. K. Yeung, "Similarity scaling of acceleration and pressure statistics in numerical simulations of isotropic turbulence," Phys. Fluids **11**, 1208 (1999).
[9] T. Gotoh and D. Fukayama, "Pressure spectrum in homogeneous turbulence," Phys. Rev. Lett. **86**, 3775 (2001).
[10] T. Gotoh, D. Fukayama, and T. Nakano, "Velocity field statistics in homogeneous steady turbulence obtained using a high-resolution direct numerical simulation," Phys. Fluids **14**, 1065 (2002).
[11] W. K. Blake, *Mechanics of Flow-Induced Sound and Vibration* (Academic, London, 1986), Vol. I.
[12] M. E. Goldstein, *Aeroacoustics* (McGraw-Hill, New York, 1976).
[13] G. W. He, M. Wang, and S. K. Lele, "On the computation of space-time correlations by large-eddy simulation," Phys. Fluids **16**, 3859 (2004).
[14] R. H. Kraichnan, "The structure of isotropic turbulence at very high Reynolds numbers," J. Fluid Mech. **5**, 497 (1959).
[15] R. H. Kraichnan, "Kolmogorov's hypotheses and Eulerian turbulence theory," Phys. Fluids **7**, 1723 (1964).
[16] H. Tennekes, "Eulerian and Lagrangian time microscales in isotropic turbulence," J. Fluid Mech. **67**, 561 (1975).
[17] G. W. He and M. Wang, "A space-time correlation theory for turbulent shear flows," Annual Research Briefs, 195-199, Center for Turbulence Research, Stanford, California, 2004.
[18] G. W. He and J. B. Zhang, "Elliptic model for space-time correlations in turbulent shear flows," Phys. Rev. E **73**, 055303 (2006).