Derivation of a Nonlinear Reynolds Stress Model Using Renormalization Group Analysis and Two-Scale Expansion Technique *

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Adopting Yoshizawa's two-scale expansion technique, the fluctuating field is expanded around the isotropic field. The renormalization group method is applied for calculating the covariance of the fluctuating field at the lower order expansion. A nonlinear Reynolds stress model is derived and the turbulent constants inside are evaluated analytically. Compared with the two-scale direct interaction approximation analysis for turbulent shear flows proposed by Yoshizawa, the calculation is much more simple. The analytical model presented here is close to the Speziale model, which is widely applied in the numerical simulations for the complex turbulent flows.

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In complex turbulent flows, the influence of extra strain rates on turbulent flow structure is profound, and it is a challenge for turbulence modelers to predict such types of flows accurately. The conventional $K-\epsilon$ model based upon the linear eddy viscosity model has been widely used in practical applications because of its simplicity and its efficiency compared to CPU time requirements. However, since it fails to include the anisotropy of the Reynolds stress, it could not predict many complex turbulent flows accurately. In the last decades, the nonlinear Reynolds stress models have been developed for overcoming this deficiency. It could generate a secondary flow in a straight square duct and yield improved results for turbulent flow past a backward-facing step,[1−3] etc. How to formulate the nonlinear Reynolds stress models receives much attention [1−9]. Many of them are constructed by the dimensional analysis and the invariance principles for the intrinsic of the Navier–Stokes equation. In these models there are some turbulent constants that are determined from the experimental data.

From the 1980s, the statistical mechanical approach has been applied to derive the turbulence closure models. In order to generalize the statistical theory of isotropic turbulence to analyse the fluctuating field of turbulent shear flows, Yoshizawa proposed a two-scale direct interaction approximation (TSDIA). In the procedure of the TSDIA, a two-scale expansion is introduced for separating the fluctuating field from the mean field so that the fluctuating field is expanded around the isotropic field, then Kraichnan's DIA theory for analysing isotropic turbulence is generalized to analyse the fluctuating field of turbulent shear flows. Using the TSDIA technique, Yoshizawa derived an eddy-viscosity representation for Reynolds stress.[10] By renormalizing an asymptotic expansion for the Reynolds stress with the eddy-viscosity approximation as the leading part, a closure model for the Reynolds stress transport equation was presented. Then, assuming an equilibrium state in the sense that the convection and diffusion effects do not play the important roles, the nonlinear Reynolds stress model was obtained.[11] Rubinstein and Barton[12,13] are pioneers to utilize the renormalization group (RNG) method for turbulence, which was developed by Yakhot and Orszag (YO)[14], to formulate the nonlinear Reynolds stress model and the second-order closure model.

In the RNG method for turbulence, Yakhot and Orszag [14] suggested that homogeneous isotropic turbulent flow could be described by the Navier–Stokes equation with the Gaussian random force $\boldsymbol{f}$, which is an explicit external force that maintains turbulence in a statistically steady state. This was called the correspondence principle. The Gaussian random force $\boldsymbol{f}$ is characterized by its correlation function:

$$\langle f_\alpha(\hat{k}) f_\beta(\widehat{k'})\rangle = 2D(2\pi)^{d+1} P_{\alpha\beta}(\boldsymbol{k}) k^{-y} \delta(\hat{k}+\widehat{k'}), \quad (1)$$

where $\hat{k} = (\boldsymbol{k}, \omega)$, $P_{\alpha m}(\boldsymbol{k}) = \delta_{\alpha m} - k_\alpha k_m/k^2$, the parameter D determines the intensity of the random force and $\langle\ \rangle$ denotes the ensemble average. With the RNG analysis, some turbulent constants, for example, the Kolmogorov constant and the velocity skewness, were calculated. They were in agreement with the experiments. Assuming that the infrared limit corresponded to the mean field of turbulent shear flows, the mode coupling approximation in the RNG method was applied to analyse turbulent shear flows. However, the derivations for the nonlinear Reynolds stress model directly using the renormalization-group method were

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very complicated and some inconsistent problems occurred.[15]

The RNG method is intrinsically appropriate for analysing the statistical self-similar system, where the characteristic length tends to infinity. It might be plausible that the RNG method is appropriate for analysing isotropic turbulence. However, the characteristic length of turbulent shear flow is limited by the scale of the large eddies. How to apply the RNG method into analysis for turbulent shear flows remains the scientific interests. In this Letter, adopting Yoshizawa's two-scale expansion technique, the fluctuating field is expanded around the isotropic field. Instead of using the DIA theory, we apply the mode coupling approximation in the YO RNG theory to study the fluctuating field of turbulent shear flows. At the lower order expansion, the nonlinear Reynolds stress model is derived and the turbulent constants inside are evaluated analytically.

For the incompressible turbulent flows, the fluctuating velocity is governed by

$$\frac{\partial u_\alpha}{\partial t}+U_k\frac{\partial u_\alpha}{\partial x_k}+u_k\frac{\partial U_\alpha}{\partial x_k}+u_k\frac{\partial u_\alpha}{\partial x_k}=-\frac{\partial p}{\partial x_\alpha}+\nu_0\frac{\partial^2 u_\alpha}{\partial x_k\partial x_k}-\frac{\partial \tau_{\alpha k}}{\partial x_k},\tag{2}$$

where $\tau_{\alpha k}=-\langle u_\alpha u_k\rangle$ is the Reynolds stress tensor. In order to separate the slow variation of the mean field from the fast variation of the fluctuating field, a scale parameter δ is introduced,[10,11]

$$\boldsymbol{x},\boldsymbol{X}(=\delta\boldsymbol{x});\quad t,T(=\delta t).\tag{3}$$

It is assumed that

$$\boldsymbol{u}=\boldsymbol{u}(\boldsymbol{x},\boldsymbol{X};t,T),\quad p=p(\boldsymbol{x},\boldsymbol{X};t,T),\quad \boldsymbol{U}=\boldsymbol{U}(\boldsymbol{X};T),\quad \tau_{\alpha k}=\tau_{\alpha k}(\boldsymbol{X};T)\tag{4}$$

Since the anisotropy appears through X and T, we expand the fluctuating field around the isotropic field:

$$\boldsymbol{u}=\boldsymbol{u}^{(0)}+\delta\boldsymbol{u}^{(1)}+\delta^2\boldsymbol{u}^{(2)}+\sum_{n\geq 3}\delta^n\boldsymbol{u}^{(n)},\quad p=p^{(0)}+\delta p^{(1)}+\delta^2 p^{(2)}+\sum_{n\geq 3}\delta^n p^{(n)},\tag{5}$$

where $\boldsymbol{u}^{(0)}$ is the isotropic part of fluctuating field and $\boldsymbol{u}^{(n)}(n\geq 1)$ denotes the anisotropic part of fluctuating field. Substituting Eq. (5) into Eq. (2), we obtain

$$\frac{\partial u_\alpha^{(0)}}{\partial t}+U_k\frac{\partial u_\alpha^{(0)}}{\partial x_k}+u_k^{(0)}\frac{\partial u_\alpha^{(0)}}{\partial x_k}=-\frac{\partial p^{(0)}}{\partial x_\alpha}+\nu_0\frac{\partial^2 u_\alpha^{(0)}}{\partial x_k\partial x_k},\tag{6}$$

for $\boldsymbol{u}^{(n)}(n\geq 1)$:

$$\frac{\partial u_\alpha^{(n)}}{\partial t}+U_k\frac{\partial u_\alpha^{(n)}}{\partial x_k}+U_k\frac{\partial u_\alpha^{(n-1)}}{\partial X_k}+\sum_{a+b=n,n\geq 1}u_k^{(a)}\frac{\partial u_\alpha^{(b)}}{\partial x_k}+\frac{\partial p^{(n)}}{\partial x_\alpha}-\nu_0\frac{\partial^2 u_\alpha^{(n)}}{\partial x_k\partial x_k}$$
$$=-\frac{\partial u_\alpha^{(n-1)}}{\partial T}-\sum_{c+d=n-1}u_k^{(c)}\frac{\partial u_\alpha^{(d)}}{\partial X_k}-u_k^{(n-1)}\frac{\partial U_\alpha}{\partial X_k}-\frac{\partial p^{(n-1)}}{\partial X_\alpha}+2\nu_0\frac{\partial^2 u_\alpha^{(n-1)}}{\partial x_k\partial X_k}.\tag{7}$$

In order to maintain isotropic turbulence in a statistically steady state, a Gaussian random force is introduced. It can be thought as the effective force felt at smaller length scales, which results from the larger scales transmitted by the nonlinear terms of the Navier–Stokes equations (correspondence principle in the YO RNG method for turbulence).[14] For removing the sweeping effect, we adopt the Taylor hypothesis with the Galilean transformation: $\boldsymbol{x}\to\boldsymbol{x}-Ut$. Then, in the Fourier space, we have

$$u_\alpha^{(0)}(\hat{k})=G_0(\hat{k})f_\alpha(\hat{k})+\frac{\lambda_0}{2i}G_0(\hat{k})P_{\alpha ml}(\boldsymbol{k})\cdot\int_{\Lambda_f<q,|k-\mathbf{q}|<\Lambda_0}u_m^{(0)(\hat{k}-\widehat{q})u_l^{(0)}(\widehat{q})}\frac{d\widehat{q}}{(2\pi)^{d+1}};\tag{8}$$

for $\boldsymbol{u}^{(n)}$ $(n\geq 1)$:

$$u_\alpha^{(n)}(\hat{k})=\frac{\lambda_0}{2i}G_0(\hat{k})\Big[\sum_{a+b=n}P_{\alpha ml}(\boldsymbol{k})\cdot\int u_m^{(a)}(\hat{k}-\widehat{q})u_l^{(b)}(\widehat{q})\frac{d\widehat{q}}{(2\pi)^{d+1}}-iP_{\alpha m}(\boldsymbol{k})u_l^{(n-1)}(\hat{k})\frac{\partial U_m}{\partial X_l}\Big]+\frac{\partial F(u_\alpha^{(n-1)})}{\partial T}+\frac{\partial H(u^{(0)},u^{(1)},\cdots,u^{(n-1)},\partial u^{(n-1)}/\partial x)}{\partial X},\tag{9}$$

where $G_0(\hat{k})=(-i\omega+\nu_0k^2)^{-1}$, $P_{\alpha mn}(\boldsymbol{k})=k_mP_{\alpha n}(\boldsymbol{k})+k_nP_{\alpha m}(\boldsymbol{k})$; λ_0 is a formal expansion parameter which will eventually be set equal to 1; Λ_f corresponds to the largest fluctuating scale and the cutoff wave number Λ_0 is beyond the dissipation wave number at which substantial modal excitations cease. For simplicity, we analyse the Reynolds stress $\tau_{\alpha\beta}=-\langle u_\alpha u_\beta\rangle$ at the lower order expansion. Up to the order of $O(\delta^2)$, the terms $\partial F/\partial T$ and $\partial H/\partial X$ have no contribution. In the following analysis, these terms are not written explicitly.

The RNG analysis for the isotropic field $\boldsymbol{u}^{(0)}$ was performed extensively in Ref. [14]. The purpose of RNG theory is weeding out of small scales. It is proposed that the equation is averaged over the fine-scale ensemble and all the terms involving only large-scale quantities are assumed to be the same for each realization in the fine-scale ensemble. After removing the velocity components with the wave number in the interval (Λ,Λ_0), according to the RNG analysis of YO, we have

$$\langle u_i^{(0)}(\hat{k})u_j^{(0)}(\hat{k'})\rangle=2D(2\pi)^{d+1}|G(\hat{k})|^2$$

$$\cdot P_{ij}(\boldsymbol{k})k^{-y}\delta(\hat{k}+\hat{k'}), \tag{10}$$

where $G(\hat{k}) = (-i\omega + \nu(k)k^2)^{-1}$ and $\nu(k) = \Big[\frac{3(d-1)}{8(d+2)}\frac{D_0S_d}{(2\pi)^d}\Big]^{1/3}k^{-4/3}$.[14,16,17]

Based upon Yoshizawa's two-scale expansion, we analyse the Reynolds stress using the YO RNG method. Substituting the two-scale expansion given in Eq. (5) into the Reynolds stress expression $\tau_{\alpha\beta} = -\langle u_\alpha u_\beta\rangle$, we obtain

$$\tau_{\alpha\beta} = -\langle u_\alpha u_\beta\rangle = \tau^{(0)}_{\alpha\beta} + \delta\tau^{(1)}_{\alpha\beta} + \delta^2\tau^{(2)}_{\alpha\beta} + O(\delta^3), \tag{11}$$

where

$$\tau^{(0)}_{\alpha\beta} = -\int\langle u^{(0)}_\alpha(\widehat{p})u^{(0)}_\beta(\widehat{q})\rangle e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}, \tag{12}$$

$$\tau^{(1)}_{\alpha\beta} = -\int\langle u^{(1)}_\alpha(\widehat{p})u^{(0)}_\beta(\widehat{q}) + u^{(0)}_\alpha(\widehat{p})u^{(1)}_\beta(\widehat{q})\rangle e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}, \tag{13}$$

$$\tau^{(2)}_{\alpha\beta} = -\int\langle u^{(2)}_\alpha(\widehat{p})u^{(0)}_\beta(\widehat{q}) + u^{(1)}_\alpha(\widehat{p})u^{(1)}_\beta(\widehat{q}) + u^{(0)}_\alpha(\widehat{p})u^{(2)}_\beta(\widehat{q})\rangle e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}. \tag{14}$$

According to Eq. (10), the term $\tau^{(0)}_{\alpha\beta}$ can be evaluated directly as follows:

$$\begin{aligned}(\tau^{(0)}_{\alpha\beta}) &= -\int_{\Lambda_f<p,q<\Lambda_0}\langle[u^{(0)}_\alpha(\widehat{p})][u^{(0)}_\beta(\widehat{q})]\rangle \cdot e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}\\ &= -\int_{\Lambda_f}^{\Lambda_0}2D|G(\widehat{q})|^2P_{\alpha\beta}(\boldsymbol{q})q^{-y}\frac{d\widehat{q}}{(2\pi)^{d+1}}\\ &= -\frac{DS_d}{(2\pi)^d}\frac{1}{\nu(\Lambda_f)\Lambda_f^2}\delta_{\alpha\beta}.\end{aligned} \tag{15}$$

The higher order terms $\tau^{(n)}_{\alpha\beta}$ $(n\geq 1)$ denote the anisotropic effect due to shear flow. In the following, we analyse the terms $\tau^{(1)}_{\alpha\beta}$ and $\tau^{(2)}_{\alpha\beta}$ using the YO RNG method. The fluctuating velocity $\boldsymbol{u}$ is divided into two parts: the fast mode $\boldsymbol{u}^>(\hat{k})[k\in(\Lambda-d\Lambda,\Lambda)]$ and the slow mode $\boldsymbol{u}^<(\hat{k})[k\in(\Lambda_f,\Lambda-d\Lambda)]$. The term $\tau^{(1)}_{\alpha\beta}$ can be rewritten as

$$\tau^{(1)}_{\alpha\beta} = (\tau^{(1)}_{\alpha\beta})^< + (\tau^{(1)}_{\alpha\beta})^>, \tag{16}$$

where

$$(\tau^{(1)}_{\alpha\beta})^< = -\int\langle[u^{(1)}_\alpha(\widehat{p})]^<[u^{(0)}_\beta(\widehat{q})]^< + [u^{(0)}_\alpha(\widehat{p})]^<[u^{(1)}_\beta(\widehat{q})]^<\rangle e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}, \tag{17}$$

$$\begin{aligned}(\tau^{(1)}_{\alpha\beta})^> = -\int\langle&[u^{(1)}_\alpha(\widehat{p})]^>[u^{(0)}_\beta(\widehat{q})]^<\\ &+ [u^{(1)}_\alpha(\widehat{p})]^<[u^{(0)}_\beta(\widehat{q})]^>\\ &+ [u^{(1)}_\alpha(\widehat{p})]^>[u^{(0)}_\beta(\widehat{q})]^>\\ &+ [u^{(0)}_\alpha(\widehat{p})]^>[u^{(1)}_\beta(\widehat{q})]^<\\ &+ [u^{(0)}_\alpha(\widehat{p})]^<[u^{(1)}_\beta(\widehat{q})]^>\\ &+ [u^{(0)}_\alpha(\widehat{p})]^>[u^{(1)}_\beta(\widehat{q})]^>\rangle e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}.\end{aligned} \tag{18}$$

Substituting Eq. (9) into Eq. (18) and adopting the conditional average $\langle u^>_\alpha u^>_\beta u^<_\gamma\rangle = \langle u^>_\alpha u^>_\beta\rangle u^<_\gamma$,[14] we obtain

$$\begin{aligned}(\tau^{(1)}_{\alpha\beta})^> &= \frac{\partial U_m}{\partial X_n}\int G(\widehat{p})\langle P_{\alpha m}(\boldsymbol{p})[u^{(0)}_n(\widehat{p})]^>[u^{(0)}_\beta(\widehat{q})]^> + P_{\beta m}(\boldsymbol{p})[u^{(0)}_n(\widehat{p})]^>[u^{(0)}_\alpha(\widehat{q})]^>\rangle \cdot e^{i(\widehat{p}+\widehat{q})\widehat{x}}\frac{d\widehat{p}d\widehat{q}}{(2\pi)^{2d+2}}\\ &= \frac{\partial U_m}{\partial X_n}\int 2DG(\widehat{q})\,|G(\widehat{q})|^2\,[P_{\alpha m}(\boldsymbol{q})P_{n\beta}(\boldsymbol{q}) + P_{\beta m}(\boldsymbol{q})P_{\alpha n}(\boldsymbol{q})]q^{-y}\frac{d\widehat{q}}{(2\pi)^{d+1}}\\ &= \frac{7}{30}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^2(\Lambda)\Lambda^5}(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha})d\Lambda.\end{aligned} \tag{19}$$

Removing the fast modes successively, we have the recursion relation

$$\frac{d\tau^{(1)}_{\alpha\beta}}{d\Lambda} = \frac{7}{30}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^2(\Lambda)\Lambda^5}\Big(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha}\Big). \tag{20}$$

Consequently, after taking the average over the fast modes in the interval $[\Lambda_f, \Lambda_0]$, we obtain

$$\tau^{(1)}_{\alpha\beta} = \frac{7}{40}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^2(\Lambda_f)\Lambda_f^4}\Big(\frac{\partial U_\alpha}{\partial X_\beta} + \frac{\partial U_\beta}{\partial X_\alpha}\Big). \tag{21}$$

Similarly to the analysis for $\tau^{(1)}_{\alpha\beta}$, after taking average of all the modes for the interval $[\Lambda_f, \Lambda_0]$, we obtain

$$\begin{aligned}\tau^{(2)}_{\alpha\beta} = -\frac{1}{8}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^3(\Lambda_f)\Lambda_f^6}\Big[&\frac{62}{105}\frac{\partial U_\alpha}{\partial X_\mu}\frac{\partial U_\beta}{\partial X_\mu}\\ &+ \frac{34}{105}\Big(\frac{\partial U_\alpha}{\partial X_\mu}\frac{\partial U_\mu}{\partial X_\beta} + \frac{\partial U_\mu}{\partial X_\alpha}\frac{\partial U_\beta}{\partial X_\mu}\Big)\\ &+ \frac{2}{35}\frac{\partial U_\mu}{\partial X_\alpha}\frac{\partial U_\mu}{\partial X_\beta} + \frac{2}{21}\frac{\partial U_\rho}{\partial X_\mu}\frac{\partial U_\rho}{\partial X_\mu}\delta_{\alpha\beta}\\ &+ \frac{2}{21}\frac{\partial U_\rho}{\partial X_\mu}\frac{\partial U_\mu}{\partial X_\rho}\delta_{\alpha\beta}\Big].\end{aligned} \tag{22}$$

Substituting Eqs. (15), (21) and (22) into Eq. (11) and utilizing the scales relation $\boldsymbol{X} = \delta\boldsymbol{x}$, at the second expansion $O(\delta^2)$, the Reynolds stress $\tau_{\alpha\beta}$ is modelled as

$$\begin{aligned}\tau_{\alpha\beta} &= \tau^{(0)}_{\alpha\beta} + \delta\tau^{(0)}_{\alpha\beta} + \delta^2\tau^{(2)}_{\alpha\beta}\\ &= -\frac{DS_d}{(2\pi)^d}\frac{1}{\nu(\Lambda_f)\Lambda_f^2}\delta_{\alpha\beta}\end{aligned}$$

$$
\begin{aligned}
&+\frac{7}{40}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^2(\Lambda_f)\Lambda_f^4}\Big(\frac{\partial U_\alpha}{\partial x_\beta}+\frac{\partial U_\beta}{\partial x_\alpha}\Big)\\
&-\frac{1}{8}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^3(\Lambda_f)\Lambda_f^6}\Big[\frac{62}{105}\frac{\partial U_\alpha}{\partial x_\mu}\frac{\partial U_\beta}{\partial x_\mu}\\
&+\frac{34}{105}\Big(\frac{\partial U_\alpha}{\partial x_\mu}\frac{\partial U_\mu}{\partial x_\beta}+\frac{\partial U_\mu}{\partial x_\alpha}\frac{\partial U_\beta}{\partial x_\mu}\Big)\\
&+\frac{2}{35}\frac{\partial U_\mu}{\partial x_\alpha}\frac{\partial U_\mu}{\partial x_\beta}+\frac{2}{21}\frac{\partial U_\rho}{\partial x_\mu}\frac{\partial U_\rho}{\partial x_\mu}\delta_{\alpha\beta}\\
&+\frac{2}{21}\frac{\partial U_\rho}{\partial x_\mu}\frac{\partial U_\mu}{\partial x_\rho}\delta_{\alpha\beta}\Big]. \qquad (23)
\end{aligned}
$$

From Eq. (23), we have

$$
\begin{aligned}
K=&-\frac{1}{2}\tau_{\alpha\alpha}=\frac{3}{2}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu(\Lambda_f)\Lambda_f^2}\\
&+\frac{7}{120}\frac{DS_d}{(2\pi)^d}\frac{1}{\nu^3(\Lambda_f)\Lambda_f^6}\Big[\frac{\partial U_\rho}{\partial x_\mu}\frac{\partial U_\rho}{\partial x_\mu}+\frac{\partial U_\rho}{\partial x_\mu}\frac{\partial U_\mu}{\partial x_\rho}\Big]. \qquad (24)
\end{aligned}
$$

Utilizing the results of the YO RNG theory: $2D/(2\pi)^d = 1.594\bar{\varepsilon}$ and $\nu(\Lambda_f){\Lambda_f}^2 = 1.195\bar{\varepsilon}/K$, we reach

$$
\begin{aligned}
\tau_{\alpha\beta}=&-\langle u_\alpha(\boldsymbol{x},t)u_\beta(\boldsymbol{x},t)\rangle\\
=&-\frac{2}{3}K\delta_{\alpha\beta}+2C_\mu\frac{K^2}{\bar{\varepsilon}}S_{\alpha\beta}\\
&-\frac{K^3}{\bar{\varepsilon}^2}\Big[C_1\Big(S_{\alpha\mu}S_{\mu\beta}-\frac{1}{3}S_{m\mu}S_{\mu m}\delta_{\alpha\beta}\Big)\\
&+C_2(S_{\alpha\mu}W_{\mu\beta}+S_{\beta\mu}W_{\mu\alpha})\\
&+C_3\Big(W_{\alpha\mu}W_{\mu\beta}-\frac{1}{3}W_{m\mu}W_{\mu m}\delta_{\alpha\beta}\Big)\Big], \qquad (25)
\end{aligned}
$$

where $S_{\alpha\beta}=\frac{1}{2}\Big(\frac{\partial U_\alpha}{\partial x_\beta}+\frac{\partial U_\beta}{\partial x_\alpha}\Big)$ is the mean rate-of-strain tensor, $W_{\alpha\beta}=\frac{1}{2}\Big(\frac{\partial U_\alpha}{\partial x_\beta}-\frac{\partial U_\beta}{\partial x_\alpha}\Big)$ is the mean-vorticity tensor, and the constants C_μ, C_1, C_2, C_3 are calculated to be

$$
C_\mu = 0.097,\quad C_1 = 0.076,\quad C_2 = -0.032,\quad C_3 = 0. \qquad (26)
$$

Table 1. Comparison for the turbulent constants of different nonlinear Reynolds stress models.

Model constants	C_1	C_2	C_3
Speziale' model[2]	0.055	−0.055	0
Demuren and Rodi's model[1]	0.209	−0.079	0
Yoshizawa's theoretical model[11]]	0.048	−0.0047	−0.057
Rubinstein and Barton's model[12]	0.228	−0.048	0.188
Huang model (Jaumann derivative)[9]	0.0234	−0.0174	0.0069
Huang model (Oldroyd derivative)[9]	0.0253	−0.0174	0.0087
The theoretical model of this Letter	0.076	−0.032	0

In Table 1, we list the turbulent constants of different nonlinear Reynolds stress models. According to the comparison, the model constants calculated in this study are close to the constants of the Speziale model. Here we should emphasize that we do not claim the superiority of the present modelling method to the conventional modelling method, the TSDIA analysis, etc. A major interest of this work is the effort on generalizing the mode coupling approximation in the RNG method into analysis of shear turbulence.

The mean field is the ensemble average of turbulence field, so it normally has much bigger characteristic length than that of the fluctuating field. Based upon this viewpoint, Yoshizawa introduced two scales that the small and large scales correspond to the fluctuating and the mean fields, respectively. In this study, adopting Yoshizawa's two-scale expansion technique, the governing equations for isotropic part $\boldsymbol{u}^{(0)}$ and anisotropic part $\boldsymbol{u}^{(n)}$ ($n \geq 1$) are derived. We use the RNG method to calculate the covariance of the fluctuating field at the lower order expansion. Compared with the TSDIA analysis for turbulent shear flows proposed by Yoshizawa,[10,11] the calculation is much more simple. Speziale presented the arguments for the realizable solutions of the nonlinear Reynolds stress models and proposed that the constant C_3 in Eq. (25) needs to be zero.[18] The theoretical model given in Eqs. (25) and (26) satisfies Speziale's arguments. In this study, the Reynolds stress $\tau_{\alpha\beta} = -\langle u_\alpha u_\beta\rangle$ is analysed at the order of $O(\delta^2)$. It is expected that the cubic model for the Reynolds stress can be derived if taking the expansion at the order of $O(\delta^3)$, where the terms $\partial F/\partial T$ and $\partial H/\partial X$ in Eq. (9) generate the higher order contributions such as $\partial^n U/\partial X^n$ ($n \geq 2$).

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