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DYNAMIC MODE I PERTURBATION SOLUTION FOR A MOVING CRACK UNSTEADILY

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Abstract—Rice *et al.* (*Journal of Mechanics and Physics of Solids* **42,** 813–843) analyze the propagation of a planar crack with a nominally straight front in a model elastic solid with a single displacement component. Using the form of Willis *et al.* (*Journal of the Mechanics and Physics of Solids* **43,** 319–341), of dynamic mode I weight functions for a moving crack, we address that problem solved by Rice *et al.* in the 3D context of elastodynamic theory. Oscillatory crack tip motion results from constructive-destructive interference of stress intensity waves. Those waves, including system of the dilatational, shear and Rayleigh waves, interact on each other and with moving edge of crack, can lead to continuing fluctuations of the crack front and propagation velocity. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Experimental measurements by Fineberg *et al.* (1992) indicate that, in at least one plastic material, the limiting fracture speed is significantly less than the Rayleigh velocity, and the approach to this limiting speed is accompanied by the onset of a dynamic instability.

Some insight regarding these issues can be obtained from the study of crack advance through brittle, locally heterogeneous materials (Fig. 1). In general, the fracture resistance (local fracture toughness) varies along the front of an advancing macroscopic crack due to microlevel heterogeneties, such as second-phase tough particles which can develop macroscopic toughing of a brittle matrix. Rice (1985), Gao and Rice (1989), Gao (1993) and Rice *et al.* (1994) developed a simplified analysis, based on linear perturbation theory for the configuration of an initially straight crack front which is trapped against forward advance by contact with arrays of obstacles. The obstacles are modeled as having the same elastic properties as the rest of the elastic medium, but with slightly higher fracture toughness. The half-plane crack results models finite-sized cracks, assuming the lengths of the cracks are large compared to other parameters such as obstacles spacing along the crack.

The motivation for our work derived for recent studies by Rice *et al.* (1994) , of the perturbation from straightness of the edge of a propagating semi-infinite crack and the associated perturbation of the stress intensity factor, and by Willis *et al.* (1995), of dynamic mode I weight functions for a moving crack. Rice *et al.* (1994) analyzed a scalar wave equation. They found how a crack front moves unsteadily through regions of locally variable fracture resistance. Although in some respects such model results may provide a mechanism for the generation of rough tensile fracture surfaces when the average propagation speed of the crack is relatively small, it is important to derive exact results for some perturbation in z about a history $v = v(t)$ of motion that has arbitrary time dependent in the context of actual elasticity theory.

In a substantial paper, Willis *et al.* (1995) found dynamic weight functions for arbitrary time-dependent loading of a plane semi-infinite crack extending at constant speed in an infinite isotropic elastic body. In the framework of first-order perturbation theory, the weight function is then employed to develop a relationship, between the Mode I stress intensity factory and a small but otherwise arbitrary time-varying deviation from straightness of the edge of a crack. The associated stress intensity factor was recognized as convolutions and the meanings of parameters in their dynamic weight functions are implicit and not very clear.

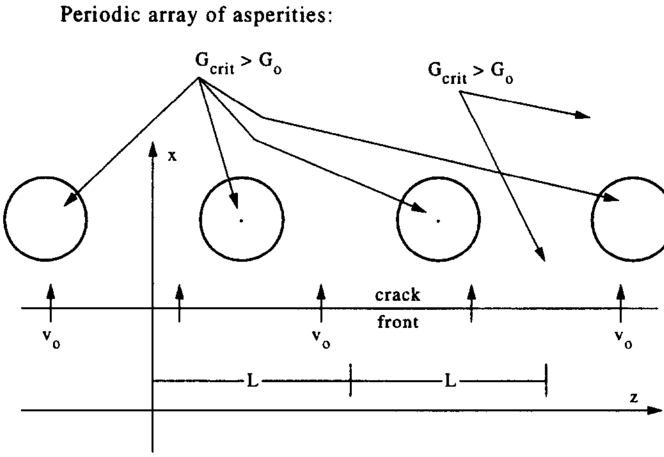

Fig. 1. A half-plane crack with an initially straight front, propagating to contact a periodic row of circular asperities of higher fracture toughness.

Our primary purpose in the investigation here is to reexamine the problem of relating a small deviation from straightness of the crack to the associated perturbation of stress intensity factor, with the view toward exposing interact phenomena between crack trapping and wave trapping. By virtue of the Mode I dynamic weight function and Rice *et al.* model methods, we hope to learn how the crack front begins to surround and penetrate into various arrays of round obstacles and the extent to which such models results remain valid in the context of actual elasticity theory, rather than the modal theory.

2. PROBLEM STATEMENT AND WEIGHT FUNCTIONS

Consider a half-plane crack propagating in an unbounded solid, nominally in the x direction along the plane $y = 0$ (Fig. 2). The crack front at time t lies along the arc $x = a(z, t)$ while we assume to have the form $x = Vt + \varepsilon\phi(t, z)$, where the function ϕ (t, z) is assumed to be bounded, and ε is a small parameter. The crack front speed thus varies along the z axis and its shape deviates from straightness.

In the framework of first-order perturbation theory, the displacement, stress and stress intensity factor fields associated with the perturbed crack are represented as $u_i + \Delta u_i$, $\sigma_{ij} + \Delta\sigma_{ij}$ and $K + \Delta K$, respectively, where u_i, σ_{ij} and K are the fields which give the unperturbed crack (straight crack) solution when $\varepsilon = 0$, i.e., they were satisfied that:

The equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i. \tag{1}$$

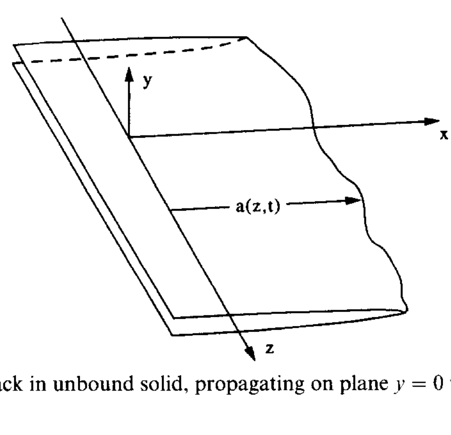

Fig. 2. A half-plane crack in unbound solid, propagating on plane $y = 0$ with non-straight front.

Initial condition:

$$u_i(t,\bar{x}) \to 0 \quad \text{as } t \to -\infty. \tag{2}$$

Boundary conditions:

$$\sigma_{yy}(t,x_1,0^{\pm},z) \to -P(t,x_1,z) \quad \text{as } x_1 < Vt. \tag{3}$$

Stress strain relation:

$$\sigma_{ij} = C_{ijkl}u_{k,l}, \tag{4}$$

where

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl}+\mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}). \tag{4$'$}$$

Local dynamic stress intensity factor along the crack edge:

$$\sigma_{yy}(t,x_1,0,z) \to K(t,z)/(\sqrt{2\pi X}) \quad \text{as } X = x_1 - Vt \to 0. \tag{5}$$

Following Willis *et al.*, the associated perturbed stress intensity factor field may be expressed as

$$\Delta K(t,z) = \varepsilon\left(Q^*(\Phi K) - \Phi(Q^*K) + \sqrt{\frac{\pi}{2}}\Phi M\right) \tag{6}$$

where "*" signifies the convolution over the variable t, X, z, interpreted in the sense of generalized functions and $\varnothing$ denotes crack perturbation, function Q, M are denoted as:

$$Q(t,z) = \frac{1}{2}Q_a(t,z) - Q_c(t,z) - \frac{\partial^2}{\partial t^2}\left(\frac{F_r(-z/t)tH(t-|z|/\alpha a)}{\pi z^2}\right) - F_i'(0)\delta'(t)\delta(z)$$

$$Q_a(t,z) = \frac{V}{\alpha^2 a^2}\delta'(t)\delta(z) - \frac{1}{\pi\alpha}\frac{\partial}{\partial t}\left(\frac{tH(t-|z|/\alpha a)}{z^2(t^2-z^2/\alpha^2a^2)^{1/2}}\right)$$

$$Q_c(t,z) = \frac{V}{\gamma^2 c^2}\delta'(t)\delta(z) - \frac{1}{\pi\gamma}\frac{\partial}{\partial t}\left(\frac{tH(t-|z|/\gamma c)}{z^2(t^2-z^2/\gamma^2c^2)^{1/2}}\right)$$

$$F\left(\frac{\omega}{|\xi_2|}\right) = \frac{\xi_b^- - \xi_a^-}{2\pi|\xi_2|}\int_0^1 \ln\left(\frac{Y+Z}{Y-Z}\right)\mathrm{d}s \tag{7a-d}$$

and

$$\alpha = \sqrt{1-\frac{V^2}{a^2}} \quad \beta = \sqrt{1-\frac{V^2}{b^2}} \quad \gamma = \sqrt{1-\frac{V^2}{c^2}}$$

$$\xi_a^{\mp} = -\frac{\omega V}{\alpha^2 a^2} \mp iq_a \quad iq_a = \alpha^{-1}\left(\frac{\omega^2}{\alpha^2 a^2} - \xi_2^2\right)^{1/2}$$

$$\xi_b^{\mp} = -\frac{\omega V}{\beta^2 b^2} \mp iq_b \quad iq_b = \beta^{-1}\left(\frac{\omega^2}{\beta^2 b^2} - \xi_2^2\right)^{1/2}$$

$$Y = (\xi_1^2+\xi_2^2+\beta^2(\xi_1-\xi_b^+)(\xi_1-\xi_b^-))^2$$

$$Z = 4(\xi_1^2+\xi_2^2)\alpha\beta((\xi_1-\xi_a^-)(\xi_1-\xi_b^-)(\xi_1-\xi_a^+)(\xi_1-\xi_b^+))^{1/2}$$

$$\xi_1 = \xi_a^- + (s-0i)(\xi_b^- - \xi_a^-)$$
$$F\left(-\frac{z}{t}+0i\right) = F_r\left(-\frac{z}{t}\right)+iF_i\left(-\frac{z}{t}\right) \tag{8a-g}$$

where a, b, c denote the speeds of dilatational, shear and Rayleigh waves, respectively.

$$M(t,z) = \{[U]^{(+)} * P'^{(-)}\}(t,0,z) \tag{9}$$

and here is $[U]^{(+)}$ the Mode I dynamic weight function defined by Willis *et al.*, its expression can be found from Appendix 1, and $P'^{(-)}$ is

$$P'^{(-)} = \frac{\partial P}{\partial X} H(-X). \tag{10}$$

The significance in fracture of M would depend upon the loading distribution on crack edge in the particular problem under discussion, and admission of a non-zero M would significance complicate the result, because the transform would no longer be an homogeneous function. As the first step toward exposing interact phenomena between crack trapping and wave trapping, we only consider the case of constant K and zero M in this paper. So the term Q^*K vanishes, and ΔK and Φ satisfy

$$\Delta K(t,z) = K(\varepsilon\Phi^*Q) = K[(a(z,t)-a(\zeta,t))^*Q(t,z)] \tag{11}$$

where ζ denotes some reference location ζ along the z axis, and for the Mode I problem, in virtue of eqn (7) and eqn (8), one finds

$$\Delta K(t,z) = K\left((a(z,t)-a(\zeta,t))^*\left(\frac{1}{2}Q_a(t,z)-Q_c(t,z)-\frac{\partial^2}{\partial t^2}\left(\frac{F_r(-z/t)tH(t-|z|/\alpha a)}{\pi z^2}\right)\right.\right.$$
$$\left.\left.-F_i'(0)\delta'(t)\delta(z)\right)\right) \tag{12}$$

$$:= \Delta K_a(t,z) - \Delta K_c(t,z) - \Delta K_f(t,z) \tag{13}$$

where

$$\Delta K_a(t,z) = \frac{1}{2}K((a(z,t)-a(\zeta,t))^*Q_a(t,z))$$
$$= PV\frac{1}{2}K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(a(z',t')-a(\zeta',t'))\left(\frac{V}{\alpha^2a^2}\delta'(t-t')\delta(z-z')-\frac{1}{\pi\alpha}\right.$$
$$\left.\times\frac{\partial}{\partial t}\left(\frac{(t-t')}{(z-z')^2}\frac{H(t-t'-|z-z'|/\alpha a)}{\sqrt{(t-t')^2-(z-z')^2/\alpha^2a^2}}\right)\right\}\mathrm{d}z'\,\mathrm{d}t'$$
$$= PV\frac{1}{2}K\left(\left(\frac{\partial a(z,t)}{\partial t}-\frac{\partial a(\zeta,t)}{\partial t}\right)\frac{V}{\alpha^2a^2}+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial a(z',t')}{\partial t'}-\frac{\partial a(\zeta,t')}{\partial t'}\right)\right.$$
$$\left.\times\frac{1}{\pi\alpha(z-z')^2}\frac{(t-t')H(t-t'-|z-z'|/\alpha a)}{\sqrt{(t-t')^2-(z-z')^2/\alpha^2a^2}}\mathrm{d}z'\,\mathrm{d}t'\right\}. \tag{14}$$

The choice of ζ in the above development is arbitrary, we may rename ζ as z in the final expression, and set $\partial a(\zeta,t)/\partial t' = V(\zeta,t') = V(z,t')$, thus

$$\Delta K_a(t,z) = \frac{K}{2\pi} PV \int_{-\infty}^{\infty} \int_{-\infty}^{t-|z-z'|/\alpha a} \frac{a(t-t')(V(z',t')-V(z,t'))}{(z-z')^2\sqrt{(t-t')^2\alpha^2 a^2-(z-z')^2}} \mathrm{d}z'\,\mathrm{d}t'. \tag{15}$$

Using same procedure, we obtain that

$$\Delta K_c(t,z) = \frac{K}{\pi} PV \int_{-\infty}^{\infty} \int_{-\infty}^{t-|z-z'|/\gamma c} \frac{c(t-t')(V(z',t')-V(z,t'))}{(z-z')^2\sqrt{(t-t')^2\gamma^2 c^2-(z-z')^2}} \mathrm{d}z'\,\mathrm{d}t' \tag{16}$$

$$\Delta K_f(t,z) = \frac{K}{\pi} PV \int_{-\infty}^{\infty} \int_{-\infty}^{t-|z-z'|/\alpha a} \frac{(V_t(z',t')-V_t(z,t'))(t-t')F_r\left(-\dfrac{z-z'}{t-t'}\right)}{(z-z')^2} \mathrm{d}z'\,\mathrm{d}t', \tag{17}$$

with PV denoting the principal value integral and $V(z,t) = \partial a(z,t)/\partial t$, $V_t(z,t) = \partial V(z,t)/\partial t$ being the local crack velocity and acceleration, respectively.

The stress intensity factor field associated with the perturbed crack is then

$$K(t,z)+\Delta K(t,z) = K+\Delta K_a-\Delta K_c-\Delta K_f. \tag{18}$$

We derive the 3D solution as a linearized perturbation about the 2D results for a crack moving at a steady speed V_0 under mode I situation, and hence for which

$$K = K_0 = k(V_0)K^* \quad G = G_0 = g(V_0)G^* \tag{19}$$

where

$$k(V_0) = \frac{1-\dfrac{V_0}{c}}{\sqrt{1-\dfrac{V_0}{a}}} \frac{1}{S_+(1/V_0)} \quad g(V_0) \approx 1-\frac{V_0}{c} \tag{20}$$

K^*, G^* being the static factors.

The energy vector factor $g(V)$ has complex functional form but it can be taken to be very simple form as eqn (20) for most practical purposes [e.g., Freund (1990)].

The result (18) can be rewritten as

$$K(t,z) = k(V)K^*(1+I_a(z,t)-I_c(z,t)-I_f(z,t)) \tag{21}$$

where

$$I_a(z,t) = \frac{1}{2\pi} PV \int_{-\infty}^{\infty} \int_{-\infty}^{t-|z-z'|\alpha a} \frac{a(t-t')(V(z',t')-V(z,t'))}{(z-z')^2\sqrt{(t-t')^2\alpha^2 a^2-(z-z')^2}} \mathrm{d}z'\,\mathrm{d}t'$$

$$I_c(t,z) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \int_{-\infty}^{t-|z-z'|/\gamma c} \frac{c(t-t')(V(z',t')-V(z,t'))}{(z-z')^2\sqrt{(t-t')^2\gamma^2 c^2-(z-z')^2}} \mathrm{d}z'\,\mathrm{d}t'$$

$$I_f(z,t) = \frac{1}{\pi} PV \int_{-\infty}^{\infty} \int_{-\infty}^{t-|z-z'|/\alpha a} \frac{(V_t(z',t')-V_t(z,t'))(t-t')F_r\left(-\dfrac{z-z'}{t-t'}\right)}{(z-z')^2} \mathrm{d}z'\,\mathrm{d}t'. \tag{22}$$

From the above expression, we can learn that the dependency of the stress intensity factor on crack front shape deviations from straightness is given in these integrals as a functional of velocity and acceleration differences along the crack tip during the entire history of the crack motion. Examination of the expression for these integrals shows that there is the system of waves produced in mode I situation, including the dilatational waves, shear waves and Rayleigh waves. Those waves interact on each other and with the moving edge of crack, so that associated stress intensity factor history shows very complex feature (e.g., Li and Liu (1994), (1995)). After the crack has propagated beyond the local heterogeneities of fracture resistance which launched them, as Rice *et al.* noted that, the constructive and destructive interference can lead to continuing fluctuations of the crack front shape and propagation velocity even when the front is moving through material of locally uniform fracture resistance.

Following Rice *et al.* (1994), the fracture criterion is

$$G(z,t) = G_{\text{crit}}(x,z) \tag{23}$$

at points $x = a(z,t)$ along the rupture front where $V(z,t) > 0$. Here $G(z,t)$ is the energy release rate per unit of new crack area and the critical energy release rate, $G_{\text{crit}}(x,z)$ is regarded as a material property.

Using the relation between the energy release rate and the dynamic stress intensity factor, we give that

$$G = G^* g(V)(1 + I_a(z,t) - I_c(z,t) - I_f(z,t))^2 \tag{24}$$

where $G^* = (1-v^2)/(2E)(K^*)^2$ is the rest energy release rate supplied to a straight crack front.

From (23) and (24), the space and time varying motion of the crack front is governed by

$$V(z,t) = \begin{pmatrix} c(1-\Lambda(z,t)), & \text{if } \Lambda(z,t) < 1 \\ 0 & \text{if } \Lambda(z,t) \geqslant 1 \end{pmatrix} \tag{25}$$

where

$$\Lambda(z,t) = \frac{G_{\text{crit}}(a,z)}{G^*(1+I_a-I_c-I_f)^2}. \tag*{(25)$'$}$$

3. FOURIER REPRESENTATION OF RESULTS

For purposes of numerical analysis of spontaneous crack growth, it is convenient to recast the results above in terms of Fourier components, in z, of $a(z,t)$.

Considering the domain of integration and changing the order and limits of the integral in (22), we have

$$I_a(z,t) = \frac{1}{2\pi}\int_{-\infty}^{t}\int_{z-\alpha a(t-t')}^{z+\alpha a(t-t')} \frac{a(t-t')(V(z',t')-V(z,t'))}{(z-z')^2\sqrt{(t-t')^2\alpha^2 a^2-(z-z')^2}}\,\mathrm{d}z'\,\mathrm{d}t'$$

$$I_c(z,t) = \frac{1}{\pi}\int_{-\infty}^{t}\int_{z-\gamma c(t-t')}^{z+\gamma c(t-t')} \frac{c(t-t')(V(z',t')-V(z,t'))}{(z-z')^2\sqrt{(t-t')^2\gamma^2 c^2-(z-x')^2}}\,\mathrm{d}z'\,\mathrm{d}t'$$

$$I_f(z,t) = \frac{1}{\pi}\int_{-\infty}^{t}\int_{z-\alpha a(t-t')}^{z+\alpha a(t-t')} \frac{(V_t(z',t')-V_t(z,t'))(t-t')F_r\left(-\dfrac{z-z'}{t-t'}\right)}{(z-z')^2}\,\mathrm{d}z'\,\mathrm{d}t' \tag{26}$$

where the principal value of the integrals is assumed implicitly.

Using in (26), the variable substitution $z' = z + \alpha a(t-t')\sin\theta$ gives

$$I_a(z,t)=\frac{1}{2\pi\alpha}\int_{-\infty}^{t}\frac{1}{\alpha a(t-t')}\int_{-\pi/2}^{\pi/2}\frac{V(z+\alpha a(t-t')\sin\theta,t')-V(z,t')}{\sin^2\theta}\mathrm{d}\theta\,\mathrm{d}t'$$

$$I_c(z,t)=\frac{1}{\pi\gamma}\int_{-\infty}^{t}\frac{1}{\gamma c(t-t')}\int_{-\pi/2}^{\pi/2}\frac{V(z+\gamma c(t-t')\sin\theta,t')-V(z,t')}{\sin^2\theta}\mathrm{d}\theta\,\mathrm{d}t'$$

$$I_f(z,t)=\frac{1}{\pi\alpha a}\int_{-\infty}^{t}\int_{-\pi/2}^{\pi/2}\frac{F_r(\alpha a\sin\theta)V_t(z+\alpha a(t-t')\sin\theta,t')-V_t(z,t')}{\sin^2\theta}\cos\theta\,\mathrm{d}\theta\,\mathrm{d}t'. \quad (27)$$

We use for $a(z,t)$ and $V(z,t)$ the Fourier representation

$$a(z,t)=\sum_{n=-N}^{N}A_n(t)e^{i2\pi n\frac{z}{\lambda}}$$

$$V(z,t)=\sum_{n=-N}^{N}\dot{A}_n(t)e^{i2\pi n\frac{z}{\lambda}}$$

$$V_t(z,t)=\sum_{n=-N}^{N}\ddot{A}_n(t)e^{i2\pi n\frac{z}{\lambda}}. \quad (28)$$

Where N is chosen as a power of 2 and the over-dot denotes a derivative with respect to time. Here $A_0(t)$ and $A_N(t)$ are real, and $A_{-n}(t)$ is the complex conjugate of $A_n(t)$ so that $\{A_n(t)\}$ involves N real functions, and similar remarks apply to the set $\{\dot{A}_n(t)\}$ and $\{\ddot{A}_n(t)\}$. One require $nA_n/\lambda \ll 1$ and $\dot{A}_n/V_0 \ll 1$, and we assume $\dot{A}_n(-\infty)=0$, $\ddot{A}_n(-\infty)=0$ for $n\neq 0$, we thus obtain

$$I_{(*)}(z,t)=\sum_{n=-N}^{N}I_n^{(*)}(t)e^{i2\pi n\frac{z}{\lambda}} \quad *:=a,c,f \quad (29)$$

where the coefficients are obtained as

$$I_n^a(t)=\frac{1}{2\pi\alpha}\int_{-\infty}^{t}\frac{\dot{A}_n(t')}{\alpha a(t-t')}\int_{-\pi/2}^{\pi/2}\frac{\exp(i\beta_a\sin\vartheta)-1}{\sin^2\vartheta}\mathrm{d}\vartheta\,\mathrm{d}t'$$

$$I_n^c(t)=\frac{1}{\pi\gamma}\int_{-\infty}^{t}\frac{\dot{A}_n(t')}{\gamma c(t-t')}\int_{-\pi/2}^{\pi/2}\frac{\exp(i\beta_c\sin\vartheta)-1}{\sin^2\vartheta}\mathrm{d}\vartheta\,\mathrm{d}t'$$

$$I_n^f(t)=\frac{1}{\pi\alpha\alpha}\int_{-\infty}^{t}\int_{-\pi/2}^{\pi/2}\frac{F_r(\alpha a\sin\vartheta)\ddot{A}_n(t')(\exp(i\beta_n\sin\vartheta)-1)}{\sin^2\vartheta}\cos\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}t' \quad (29)'$$

and where

$$\beta_a=\frac{2\pi n\alpha a(t-t')}{\lambda} \quad \beta_c=\frac{2\pi n\gamma c(t-t')}{\lambda} \quad (30)$$

set

$$Q(\beta)=\int_{-\pi/2}^{\pi/2}\frac{\exp(i\beta\sin\vartheta)-1}{\pi\sin^2\vartheta}\mathrm{d}\vartheta \quad (31)$$

note that

$$Q''(q) = -J_0(\beta), \quad Q(0) = Q'(0) = 0. \tag{31$'$}$$

Consider new function $F(q) = -Q(q)/q$, by virtue of the relations between $Q(q)$ and $F(q)$, we may obtain (see Appendix 2)

$$F(q) = \int_0^q \frac{J_1(p)}{p}\,\mathrm{d}p = \int_0^q J_0(p)\,\mathrm{d}p - J_1(q). \tag{32}$$

Using the expression of $F(q)$, eqn (29)$'$ may be rewritten as

$$\begin{aligned} I_n^q(t) &= -\frac{n\pi}{\alpha\lambda}\int_{-\infty}^{t} \dot{A}_n(t')F\left(2\pi n\frac{\alpha a(t-t')}{\lambda}\right)\mathrm{d}t' \\ I_n^c(t) &= -\frac{2n\pi}{\gamma\lambda}\int_{-\infty}^{t} \dot{A}_n(t')F\left(2\pi n\frac{\gamma c(t-t')}{\lambda}\right)\mathrm{d}t' \end{aligned} \tag{33}$$

and $I_n^f(t)$ can be rewritten as

$$I_n^f(t) = \frac{1}{\pi\alpha a}\int_{-\infty}^{t} \ddot{A}(t')D(t-t')\,\mathrm{d}t'. \tag{34}$$

Here

$$D(t-t') = \int_{-1}^{1} \frac{F_r(\alpha a u)}{u^2}\left(\exp(i\beta_a u) - 1\right)\mathrm{d}u \tag{35}$$

we easily verify that $F_r(x)$ is an even function, that is

$$F_r(x) = F_r(-x) \tag{36}$$

therefore, the result (35) may be simplified as

$$D(t-t') = \int_0^1 \frac{2}{u^2}F_r(\alpha a u)\cos(\beta_a u)\,\mathrm{d}u \tag{37}$$

or $I_n^f(t)$ may be written as

$$I_n^f(t) = \frac{1}{\pi\alpha a}\int_{-\infty}^{t} \ddot{A}_n(t')\,\mathrm{d}t'\int_0^1 \frac{2}{u^2}F_r(\alpha a u)\cos 2\pi n\alpha a\left(\frac{t-t'}{\lambda}\right)u\,\mathrm{d}u. \tag{38}$$

4. MODAL ANALYSIS OF RECOVERY OF THE STRAIGHT CRACK FRONT FROM A PERTURBATION

According to strictly linearized analysis concept proposed by Rice *et al.* (1994), the strictly linearized form of energy release rate, most conveniently given for

$$\sqrt{G(z,t)} \approx \sqrt{G_0}\left(1 - \frac{1}{2}\frac{V-V_0}{c-V_0} + I_a(z,t) - I_c(z,t) - I_f(z,t)\right) \tag{39}$$

where G_0 is given by (19).

Substituting in (39) the Fourier representation of $V(z,t)$, I_a, I_c and I_f, we have

$$\sqrt{G(z,t)} = \sqrt{G_0}\left(1+\frac{1}{2}\frac{V_0}{c-V_0}+\sum_{n=-N}^{N}\left(\frac{-1}{2(c-V_0)}\dot{A}_n(t)-\frac{n\pi}{\alpha\lambda}\int_{-\infty}^{t}\dot{A}_n(t')F(\beta_a)\,\mathrm{d}t'\right.\right.$$

$$\left.\left.+\frac{2n\pi}{\gamma\lambda}\int_{-\infty}^{t}\dot{A}_n(t')F(\beta_c)\,\mathrm{d}t'-\frac{1}{\pi\alpha a}\int_{-\infty}^{t}\ddot{A}_n(t')\int_0^1\frac{2}{u^2}F_r(\alpha au)\cos(\beta_a u)\,\mathrm{d}u\,\mathrm{d}t'\right)e^{i2\pi nz}\right). \quad (40)$$

The material property $\sqrt{G_{\text{crit}}(x,z)}$, assumed to be only slightly inhomogeneous for consistency with a first-order perturbation, can be written as a Fourier series:

$$\sqrt{G_{\text{crit}}(x,z)} = \sum_{n=-N}^{N} g_n(x)e^{i2\pi n\frac{z}{\lambda}}. \quad (41)$$

In using the fracture criterion (23), we may approximate x as V_0t as is consistent with a strictly linearized analysis in perturbation amplitude. Thus, the equation governing the crack tip motion is given by the requirement

$$\sqrt{G(z,t)} = \sqrt{G_{\text{crit}}(x,z)}. \quad (42)$$

We then get for $n = 0$

$$\sqrt{G_0}\left(1+\frac{V_0-\dot{A}_0(t)}{2(c-V_0)}\right) = g_0(V_0t) \quad (43)$$

and for $n \neq 0$

$$\sqrt{G_0}\left(\frac{-\dot{A}_n(t)}{2(c-V_0)}-\frac{n\pi}{\alpha\lambda}\int_{-\infty}^{t}\dot{A}_n(t')F(\beta_a)\,\mathrm{d}t'+\frac{2\pi n}{\gamma\lambda}\int_{-\infty}^{t}\dot{A}_n(t')F(\beta_c)\,\mathrm{d}t'\right.$$

$$\left.-\frac{1}{\pi\alpha a}\int_{-\infty}^{t}\ddot{A}_n(t')\int_0^1\frac{2}{u^2}F_r(\alpha au)\cos(\beta_a u)\,\mathrm{d}u\,\mathrm{d}t'\right) = g_n(V_0t). \quad (44)$$

Applying the Laplace transform to both sides of (44) and set $\dot{A}_n(t) = B_n(t)$, we find that

$$B_n(s) = \frac{g_n(s)}{\sqrt{G_0}C_n(s)} \quad (45)$$

where

$$C_n(s) = \frac{-1}{2(c-V_0)}-\frac{n\pi}{\alpha\lambda}\left(\frac{1}{s}\sqrt{1+\frac{s^2}{q_a^2}}-\frac{1}{q_a^2}\right)+\frac{2\pi n}{\gamma\lambda}\left(\frac{1}{s}\sqrt{1+\frac{s^2}{q_c^2}}-\frac{1}{q_c^2}\right)$$

$$-\frac{s}{\pi\alpha a}\int_0^1\frac{2}{u^2}F_r(\alpha au)\frac{s^2}{s^2+q_u^2}\,\mathrm{d}u \quad (46)$$

and

$$q_a = 2\pi n\alpha\frac{a}{\lambda} \quad q_c = 2\pi n\gamma\frac{c}{\lambda} \quad q_u = 2\pi n\alpha u\frac{a}{\lambda}. \quad (47)$$

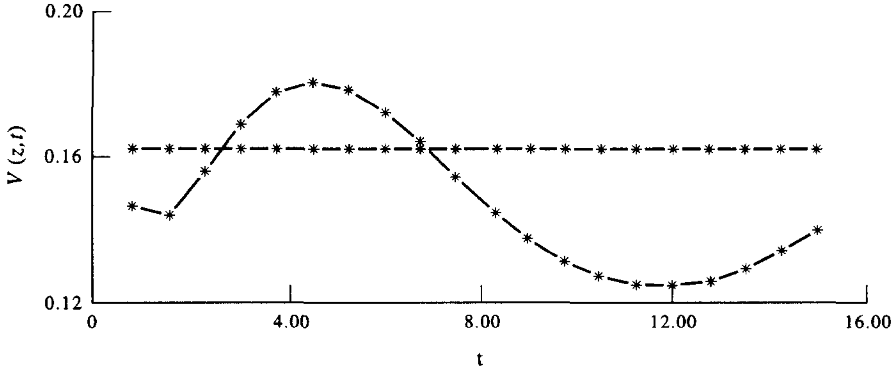

Fig. 3. Modal propagation velocity for a straight crack after a heterogeneous strip, with $V_0 = 0.3c$, $p = 0.1$, $t_1 = 0$, $t_2 = 0.5$, $\hat{\lambda} = 6.0$, $z = 5.7$, $c = 0.53851a = 0.93273b$.

Applying an inverse Laplace transform to (45), we obtain the modal velocity response to a toughness heterogeneity as

$$B_n(t) = \dot{A}_n(t) = L^{-1}\left(\frac{g_n(s)}{\sqrt{G_0}C_n(s)}\right). \tag{48}$$

Inversion of the transform for $B_n(s)$ less straightforward but it can be carried out by numerical inversion method.

Consider a simple case of a finite-width strip, having a single $m \neq 0$ Fourier component of heterogeneity, embedded in an otherwise homogeneous medium. Specifically, we assume that besides g_0, which is constant at $\sqrt{G_0}$, only $g_m(\tau)$ and $g_{-m}(\tau)$ are non-zero (they are complex conjugate of each other), we write

$$g_m(\tau) = g_{-m}(\tau) = \left(\frac{p\sqrt{G_0}}{2}\right)(H(\tau-\tau_1)-H(\tau-\tau_2)) \tag{49}$$

where H is the Heaviside unit step function and p is the (small) amplitude.

That is, writing $\hat{\lambda}$ for $\lambda/|m|$, the result is that the response to

$$\sqrt{G_{\text{crit}}(x,z)} = \sqrt{G_0}\left(1+p\cos\left(2\pi\frac{z}{\hat{\lambda}}\right)\right) \tag{50}$$

for x within the strip, with $\sqrt{G_{\text{crit}}(x,z)} = \sqrt{G_0}$ for x outside it, that the propagation velocity is

$$V(z,t) = V_0 + 2\dot{A}_m(\tau)\cos\left(2\pi\frac{z}{\hat{\lambda}}\right). \tag{51}$$

Figure 3 gives calculations for $V(z,t)$ when $V_0 = 0.3c$, $p = 0.1$, $t_1 = 0$, $t_2 = 0.5$ and $\hat{\lambda} = 2L$. From the figure, we see that when a straight crack enters a heterogeneous region its motion is modulated by a set of decaying oscillations.

5. NUMERICAL SIMULATION ON DYNAMIC GROWTH OF A CRACK

In this section the previous results are used to simulate the dynamic growth of a crack along a plane having a non-uniform distribution of critical energy release rate. The plane of the crack is characterized by a homogeneous "background" critical energy release rate $G_{\text{crit}0}$, from which there are local perturbations where $G_{\text{crit}}(x,z) \neq G_{\text{crit}0}$.

Following Rice *et al.*, our simulations begin with a straight crack propagating with a uniform velocity V_0 in the region $x < 0$. The calculation of space and time varying dynamic crack propagation in the heterogeneous region $x > 0$ (correct to first order) is done using the following procedure:

(1) Having crack front positions, velocities and accelerations at a general (discrete) time step $m\Delta t$

$$a(z, m\Delta t) = \sum_{n=-N}^{N} A_n(m\Delta t)e^{i2\pi n\frac{z}{\lambda}}$$

$$V(z, m\Delta t) = \sum_{n=-N}^{N} \dot{A}_n(m\Delta t)e^{i2\pi n\frac{z}{\lambda}}$$

$$V_t(z, m\Delta t) = \sum_{n=-N}^{N} \ddot{A}_n(m\Delta t)e^{i2\pi n\frac{z}{\lambda}}. \tag{52}$$

Use the FFT procedure to calculate from current velocities $V(z, m\Delta t)$, accelerations $V_t(z, m\Delta t)$ the Fourier coefficients $\dot{A}_n(m\Delta t)$, $\ddot{A}_n(m\Delta t)$; we first FFT the set $\{v(z_j, t)\}$ to get

$$\hat{A}_n(t) = \sum_{j=0}^{m-1} V(z_j, t)e^{-2\pi in\frac{j}{N}} \quad (m = 2N) \tag{53}$$

this coefficient set $\{\hat{A}_n(t)\}$ is related to

$$\dot{A}_n = \hat{A}_n/m \quad \text{for } n = 0 \quad \text{to } m/2-1$$
$$\dot{A}_{m/2} = \hat{A}_{m/2}/2m \quad \text{for } n = m/2, -m/2$$
$$\dot{A}_n = \hat{A}_{n+m/2}/m \quad \text{for } n = -m/2+1 \quad \text{to } -1.$$

Verify that first order perturbation conditions $\dot{A}_n/V_0 \ll 1$, $n\dot{A}_n\Delta t/\lambda \ll 1$ are satisfied.
(2) Calculation local crack front velocities for the next time increment as follows:
(2.1) using (33) and (38) and history of $\dot{A}_n$, $\ddot{A}_n$ to calculate the coefficient I_n^a, I_n^c, I_n^f $(m\Delta t)$, where the integral calculations of eqns (33) and (38) by use of Gaussian quadrature formulas are:

$$I_n^a(t_i) = -\frac{n\pi}{2\alpha}\sum_{i=1}^{m}\sum_{j=1}^{12}\left\{\left(\frac{\dot{A}_n(t_i)+\dot{A}_n(t_{i-1})}{2}+\frac{\dot{A}_n(t_i)-\dot{A}_n(t_{i-1})}{2}y_j\right)\right.$$
$$\left.\times F\left[2\pi n\alpha\left(m\Delta T-\frac{t_i+t_{i-1}}{2}+\frac{t_i-t_{i-1}}{2}y_j\right)\right]\right\}DTw_j$$

$$I_n^c(t_i) = -\frac{n\pi}{\gamma}\sum_{i=1}^{m}\sum_{j=1}^{12}\left\{\left(\frac{\dot{A}_n(t_i)+\dot{A}_n(t_{i-1})}{2}+\frac{\dot{A}_n(t_i)-\dot{A}_n(t_{i-1})}{2}y_j\right)\right.$$
$$\left.\times F\left[2\pi n\gamma c\left(m\Delta T-\frac{t_i+t_{i-1}}{2}+\frac{t_i-t_{i-1}}{2}y_j\right)\right]\right\}DTw_j$$

$$I_n^f(t_i) = -\frac{1}{\pi\alpha}\sum_{i=1}^{m}\sum_{k=1}^{12}\ddot{A}_n(t_i)\frac{30^*y_k^2}{\mu_k^2}F_r(\alpha\mu_k)\int_{t_{i-1}}^{t}\cos(\beta_a\mu)\,dt^*w_k. \tag{54}$$

Here y_j, w_j are the jth Gaussian zeros and weights of order 12, respectively. The third nonlinear polynomials transform has been used to delete the singular point "0" in the equation of I_n^f.

Next, we rearrange from $\{\hat{I}_n^*(t)\}$ to $\{I_n^*(t)\}$, following the same rules as for rearrange from $\{\hat{A}_n(t)\}$ to $\{\dot{A}_n(t)\}$ above.

(2.2) use FFT to invert $I_n^\alpha(m\Delta t)$, ($\alpha = a, c, F$) to $I(z, m\Delta t)$

(2.3) use (25) and current crack front positions to calculate velocities $V(z,(m+1)\Delta t)$ during the next time step.

(3) use difference formula to calculate accelerations $V_t(z,(m+1)\Delta t)$ during the next time step:

$$V_t(z,(m+1)\Delta t) = (V(z,(m+1)\Delta t) - V(z, m\Delta t))/\Delta t.$$

(4) Calculate the local crack front positions at the end of the next time step as

$$a(z,(m+1)\Delta t) = a(z, m\Delta t) + V(z,(m+1)\Delta t)\Delta t$$

(5) Write output, check exit criteria (location of crack front or violation of first order conditions); increase time index m by 1, go to step (1).

6. RESULTS AND DISCUSSION

Figure 1 shows a periodic array of circular asperities with radius R and center to center spacing L. Following the conditions of Fares (1989) for the validity of quasi-static first-order analysis and by Gao and Rice (1989), Rice *et al.* (1994), we choose $R/L = 0.1$, and the speed of Rayleigh waves $c = 0.53851a = 0.93273b$. All calculations have been non-dimensionalized.

In order to check correctness for our numerical computational program, we firstly calculated several problems solved by Rice *et al.* (1994) and obtained same results. Figure 4 shows one of those results using Rice *et al.* model, calculated for the case $V_0 = 0.3c$, hitting an infinite row of asperities with $G_{\text{crit}}(\text{left}) = G_{\text{crit}}(\text{right}) = 4G_0$.

Figure 5 shows crack front profiles in the regions at successives times and $G_{\text{crit}}(\text{left})/G_0 = 4.0$ and $G_{\text{crit}}(\text{right})/G_0 = 2.0$, where $G_{\text{crit}}(\text{left})$ and $G_{\text{crit}}(\text{right})$ denote, respectively, the critical energy release rates of the left and right asperities in a fundamental wavelength $\lambda = 2L$. G_0 denote non-asperity regions. The computations are done using $m = 2N = 256$ and $\Delta t = \lambda/5Nc$. At the initial instant, a straight crack was propagating with a uniform velocity V_0 in the region $x < 0$. The asperities block the crack advancement after it penetrates into the inter-asperity G_0 regions. As a result, the distribution of velocity initial uniform, turns wavy and shows instability feature in a successive instants. Then the weaker right asperity broke, the left asperity also broke after some further crack front motion.

Figure 6 shows calculations for the case $V_0 = 0.45c$, hitting an infinite elliptical asperities $(z-0.5)^2/3+(x-0.1)^2 \leqslant 0.1^2$ and $(z-1.5)^2/3+(x-0.1)^2 \leqslant 0.1^2$ with $G_{\text{crit}}(\text{left})/G_0 = 3.0$ and $G_{\text{crit}}(\text{right})/G_0 = 2.0$, at this relatively large incoming crack velocity, the asperities break after causing a small retardation in crack front positions. By virtue of the Mode I dynamic

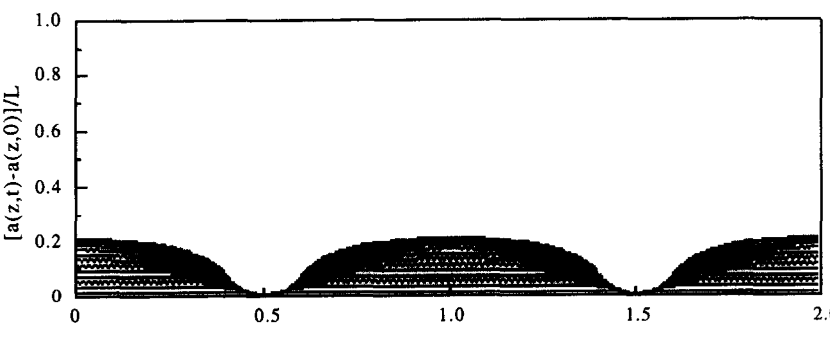

Fig. 4. Positions $a(z, t)$ vs z at successive times, for a crack at incoming speed $V_0 = 0.3c$, hitting an infinite row of asperities with $G_{\text{crit}}(\text{left}) = 4G_0$ and $G_{\text{crit}}(\text{right}) = 4G_0$, using Rice *et al.* model results.

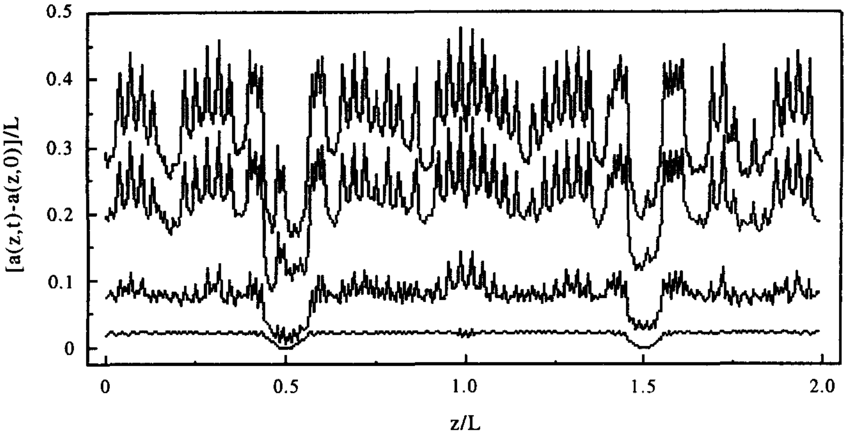

Fig. 5. Positions $a(z, t)$ vs z at successive times, for a crack at incoming speed $V_0 = 0.3c$, hitting an infinite row of asperities with $G_{crit}(\text{left}) = 4G_0$ and $G_{crit}(\text{right}) = 2G_0$. (a) $t = 45\Delta t$; (b) $t = 120\Delta t$; (c) $t = 240\Delta t$; (d) $t = 320\Delta t$. Here $\Delta t = 1/640$.

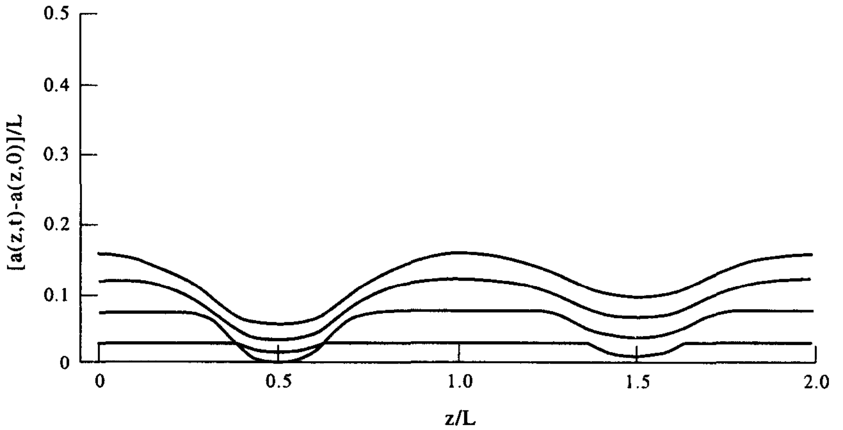

Fig. 6. Positions $a(z, t)$ vs z at successive times, for a crack at incoming speed $V_0 = 0.45c$, hitting an infinite row of elliptical asperities $(z-0.5)^2/3+(x-0.1)^2 \leqslant 0.1^2$ and $(z-1.5)^2/3+(x-0.1)^2 \leqslant 0.1^2$ with $G_{crit}(\text{left}) = 3G_0$ and $G_{crit}(\text{right}) = 2G_0$. (a) $t = 50\Delta t$; (b) $t = 100\Delta t$; (c) $t = 160\Delta t$; (d) $t = 220\Delta t$. Here $\Delta t = 1/640$.

weight function and its numerical simulations in a first order perturbation analysis of the deviation from straightness of the crack edge, we observe that, oscillatory effects in crack motion, as denoted by Rice *et al.*, are found to follow encounter of the crack front with regions of variable toughness and these may also be interpreted in terms of constructive-destructive interference of stress intensity waves initialed by encounters of the crack front with asperities and then propagating along the front. These waves, including system of the dilational, shear and Rayleigh waves, interact on each other and with moving edge of crack, lead to oscillatory feature of crack front profiles. It seems that the type of oscillatory crack motion could be the basis for careful recent measurements by Gross *et al.* (1993).

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APPENDIX 1

Mode I dynamic weight function $[U]^{(+)}$ defines as

$$[U]^{(+)} = \Phi_+ * W_+ \tag{A.1}$$

where "*" denotes the convolution over the variable t, X, z

$$\Phi_+(t,X,z) = -i\pi^2 C\left(\frac{\pi i}{X}\right)^{1/2}\left(\frac{X/\alpha}{X^2/\alpha^2+z^2}\right)H(X)\times\frac{\partial}{\partial t}\left[\left(t-\frac{VX}{\alpha^2 a^2}\right)\mathrm{Re}\left\{\left\{\left[t-\frac{VX}{\alpha^2 a^2}\right]^2-\frac{X^2/\alpha^2+z^2}{\alpha^2 a^2}\right\}^{-1/2}\right\}\right] \tag{A.2}$$

$$W_+(t,X,z) = -(2\pi)^{-2}\frac{\partial^2}{\partial t^2}\int_{|\eta|=1}\frac{\mathrm{d}s}{(\eta\cdot x/t-0i)^2}\frac{1}{t}\frac{\left[\eta_1-\dfrac{\eta\cdot x}{t}\dfrac{V}{\alpha^2 a^2}+iq_a\right]}{\left[\eta_1+\left(\dfrac{-\eta\cdot x}{t}+0i\right)\dfrac{V}{\gamma^2 c^2}+iq_c\right]T_+(-\eta\cdot x/t+0i,\eta_1,\eta_2)}. \tag{A.3}$$

Here $\eta\cdot x = \eta_1 X+\eta_2 x_2$ and $T_+(\omega,\xi_1,\xi_2)$ is

$$T(\omega,\xi_1\xi_2) = T_+(\omega,\xi_1,\xi_2)T_-(\omega,\xi_1,\xi_2). \tag{A.4}$$

Detail expression of $T(\omega,\xi_1,\xi_2)$ can be found from Willis *et al.* (1995).

APPENDIX 2

The function $F(q)$ satisfies equation

$$F''(q)+\frac{2}{q}F'(q) = \frac{1}{q}J_0(q) \tag{B.1}$$

easily obtain:

$$F'(q) = \frac{1}{q^2}\left(\int_0^q pJ_0(p)\,\mathrm{d}p+C\right). \tag{B.2}$$

One notes that

$$\operatorname*{Lim}_{q\to 0}F'(q) = \operatorname*{Lim}_{q\to 0}\left(-\frac{Q'(q)}{q}+\frac{Q(q)}{q^2}\right) = \operatorname*{Lim}_{q\to 0}\left(-Q''(q)+\frac{Q''(q)}{2}\right) = \frac{1}{2}J_0(0) = 0.5.$$

While

$$\lim_{q\to 0}\frac{1}{q^2}\int_0^q pJ_0(p)\,\mathrm{d}p = \frac{1}{2}J_0(0) = 0.5$$

thus $C = 0$.

One has

$$F(q) = \frac{1}{q^2}\int_0^q J_0(p)p\,\mathrm{d}p. \tag{B.3}$$

By virtue of the relation between the Bessel functions $J_0(\beta)$ and $J_1(\beta)$

$$\beta J_0(\beta) = \frac{\mathrm{d}}{\mathrm{d}\beta}(\beta J_1(\beta))$$

one has

$$F(q) = \frac{1}{q^2}\int_0^q \frac{\mathrm{d}}{\mathrm{d}p}(pJ_1(p))\,\mathrm{d}p = \frac{J_1(q)}{q}. \tag{B.4}$$

So one obtains eqn (33).