Algorithm for simulating fracture processes in concrete by lattice modeling

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ABSTRACT

To simulate fracture behaviors in concrete more realistically, a theoretical analysis on the potential question in the quasi-static method is presented, then a novel algorithm is proposed which takes into account the inertia effect due to unstable crack propagation and meanwhile requests much lower computational efforts than purely dynamic method. The inertia effect due to load increasing becomes less important and can be ignored with the loading rate decreasing, but the inertia effect due to unstable crack propagation remains considerable no matter how low the loading rate is. Therefore, results may become questionable if a fracture process including unstable cracking is simulated by the quasi-static procedure excluding completely inertia effects. However, it requires much higher computational effort to simulate experiments with not very high loading rates by the dynamic method. In this investigation which can be taken as a natural continuation, the potential question of quasi-static method is analyzed based on the dynamic equations of motion. One solution to this question is the new algorithm mentioned above. Numerical examples are provided by the generalized beam (GB) lattice model to show both fracture processes under different loading rates and capability of the new algorithm.

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1. Introduction

Numerical analysis of fracture processes in concrete is an interesting research topic. Many numerical investigations can be found in the literature [1–6]. Finite elements or remeshing approaches have been used to study fracture [7,8]. Inertia effects on fracture behavior [9] were found to be important when unstable crack propagation occurred. However, the work in [9] was based on approximated numerical examples. Further elaboration is considered presently by making a theoretical analysis to explain why cracking-induced inertia effect can influence the fracture behavior. Then a novel numerical algorithm combining the quasi-static method and the dynamic method is proposed such that inertia effect is included.

Microstructure in concrete influences greatly its fracture processes. The characteristic inelastic response of concrete is very hard to interpret without appealing to their microstructure [10]. Concrete has been described by many investigators as a three-phase material, where matrix, aggregate and interface zones are distinguished [1,2]. Among different kinds of microstructural models, lattice-type models are being applied by more and more investigators. One important reason is that the lattice model allows a straightforward implementation of the material heterogeneity [3]. Lattice-type material modeling has been successfully used for

solving classical problems of elasticity [11]. More recently, it has been employed for simulating the progressive failure in heterogeneous media, especially by theoretical physicists [12]. To investigate fracture processes in concrete, many different types of meso-level lattice models have been developed in the past. The material structure can be projected directly on to the lattice in order to obtain more realistic results [1–3]. The works in [13,14] used truss elements that require some numerical measures to exclude instability during fracture propagation. Adopted in [15] is the Euler–Bernoulli beams of [12,16] that considered a kind of spring element, equivalent to the Euler–Bernoulli beam element for the special case. Established in [4] is a lattice-type fracture model called GB lattice model. Some interesting results have been obtained by using the GB lattice model [5,6,9].

Quasi-static method and dynamic method are two main approaches to deal with fracture of concrete, respectively, applied in the quasi-static loading cases and the dynamic loading cases. In the present investigation, only ideally brittle lattices are discussed. As long as a strength criterion is adopted for failure of the elements, a typical procedure for the quasi-static simulation is as follows [17–19]. The load is applied gradually and linear elastic analysis is performed until the element with the highest stress-to-strength ratio reaches the prescribed threshold value. It is then eliminated and a new analysis is performed after updating the stiffness matrix while the applied load is kept unchanged to check whether another element will fail. If no more elements fail, the calculation is restarted from zero loads again until the complete

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failure of the specimen [3,4]. Obviously, the fracture process obtained by the quasi-static method is composed of a series of static-equilibrium states. In the dynamic method, critical elements are also eliminated immediately, just as what has been done in the double cantilever beam (DCB) test [20,21]. The essential difference between two methods is: fracture processes are simulated by solving the dynamic equations of motion of the system in the dynamic method, but they are obtained by solving the static-equilibrium equations in the quasi-static method. Additionally, the quasi-static method is much more computationally efficient than the dynamic method because the time step has to be set adequately small to guarantee the stability of the simulation in the dynamic method [22,23].

Then, what is the difference between results obtained by the two methods? Does the two kinds of results are consistent with each other when the loading rate is enough low? The first question has been investigated by many researchers. Shown in [10] are the inertial effects on the crack pattern. The second question is equivalent to the assumption of the quasi-static method. In this paper, cases where the loading rate is, respectively, 1.0 m/s, 10^{-1} m/s, 10^{-2} m/s and 10^{-3} m/s are simulated. Moreover, they are, respectively, compared with the quasi-static result. In a dynamic fracture process, there are two kinds of inertia effects: that due to load increasing, and that due to crack propagation. The two inertia effects are discussed in detail. It is found that the inertia effect due to unstable crack propagation still keeps a considerable influence on fracture processes even though the loading rate is very low. The quasi-static method is more computationally efficient but cannot take the inertia effect into consideration, and the dynamic method can include the inertia effect but requires much larger computation effort. This newly proposed method is a combination of the two methods. The method can include the inertia effect. Moreover, it is much efficient than the dynamic method.

2. GB lattice model [4]

2.1. The geometry of the model

The material structure is projected directly on to the lattice [1,3]. In Fig. 1, if both ends of a beam fall in the matrix (aggregate) phase, it is assigned the equivalent matrix (aggregate) properties. If however one of its ends is in the matrix-phase and the other is in the aggregate phase, then it is assigned the equivalent interface properties. The relationship between the properties of a continuum phase and its representative beams in the lattice is obtained by the equivalence of strain energy stored in a unit cell under constant strains [2].

The GB lattice corresponds to the material structure. In Fig. 2a, an aggregate is projected on to a matrix GB lattice, with its center lying on node i . The six elements around node i are partly overlain by the aggregate. These parts are assigned the equivalent aggregate properties. The interface is simplified as a thin layer with a rectangular cross-section clinging to the aggregate. The parts of the six elements overlain by the interface are assigned the interfacial properties. Finally, the GB lattice including the particle structure is shown in Fig. 2b.

The GB element is apparently a two-node and three-phase element. Every phase of the element is represented by a beam of corresponding (equivalent) properties. For example, in Fig. 2b, the six GB elements around node i are all composed of an aggregate beam, an interface beam and a matrix beam. However, it can be imagined that the GB element which is not overlain by any aggregate is actually one-phase, i.e., matrix-phase. Nevertheless, in order to make up the deficiency of the relatively coarser GB

(a) General particle structure

(b) Matrix, interface and aggregate element

Fig. 1. Particle structure modeled by the beam lattice model.

lattice, the one-phase GB elements are divided into three matrix beams of the same span. Analogously, the elements overlain wholly by an aggregate are divided into three aggregate beams of the same span. In a word, all GB elements are composed of three beams.

Fig. 1b is compared with Fig. 2b for an aggregate that usually covers 10–100 nodes in the beam lattice, but an aggregate just overlays a single node in the GB lattice. As a result, computational effort is reduced in the GB lattice model.

If three beams in each GB element are assumed to cling firmly to each other and deform together without sliding, the displacements of two ends of the middle beam in the GB element are completely determined by the displacements of the two nodes of the GB element. Then, the degrees of freedom of two ends of the middle beam are not necessarily included in the discrete system of equations of equilibrium, with the result that computational effort is further reduced.

(a) Inclusion of the aggregate

(b) GB lattice

Fig. 2. A GB lattice including particle structure.

2.2. Stiffness matrices of matrix, aggregate and interface beams

Matrix beams and aggregate beams are described by Timoshenko beam theory. However, Timoshenko beam theory is not very suitable to describe interface beams, because the aspect ratio of interface beams is usually too high. In this paper, interface beams are described by Eq. (A1) in Appendix A of [16]. These two kinds of stiffness matrices have the following common expression (Fig. 3):

$$\begin{Bmatrix} Q_1 \\ N_1 \\ M_1 \\ Q_2 \\ N_2 \\ M_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} M_{11} & 0 & -M_{34} & -M_{11} & 0 & -M_{34} \\ & M_{22} & 0 & 0 & -M_{22} & 0 \\ & & M_{33} & M_{34} & 0 & M_{36} \\ & & & M_{11} & 0 & M_{34} \\ & & & & M_{22} & 0 \\ & & \text{SYM} & & & M_{33} \end{bmatrix}}_{\mathbf{M}} \begin{Bmatrix} u_1 \\ v_1 \\ \varphi_1 \\ u_2 \\ v_2 \\ \varphi_2 \end{Bmatrix} \quad (1)$$

where $\mathbf{F}_{12} = \{Q_1 \ N_1 \ M_1 \ Q_2 \ N_2 \ M_2\}^T$ and $\mathbf{u}_{12} = \{u_1 \ v_1 \ \varphi_1 \ u_2 \ v_2 \ \varphi_2\}^T$ are the generalized force vector and the generalized displacement vector, respectively.

For a Timoshenko beam, the independent elements in Eq. (1) can be expressed in the form [22,24]

Fig. 3. Kinematics and statics of a beam.

$$\begin{aligned} M_{11} &= \frac{12E^{(b)}I}{(1+b)h^3}, & M_{34} &= \frac{6E^{(b)}I}{(1+b)h^2}, & M_{22} &= \frac{E^{(b)}A}{h}, \\ M_{33} &= \frac{E^{(b)}I(4+b)}{h(1+b)}, & M_{36} &= \frac{E^{(b)}I(2-b)}{h(1+b)} \end{aligned} \quad (2)$$

where, $E^{(b)}$ is the Young's modulus; $t^{(b)}$, h and l are, respectively, the thickness, the span and the height of the Timoshenko beam; $A = t^{(b)}l$ is the cross-section area; $I = t^{(b)3}l^3/12$ is the moment of inertia; $b = 6E^{(b)}I^2/5G^{(b)}h^2$ is the dimensionless parameter in Timoshenko beam theory; $G^{(b)} = E^{(b)}/2(1+\nu^{(b)})$ is the modulus of rigidity, where $\nu^{(b)}$ is the Poisson's ratio.

By comparing Eq. (A1) in Appendix A [16] with Eq. (1), the independent elements in Eq. (1) can be expressed in the form

$$\begin{aligned} M_{11} &= \frac{G^{(b)}A}{h}, & M_{34} &= \frac{G^{(b)}A}{2}, & M_{22} &= \frac{E^{(b)'}A}{h}, \\ M_{33} &= \frac{G^{(b)}Ah}{4} + \frac{E^{(b)'}I}{h}, & M_{36} &= \frac{G^{(b)}Ah}{4} - \frac{E^{(b)'}I}{h} \end{aligned} \quad (3)$$

where $E^{(b)'} = E^{(b)}/[1 - (\nu^{(b)})^2]$.

In the following subsections, a superscript is added to $\mathbf{M}$ in order to declare the material property, i.e., $\mathbf{M}^m$, $\mathbf{M}^a$ and $\mathbf{M}^i$ are, respectively, the stiffness matrix of matrix, aggregate and interface beams.

2.3. Parameter calibration of the regular triangular GB lattice

Spans of three beams of each GB element are determined as soon as the particle overlay [1,2] is projected on top of the GB lattice. To an interface beam, the properties of interface material themselves instead of its equivalent properties are assigned. Moreover, the depth of the interface beam can be calculated in the form

$$l = \begin{cases} 2d_i/\sqrt{3} & (d_i \leq L/2) \\ 2(L - d_i)/\sqrt{3} & (L/2 \leq d_i \leq L) \end{cases} \quad (4)$$

where L is the length of the GB element, and d_i denotes the distance between node i and the middle point of the interface beam along its span-direction, in the GB element $i - j$.

Then we introduce how to calculate the geometrical and material properties of both matrix and aggregate beams. The basic idea is based on the equivalence of strain energy stored in a unit cell (Fig. 4) of a lattice with its continuum counterpart under constant strains

$$U_{\text{cell}} = U_{\text{continuum}} \quad (5)$$

The detailed process has been presented in [2,17] to obtain the relationship between the triangular Timoshenko beam lattice and its micropolar continuum equivalent. In the present investigation, the result of [2] is used directly in our parameter system (Fig. 4):

$$\frac{E}{E^{(b)}} = \sqrt{3} \left(\frac{l}{h} \right) \frac{t^{(b)}}{t} \frac{4 + \left(\frac{l}{h} \right)^2 \frac{1}{1+b}}{12 + \left(\frac{l}{h} \right)^2 \frac{1}{1+b}}, \quad \nu = \frac{4 - \left(\frac{l}{h} \right)^2 \frac{1}{1+b}}{12 + \left(\frac{l}{h} \right)^2 \frac{1}{1+b}} \quad (6)$$

where $E^{(b)}$ and E are the Young's modulus of the Timoshenko beams and its continuum equivalent. h and l are the span and depth of the

Fig. 4. A triangular matrix/aggregate GB lattice with a hexagonal unit cell.

Timoshenko beam. $t^{(b)}$ and t are the thickness of the Timoshenko beam and its continuum equivalent. Moreover,

$$b = \frac{12 + 11\nu^{(b)}}{20} \left(\frac{l}{h} \right)^2 \quad (7)$$

The second equation of Eq. (6) can be rewritten in the form

$$b = \frac{1 + \nu}{4(1 - 3\nu)} \left(\frac{l}{h} \right)^2 - 1 \quad (8)$$

Substitution of Eq. (7) into the second equation of Eq. (6) yields the result

$$\nu^{(b)} = \frac{5(1 + \nu)}{11(1 - 3\nu)} - \frac{20}{11} \left(\frac{l}{h} \right)^{-2} - \frac{12}{11} \quad (9)$$

Substitution of Eq. (8) into the first equation of Eq. (6) yields the result

$$E^{(b)} = \frac{2}{\sqrt{3}} \left(\frac{l}{h} \right)^{-1} \frac{t}{t^{(b)}} \frac{E}{1 - \nu} \quad (10)$$

In this investigation, all matrix beams and aggregate beams have rectangular cross-sections, the same aspect ratio $l/h = 1$, and the same thickness $t^{(b)} = t$. The Poisson's ratio of both matrix and aggregate is $11/40$. Then, from Eqs. (9) and (10), their equivalent material properties $\nu^{(b)}$ and $E^{(b)}$, respectively, become

$$\nu^{(b)} = \frac{31}{77} \quad (11)$$

$$E^{(b)} = \frac{80}{29\sqrt{3}} E \quad (12)$$

2.4. The stiffness matrix of a GB element

This section is going to answer how to determine the stiffness matrix of a GB element from its three beams. For the sake of simplification and without loss of generality, a GB element composed of one aggregate beam, one interface beam and one matrix beam is under investigation (Fig. 5).

In the GB element $i-j$ (Fig. 5a), when external generalized forces are applied on node i and j , the relationship between the generalized nodal forces and the generalized nodal displacements can be expressed in the form

$$\mathbf{F}_{ij} = \mathbf{K} \mathbf{u}_{ij}$$

$$\mathbf{F}_{ij} = \{ Q_i \quad N_i \quad M_i \quad Q_j \quad N_j \quad M_j \}^T \quad (13)$$

$$\mathbf{u}_{ij} = \{ u_i \quad v_i \quad \phi_i \quad u_j \quad v_j \quad \phi_j \}^T$$

where $\mathbf{K}$, a symmetric matrix, denotes the stiffness matrix of the GB element.

Analogously, for the aggregate beam $i-I$ (Fig. 5b), the interface beam $I-J$ (Fig. 5c), and the matrix beam $J-j$ (Fig. 5d), the relationships between the kinematics and statics can be expressed in the forms, respectively

$i-I$:

$$\mathbf{F}_{iI} = \mathbf{M}^a \mathbf{u}_{iI}$$

$$\mathbf{F}_{iI} = \{ Q_i \quad N_i \quad M_i \quad Q_I \quad N_I \quad M_I \}^T$$

$$\mathbf{u}_{iI} = \{ u_i \quad v_i \quad \phi_i \quad u_I \quad v_I \quad \phi_I \}^T \quad (14)$$

$I-J$:

$$\mathbf{F}_{IJ} = \mathbf{M}^i \mathbf{u}_{IJ}$$

$$\mathbf{F}_{IJ} = \{ -Q_I \quad -N_I \quad -M_I \quad Q_J \quad N_J \quad M_J \}^T$$

$$\mathbf{u}_{IJ} = \{ u_I \quad v_I \quad \phi_I \quad u_J \quad v_J \quad \phi_J \}^T \quad (15)$$

$J-j$:

$$\mathbf{F}_{Jj} = \mathbf{M}^m \mathbf{u}_{Jj}$$

$$\mathbf{F}_{Jj} = \{ -Q_J \quad -N_J \quad -M_J \quad Q_j \quad N_j \quad M_j \}^T$$

$$\mathbf{u}_{Jj} = \{ u_J \quad v_J \quad \phi_J \quad u_j \quad v_j \quad \phi_j \}^T \quad (16)$$

In consideration of the equilibrium conditions at I and J , a system of algebraic equations expressed in the generalized displacement vector $\mathbf{u}_{ij}$ can be obtained when $\mathbf{u}_{iI}$ are assumed to be known

$$\underbrace{\begin{bmatrix} \mathbf{M}_{ii}^a + \mathbf{M}_{ii}^i & \mathbf{M}_{ii}^i \\ (\mathbf{M}_{ii}^i)^T & \mathbf{M}_{ii}^i + \mathbf{M}_{ii}^m \end{bmatrix}}_A \mathbf{u}_{ij} = \underbrace{\begin{bmatrix} -(\mathbf{M}_{ii}^a)^T & \mathbf{0} \\ \mathbf{0} & -\mathbf{M}_{ii}^m \end{bmatrix}}_B \mathbf{u}_{ij} \quad (17)$$

Fig. 5. A GB element composed of an aggregate beam, an interface beam and a matrix beam.

where $\mathbf{M}_I^a$, $\mathbf{M}_{II}^a$ and $\mathbf{M}_{III}^a$ denote the top-left, top-right and bottom-right 3×3 sub-matrix of $\mathbf{M}^a$, respectively, and the same denotation rule is also used to $\mathbf{M}^i$ and $\mathbf{M}^m$. Both $\mathbf{A}$ and $\mathbf{B}$ are 6×6 matrices.

By solving the system of Eq. (17), the relationship between $\mathbf{u}_{ij}$ and $\mathbf{u}_{ij}$ can be expressed in the form

$$\mathbf{u}_{ij} = \underbrace{\mathbf{A}^{-1}\mathbf{B}}_{\mathbf{R}} \mathbf{u}_{ij} \quad (18)$$

where $\mathbf{R} = \mathbf{A}^{-1}\mathbf{B}$.

$\mathbf{R}_I$, $\mathbf{R}_{II}$ and $\mathbf{R}_{III}$, respectively, denote the top-left, top-right, bottom-right 3×3 sub-matrix of $\mathbf{R}$. After a lengthy but elementary calculation, the stiffness matrix of the GB element $\mathbf{K}$ can be expressed in the form

$$\mathbf{K} = \begin{bmatrix} \mathbf{M}_I^a + \mathbf{M}_{II}^a \mathbf{R}_I & \mathbf{M}_{II}^a \mathbf{R}_{II} \\ \text{SYM} & (\mathbf{M}_{II}^m)^T \mathbf{R}_{III} + \mathbf{M}_{III}^m \end{bmatrix} \quad (19)$$

3. Quasi-static method and dynamic method to simulate fracture processes [9]

3.1. Quasi-static method

As pointed out in the introduction, fracture is simulated by subsequent removal of critical elements from a lattice. Three beams in every GB element are judged whether to be critical independently. Once a beam approaches criticality, the corresponding GB element becomes a critical element. A criterion must be set to decide when a beam must be removed. In this paper the maximum tensile stress in each beam is computed in the following form [1]

$$\sigma_{eff} = \frac{N}{A} + \alpha \frac{(|M_i|, |M_j|)_{\max}}{W} < f_t \quad (20)$$

where N is the normal force in the considered beam, M_i and M_j are the bending moments at the nodes i and j of the beam, and $W = t^{(b)}l^2/6$ is the section modulus. The coefficient α regulates what part of the bending moment is considered. In this paper, α is 0.005 [1–4]. When the effective stress of a particular beam reaches its tensile strength f_t , brittle fracture is simulated by instantaneously removing the beam from the lattice.

3.2. Dynamic method

The dynamic equations of motion of the system can be expressed in the form

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{Q}(t) \quad (21)$$

The symbols in Eq. (21) are explained one by one in the following.

- (1) $\mathbf{M}$ is the mass matrix of the lattice. It is assumed that mass of each GB element concentrates on its ends in the triangular GB lattice. As a result, $\mathbf{M}$ can be written as

$$\mathbf{M} = \text{diag}(m_1, m_1, J_1, m_2, m_2, J_2, \dots, m_n, m_n, J_n) \quad (22)$$

where n is the total of nodes. Then, $\mathbf{M}$ is a $3n \times 3n$ diagonal matrix. For the sake of simplicity, it is assumed that the three phases have the same density ρ . Then, matrix can be written as

$$m_i = \frac{\sqrt{3}\rho L^2 t}{2}, \quad J_i = \frac{5\rho L^4 t}{8} \left(\frac{\sqrt{3}}{9} - \frac{3\pi}{64} \right) \quad (i = 1, 2, \dots, n) \quad (23)$$

- (2) $\mathbf{u}(t)$ is the vector of nodal displacements at time t :

$$\mathbf{u}(t) = \{u_1(t) \ v_1(t) \ \phi_1(t) \ u_2(t) \ v_2(t) \ \phi_2(t) \ \dots \ u_n(t) \ v_n(t) \ \phi_n(t)\}^T \quad (24)$$

Analogously, $\dot{\mathbf{u}}(t)$ and $\ddot{\mathbf{u}}(t)$ are the vector of nodal velocities and the vector of nodal accelerations, respectively:

$$\dot{\mathbf{u}}(t) = \{\dot{u}_1(t) \ \dot{v}_1(t) \ \dot{\phi}_1(t) \ \dot{u}_2(t) \ \dot{v}_2(t) \ \dot{\phi}_2(t) \ \dots \ \dot{u}_n(t) \ \dot{v}_n(t) \ \dot{\phi}_n(t)\}^T \quad (25)$$

$$\ddot{\mathbf{u}}(t) = \{\ddot{u}_1(t) \ \ddot{v}_1(t) \ \ddot{\phi}_1(t) \ \ddot{u}_2(t) \ \ddot{v}_2(t) \ \ddot{\phi}_2(t) \ \dots \ \ddot{u}_n(t) \ \ddot{v}_n(t) \ \ddot{\phi}_n(t)\}^T \quad (26)$$

- (3) $\mathbf{K}$ is the stiffness matrix of the lattice,

- (4) $\mathbf{C}$ is the Rayleigh damping matrix of the lattice and can be written as

$$\mathbf{C} = \alpha \mathbf{M} + \beta \mathbf{K} \quad (27)$$

where $\alpha = 0.5 \text{ s}^{-1}$, and $\beta = 10^{-4} \text{ m s}$.

- (5) $\mathbf{Q}(t)$ is the total load vector.

In this work, the differential equations of motion in Eq. (21) are solved by using the central difference method. In the method, velocity and acceleration can be expressed as

$$\dot{\mathbf{u}}(t) = \frac{1}{2\Delta t} [\mathbf{u}(t + \Delta t) - \mathbf{u}(t - \Delta t)] \quad (28)$$

$$\ddot{\mathbf{u}}(t) = \frac{1}{\Delta t^2} [\mathbf{u}(t - \Delta t) - 2\mathbf{u}(t) + \mathbf{u}(t + \Delta t)] \quad (29)$$

where Δt is the time step.

Substitution of Eqs. (27)–(29) into Eq. (21) yields the result as

$$\begin{aligned} & \left(\frac{1}{\Delta t^2} \mathbf{M} + \frac{\alpha}{2\Delta t} \mathbf{M} + \frac{\beta}{2\Delta t} \mathbf{K} \right) \mathbf{u}(t + \Delta t) \\ &= \mathbf{Q}(t) + \frac{2}{\Delta t^2} \mathbf{M} \mathbf{u}(t) - \mathbf{K} \mathbf{u}(t) - \left(\frac{1}{\Delta t^2} - \frac{\alpha}{2\Delta t} \right) \mathbf{M} \mathbf{u}(t - \Delta t) \\ & \quad + \frac{\beta}{2\Delta t} \mathbf{K} \mathbf{u}(t - \Delta t) \end{aligned} \quad (30)$$

In the present dynamic simulation, the strength criterion in Eq. (20) is adopted. When the effective stress of a particular beam reaches its tensile strength f_t , brittle fracture is simulated by instantaneously removing the beam from the lattice. Then a new time step is solved, which is the same as simulations of DCB tests [20,21].

4. Comparison of results with different loading rates

Uniaxial tensile experiments conducted on a rectangle concrete panel were analyzed [1,2] by using the beam lattice model. The same experiment on a GB lattice will be simulated by the quasi-static method and dynamic method, respectively. Then the difference between two kinds of results is discussed in detail.

The plate is $210\sqrt{3} \times 210\sqrt{3} \text{ cm}^2$. A regular triangular GB lattice having a total of 1593 GB elements and 562 nodes is used in simulation. All elements are $10\sqrt{3} \text{ cm}$ long. The span of all interface beams is $\sqrt{3} \text{ cm}$. The time step Δt is $= 10^{-6} \text{ s}$. In fact, the choice of the time step depends on the characteristic motion time of the smallest element [22,23]. Therefore, to avoid a too small time step, the adopted lattice is very coarse and the sizes of both particles and interface are much bigger than realistic. Nevertheless, such a numerical experiment is useful for a qualitative understanding of dynamic fracture processes. Material properties are shown in Table 1. Three phases have the common density $\rho = 10^3 \text{ kg/m}^3$. All

Table 1
Elastic properties of phases.

	E (Mpa)	f_t (Mpa)
Aggregate	70,000	10.0
Interface	25,000	1.25
Matrix	25,000	5.0

translational degrees of freedom are fixed along the left edge, while a uniaxial tension is applied to the right edge through a controlled displacement.

Five cases are analyzed. Case 1 is simulated by the quasi-static method. The other four cases, i.e., cases 2–5 where the loading rate is 1.0 m/s, 10^{-1} m/s, 10^{-2} m/s and 10^{-3} m/s, respectively, are done by the dynamic method.

4.1. Results

Fig. 6 shows the P – δ curve and crack patterns at typical load levels in case 1. Figs. 7–10 show the comparison between the P – δ curves of the four dynamic processes and the quasi-static curve, and also the crack patterns at typical load levels.

4.2. Effect of inertia effects on the fracture process

Both the dynamic load and unstable cracking can lead to inertia effects. They are observed by tests for different loading rates. Although similar analyses have been conducted in [9], more attention are given to data scattering, typical of concrete experiments that can entail microstructure effects modeled by specimen lattice. Numerical tests of the lattice are very helpful to further clarify the explanation of results in [9]. The five cases analyzed are:

(1) Case 1.

Many interface beams fail before point a in Fig. 6a (Fig. 6b). There is a steep drop in load from point a to point b, which corresponds to the appearance of localized macro-crack

Fig. 6. Case 1: P – δ curve and crack patterns at five load levels.

Fig. 7. Case 2: P – δ curve and crack patterns at five load levels.

(Fig. 6c). While new micro-cracks are formed, part of the previously developed cracks are incorporated into the macro-cracks (Fig. 6d). Finally, the specimen is separated into two parts by the macro-crack (Fig. 6e and f). On the path of the macro-crack, most failed beams are interface beams, only a few matrix beams fail, and no aggregate beams fail. For the quasi-static case, the main crack progresses through by connecting the jagged profile of the successive weakest links [10]. In other words, the crack path is the weakest path in the specimen.

(2) Case 2.

In the beginning of the curve, the slope of the elastic regime is much larger than that in case 1 (Fig. 7a), which is also observed in dynamic physical experiments [25]. The peak load is nearly four times bigger than that in case 1. Moreover, at the peak load level (point b in Fig. 7a), the controlled displacement is 5.50×10^{-5} m. After the peak load, the load drops steeply. The above features can be explained as fol-

lows. The loading rate, 1.0 m/s, is so high that the deformation has not enough time to distribute throughout the specimen. Therefore, deformation concentration appears near the displacement-controlled edge, leading to the failure of GB elements near this edge (Fig. 7b and c). On the other hand, this kind of failures makes the propagation of deformation from the controlled edge into the specimen more difficult. In the quasi-static case, i.e., case 1, the crack path depends only on three-phase material structure. In the present case, however, crack pattern depends on both the material structure and the inertia effect due to load increasing, and the latter plays a more important role than the former. Elements fail near the controlled edge (Fig. 7b). The local crack appears in the pure matrix region (Fig. 7c and d). The crack path is smoother than that in case 1 because the inertial effects appear to even out the material structure [1] (Fig. 7e). Fig. 7f shows the crack pattern at the load level e (Fig. 7a).

Fig. 8. Case 3: P - δ curve and crack patterns at four load levels.

(3) Case 3.

In Fig. 8a, both the elastic slope and the peak load are still larger than those of case 1. However, the differences have reduced obviously when compared with the differences between case 2 and case 1 in Fig. 7a. P - δ curves of cases 3 and 1 have the approximately similar up-down tendency except that the curve of case 3 has a strong vibration throughout the whole experimental process (Fig. 8a). Now, material structure plays a dominant role in the crack path instead of deformation concentration due to load increasing as in case 2. Therefore, in Fig. 8b-e, the path of the localized crack mainly runs through the weakest interface regions. However, cases 3 and 1 have different crack paths, which

can be found when comparing Fig. 8e with Fig. 6f. When the crack runs through the specimen, a compression wave [25] begins propagating from the newly-produced crack surface to the displacement-controlled edges. When the compression wave reaches the edge, load P will be reduced. In our simulations, it has been found that the compression wave can even lead to a negative value of load P in some conditions. Additionally, from point a to point b, the duration is just 1.64×10^{-3} s. At point a (Fig. 9b) no obvious crack can be found at the bottom-right region of the specimen, but during the duration 1.64×10^{-3} s a crack-like band has almost run through half of the whole specimen at point b (Fig. 9c). It suggests that the local crack propagates unstably and quickly.

Fig. 9. Case 4: P – δ curve and crack patterns at five load levels.

(4) Case 4.

This case has an elastic slope agreeing to that in case 1 (Fig. 9a). Moreover, cases 4 and 1 have nearly equivalent peak load values: 2.09 MN and 2.10 MN, respectively. However, an obvious difference can be found in the long tail. A load vibration directly follows every steep drop, e.g., at point b and c in Fig. 9a. The macro-crack forms at roughly the same position as case 1 (Fig. 6b), but the failure details along the crack path are very different. A crack-like band has almost run through the whole specimen at point c (Fig. 9d). Inertia effect due to the unstable cracking propagation leads to the load vibration directly following every steep drop. In the long tail regime, case 4 (Fig. 9e and f) and case 1 (Fig. 6e and f) have different crack paths. Found from Fig. 9a, the load vibration period is roughly 1.5×10^{-3} s. In case 2 (Fig. 7), the total experiment time is 10^{-3} s, which is smaller than the vibration period. Therefore, although the loading rate in case 2 is the highest, load vibration due to unstable propagation cannot yet be found in Fig. 7a. In case 3

(Fig. 8), the total experiment time is 10^{-2} s, which corresponds to several vibration periods, leading to an obvious vibration feather throughout its P – δ curve. In case 4, the total experiment time is 10^{-1} s, which is much larger than the vibration period, 1.5×10^{-3} s. As a result, in Fig. 9a, obvious vibration appears mainly directly following the steep drops, e.g., after point a.

(5) Case 5.

In Fig. 10a, the two P – δ curves have a good agreement. Moreover, the curve of case 5 also has the zigzag feather similar to the quasi-static curve. However, it is still hard to say that case 5 can be also simulated correctly by the quasi-static method. Actually, in case 5, the inertia effect due to unstable crack propagation remains a considerable influence. From the inset of Fig. 10a, an obvious load vibration occurs after the steep drop near point d. This kind of vibrations leads to a deviation of the deformation field from the corresponding equilibrium field. Because of the same reason, the crack pattern can become different from the

Fig. 10. Case 5: P – δ curve and crack patterns at four load levels.

quasi-static one. Fig. 10b–e shows a different failure details along roughly the same crack path from the quasi-static case (Fig. 6b–e).

4.3. Difference of the two methods in simulating unstable crack propagation

When the loading rate is very low (e.g., 10^{-3} m/s in case 5), the inertia effect due to load increasing becomes ignorable, but the inertia effect due to unstable crack propagation remains considerable. During the steep drops, the controlled displacement keeps unchanged in the quasi-static case, but keeps increasing at the loading rate in the four dynamic cases. Although the increase of

the controlled displacement is very small due to the steep drops being quick, it is hard to say the difference between quasi-static and dynamic results is not caused by this increase of the controlled displacement.

To clarify the above question, the following experiment is conducted. The state at point a in Fig. 6a is chosen as the initial state. The dynamic method is employed to simulate the process directly following point a until reaching a new equilibrium state. The controlled displacement keeps unchanged, i.e., the value at point a, 2.35×10^{-4} m.

The load P is shown versus the time in Fig. 11a. In the beginning, the load reduces sharply due to the appearance of failures in succession. In fact, seven elements fail in the whole process. All the failures happen during the period of 0–1400 μ s. Then, the load

Fig. 11. Results at final equilibrium state for quasi-static and dynamic method.

vibration regime follows, and trends to equilibrium under the action of the damping. After 0.03 s, the load becomes nearly unchanged with time, and the unchanged value is 1.18 MN. It indicates that the system has reached a new equilibrium state. In the quasi-static simulation (Fig. 6a), the load of the new equilibrium state denoted by point b is 0.72 MN. Therefore, final equilibrium loads obtained by the quasi-static method and the dynamic method are very different.

Fig. 11b shows the crack pattern at the equilibrium load level where the dynamic process finally reaches. Fig. 11b is different from Fig. 6c obtained by the quasi-static method. Moreover, the total of failed elements is 7 during the dynamic process directly following point a, but it is 29 from point a to point b in the quasi-static method.

5. General analysis of the crack process

The above numerical examples have indicated the differences between the dynamic method and the quasi-static method. Here, a general analysis is given based on the dynamic equations of motion and in obedience to the second Newton law.

The basic hypothesis of the quasi-static method is that all inertia effects can be neglected when the loading rate is low enough. Here, a case whose loading rate can be taken as quasi-static loading rate is analyzed. In the system composed of elements, it can be as-

sumed without loss of generality that some element reaches its critical strength state at time t^- . In other words, this element will crack immediately. Because inertia effect due to the quasi-static loading rate can be neglected, the system lies in an equilibrium state at time t^- , then

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}}(t^-) + \mathbf{C}\dot{\mathbf{u}}(t^-) + \mathbf{K}(t^-)\mathbf{u}(t^-) = \mathbf{Q}(t^-) \\ \ddot{\mathbf{u}}(t^-) = 0 \\ \dot{\mathbf{u}}(t^-) = 0 \end{cases} \quad (31)$$

Eq. (31) can be simplified as

$$\mathbf{K}(t^-)\mathbf{u}(t^-) = \mathbf{Q}(t^-) \quad (32)$$

Obviously, Eq. (32) is exactly the static-equilibrium equation adopted in the quasi-static method.

In time t^+ , at which the cracking element is being deleted, the critical element has been deleted from the system, then

$$\begin{cases} \mathbf{M}\ddot{\mathbf{u}}(t^+) + \mathbf{C}\dot{\mathbf{u}}(t^+) + \mathbf{K}(t^+)\mathbf{u}(t^+) = \mathbf{Q}(t^+) \\ \ddot{\mathbf{u}}(t^+) \neq 0 \\ \dot{\mathbf{u}}(t^+) = 0 \\ \mathbf{u}(t^+) = \mathbf{u}(t^-) \\ \mathbf{Q}(t^+) = \mathbf{Q}(t^-) \end{cases} \quad (33)$$

The first equation in Eq. (33) is the motion equation at time t^+ . Non-equilibrium force will appear due to deleting the critical element, and the non-zero acceleration arises at the same time according to the second Newton law. Because the time gap between t^- and t^+ is infinitesimal, the velocity field and the displacement field at time t^+ are the same as those at time t^- . Analogously, the vector of external actions $\mathbf{Q}(t^+)$ is equal to the counterpart at time t^- , namely $\mathbf{Q}(t^-)$.

However, in the quasi-static method, the whole material degrading process is obtained by solving statically equilibrium equations successively. At time t^+ , there exists in principle non-zero inertia force vector $\mathbf{M}\ddot{\mathbf{u}}(t^+)$ in the system. The common treatment in different quasi-static methods is to take $\mathbf{M}\ddot{\mathbf{u}}(t^+)$ as a static force, and then to find the next equilibrium by iterations. This is the essential difference between the two methods in our current topic. The non-equilibrium process is simulated by solving static-equilibrium equations in the quasi-static method. As a result, inertia effects due to successive elemental failures are neglected. Therefore, it can be imagined that considerable error can arise when simulating the fracture processes including unstable crack propagation.

6. New algorithm and numerical results

Since the quasi-static method can become questionable in simulating fracture behaviors under the quasi-static loading rate, it is worthy of improvement. This is the task of the present section.

6.1. New method

In order to take the inertia effect due to unstable crack propagation into consideration with a lower computational effort requirement than the pure dynamic method, a new method is proposed. The new method is mainly used in simulating experiments with low loading rates, where the inertia effect due to load increasing can be ignored.

At first, both the quasi-static curve (Fig. 12a) and the dynamic curve of cases with low loading rates (Fig. 12b) are investigated. The quasi-static curve, which is composed of ascent segments and descent segments, is strictly zigzagged. Every corner denotes an equilibrium state. Throughout every ascent segment, the

Fig. 12. Comparison of quasi-static and dynamic method for very low loading rate.

specimen is under equilibrium. However, it is not under equilibrium during every descent segment except its two ends. Particularly, at the end of an ascent segment (the beginning of the descent segment), the element with the highest stress-to-strength ratio reaches the prescribed threshold value, so this state is called a critical equilibrium state. Directly following the critical state, steep drops of load happens. Then, the specimen reaches a new equilibrium state. In the following, the dynamic curve is analyzed. In the dynamic curve (Fig. 12b), ascent segments and corners denoting critical equilibrium states can also be distinguished. However, directly after the critical equilibrium states there are vibrating segments. Because of the very low loading rate, inertia effect due to load increasing can be ignored. As a result, during every ascent segment following vibrating segments, the specimen is approximately under equilibrium. However, inertia effect due to unstable crack propagation plays an important role in a short time following new failures, which can be found from the appearance of vibrating segments following the critical states. Then, the specimen reaches new equilibrium states under the action of damping on caused vibration.

The new method is the combination of the quasi-static method and the dynamic method (Fig. 12b). At first, by solving the equilibrium equations, a critical equilibrium state is found, where the element with the highest stress-to-strength ratio reaches the prescribed threshold value. The following vibrating segment is simulated by using the dynamic method, taking the critical equilibrium state as the initial state and removing the critical element

Fig. 13. Comparison of P – δ curves for dynamic method (case 5) and new method (case 6).

from the system at the first time step. In this kind of dynamic processes, the controlled displacement increases at the loading rate. If more failures occur, they are dealt with like in the dynamic method in Section 3.2. The dynamic simulation ends until the specimen reaches a new equilibrium state approximately. Then, the next crit-

ical equilibrium state can be found by the quasi-static method, and so on.

It should be guaranteed that the specimen reaches an equilibrium state at the end of every dynamic simulation. In the current procedure, it is implemented by introducing a controls parameter N_e . In some critical equilibrium state, element T_0 has the highest stress-to-strength ratio. In the dynamic simulations directly following the critical equilibrium state, element T_0 is removed from the system and the global stiffness matrix is updated. If there is no more failed element except T_0 , the state at the end of the N_e -th time step is taken as the equilibrium state that the dynamic sim-

ulation finally reaches. Otherwise, if new failure occurs at the k th time step ($k \leq N_e$), time steps with a total of N_e at least following the k th time step should be computed. In a word, the dynamic simulation ends after N_e time steps following the last failure. Obviously, as long as the value of N_e is large enough, the inertia effect due to the failed elements can be ignored at the final state. In consideration of computational effort, a proper value of N_e can be set.

6.2. Numerical results

The same experiment as case 5 is simulated by the new method and is called case 6, where $N_e = 4000$. Fig. 13 shows that the P – δ curves of cases 5 and 6 are almost the same. No obvious difference can be found in crack patterns of the two cases, so they are not given here again. Particularly, the inertia effect due to unstable crack propagation is considered correctly in the new method. In case 5, 1,000,000 steps were conducted. In case 6, however, only about 300,000 steps were done.

The experiment with the loading rate 10^{-4} m/s is simulated by the new method and denoted case 7. Fig. 14a shows the P – δ curve, whilst Fig. 14b and c shows the variation of P at the neighborhood of point a and b, respectively, which indicate an obvious vibration after steep drops. Therefore, unstable crack propagation still produces a considerable inertia effect even in such a low loading rate.

7. Conclusions

Results obtained by the quasi-static method and the dynamic method were compared and analyzed in detail. Numerical experiments were conducted on a concrete plate subjected to uniaxial tension. This specimen has a different microstructure from that in [9]. Case 1 was simulated by using the quasi-static method. Cases 2–5 with the loading rate 1.0 m/s, 10^{-1} m/s, 10^{-2} m/s and 10^{-3} m/s, respectively, were simulated by the dynamic method. The numerical results show that the inertia effect due to load increasing becomes weaker and weaker with the loading rate decreasing. However, the inertia effect due to unstable crack propagation remains considerable in spite of the very low loading rate. Numerical experiment shows that the quasi-static method cannot obtain the same result as the dynamic method in simulating unstable crack propagation.

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Fig. 14. Case 7 for 10^{-4} m/s from new method.

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