Acta Mechanica Solida Sinica, Vol. 22, No. 6, December, 2009 ISSN 0894-9166
Published by AMSS Press, Wuhan, China.

SELF-INSTABILITY AND BENDING BEHAVIORS OF NANO PLATES**

Zhiqiao Wang **Yapu Zhao***

(*State Key Laboratory of Nonlinear Mechanics, Institute of Mechanics, Chinese Academy of Sciences, Beijing* 100190, *China*)

Received 24 June 2009, revision received 22 August 2009

ABSTRACT This paper aims at investigating the size-dependent self-buckling and bending behaviors of nano plates through incorporating surface elasticity into the elasticity with residual stress fields. In the absence of external loading, positive surface tension induces a compressive residual stress field in the bulk of the nano plate and there may be self-equilibrium states corresponding to the plate self-buckling. The self-instability of nano plates is investigated and the critical self-instability size of simply supported rectangular nano plates is determined. In addition, the residual stress field in the bulk of the nano plate is usually neglected in the existing literatures, where the elastic response of the bulk is often described by the classical Hooke's law. The present paper considered the effect of the residual stress in the bulk induced by surface tension and adopted the elasticity with residual stress fields to study the bending behaviors of nano plates without buckling. The present results show that the surface effects only modify the coefficients in corresponding equations of the classical Kirchhoff plate theory.

KEY WORDS nano plates, self-buckling, surface tension, residual stress

LIST OF SYMBOLS

$\mathbf{1}$	identity tensor on 3D Euclidean space
a, b, h	dimensions of a plate along the x_1, x_2 and x_3 direction
$\boldsymbol{a}_\alpha$	covariant base vectors on the tangent plane of a surface after deformation
$\boldsymbol{A}_\alpha$	covariant base vectors on the tangent plane of a surface before deformation
A_0	surface in the reference configuration
A	surface in the current configuration
$\boldsymbol{b}_0, \boldsymbol{b}$	surface curvature tensors in the reference and the current configurations
C	boundary of the plate
D	flexural stiffness of a plate
D_e	effective flexural stiffness of a nano plate

* Corresponding author. E-mail: yzhao@imech.ac.cn

** Project supported by the National Basic Research Program of China (973 Program, Grant No. 2007CB310500), the National High-tech R&D Program of China (863 Program, Grant No. 2007AA04Z348), the National Natural Science Foundation of China (NSFC, Grant No. 10772180) and the Postdoctoral Science Foundation of China (Grant No. 20080440530).

$\boldsymbol{E}$	infinitesimal strain tensor
$\boldsymbol{E}_s$	infinitesimal surface strain tensor
$\boldsymbol{F}_s, \boldsymbol{F}_s^{-1}$	surface deformation gradient and its inverse
$\boldsymbol{F}_s^{(o)}$	out-of-plane component of $\boldsymbol{F}_s$
$\boldsymbol{F}_s^{-1(o)}$	out-of-plane component of $\boldsymbol{F}_s^{-1}$
$\boldsymbol{H}$	displacement gradient calculated from the reference configuration
i	index ranging over the integers 1, 2 and 3
$\boldsymbol{I}$	identity tensor on the tangent plane of the surface in the current configuration
$\boldsymbol{I}_0$	identity tensor on the tangent plane of the surface in the reference configuration
$\bar{m}, \bar{n}$	integers
M_n, M_{ns}	bending and twisting moments per unit length
$\boldsymbol{N}, \boldsymbol{n}$	unit normal vectors of the surface before and after deformations
$p\,(x, y)$	lateral load along x_3-axis
$\boldsymbol{P}_0, \boldsymbol{P}$	perpendicular projection in the reference and the current configurations
Q_n	shear force per unit length
S	surface area of the plate in the reference configuration
$\boldsymbol{S}$	first kind Piola-Kirchhoff stress
$S_{i\beta}$	components of $\boldsymbol{S}$ in the reference configuration
$\boldsymbol{S}_s$	first kind Piola-Kirchhoff stress of the surface
$S_{i\beta}^s$	components of $\boldsymbol{S}_s$ in the reference configuration
S_m	middle neutral surface of the plate
S^+, S^-	upper and lower surfaces of the plate
s, n	local coordinates taken along and normal to the boundary C
$\hat{\boldsymbol{T}}$	residual stress in the bulk
$\hat{T}_{\alpha\beta}$	components of $\hat{\boldsymbol{T}}$ under the frame of the reference configuration
$\boldsymbol{u}$	displacement vector
u_i	components of displacement vector
u_i^0	components of displacement vector for a point on the middle neutral surface
U_B	strain energy for the bulk in the reference configuration
U_S	surface energy in the reference configuration
W	work of external forces
x_1, x_2, x_3	Cartesian coordinate system
$\boldsymbol{x}, \boldsymbol{x}'$	position vectors in the reference and the current configurations
Y	bulk Young's modulus
Y_s	surface Young's modulus
α, β, κ	indexes ranging over the integers 1 and 2
α_A	aspect ratio of a plate
δ	variation operator
$\delta_{\alpha\beta}$	Kronecker delta symbol
γ	surface energy per unit area of a surface in the current configuration
γ_0^*	surface tension
γ_1^*, γ_1	surface Lame constants
ε_{ij}	components of $\boldsymbol{E}$ in the reference configuration
$\varepsilon_{\alpha\beta}^0$	strain of the middle surface in the reference configuration
$\varepsilon_{\alpha\beta}^s$	components of $\boldsymbol{E}_s$ in the reference configuration
λ_1	extension of a plate along x_1 direction
λ, μ	Lame constants of the bulk
ν	Poisson's ratio of the bulk
θ	angle between the tangent to periphery and the x_1-axis
$\boldsymbol{\sigma}_s$	surface Cauchy stress
$\sigma_{i\alpha}^s$	components of $\boldsymbol{\sigma}_s$under the frame of the reference configuration
ζ, ξ	variables
$\bar{\nabla}_{0s}$	gradient operator defined in Eq.(51)
∇_{0s}	gradient operator defined in Eq.(64)
$\bar{\nabla}_s$	gradient operator defined in Eq.(52)
∇_s	gradient operator defined in Eq.(65)

I. INTRODUCTION

Nano plates have been widely used as the building blocks for ultra-sensitive and ultrafine resolution applications in the field of nanoelectromechanical systems (NEMS)[1,2], due to their potentially remarkable mechanical properties, which deviate from macroscopic counterparts and depend on their characteristic size[3,4]. The accurate analysis of mechanical behaviors is currently of particular interest in the function design and reliability analysis of those nano devices. Due to the large surface-to-bulk ratio of nanostructures, the size effects of mechanical responses are generally attributed to surface effects. Besides the numerical simulations[5–7], three-dimensional (3D) lattice model with the relaxation on the surface[4,8–11] and nonlocal elasticity theory[12], the surface elasticity theory[13–19] has attracted considerable attention and is used to model the size dependence of the elastic behaviors of nano plates[3,20–28]. In the theory of surface elasticity, the surface regions are modeled, as done by Gibbs[29], as mathematical surfaces; that is, the boundaries of the plate are regarded as two-dimensional continuous body, endowed with surface energy, surface tension and surface stress which reflect the behaviors of the surface regions. Since it is not convenient to work in the current configuration, the surface elasticity models except Shuttleworth's relations are proposed in the Lagrangian descriptions[14–18]. The existence of the surface tension in nano structures results in a difference between the Lagrangian and the Eulerian descriptions of the surface elasticity at infinitesimal strains[18]. However, the size-dependent mechanical behavior analyses of nano structures are often modeled by continuum mechanics including the Eulerian surface elasticity, where the current configuration is assumed to be the same as the configuration without external loading and the out-of-plane terms associated with the surface tension are omitted[22,27,30,31]. In the present paper, the complete form of surface elasticity will be considered.

In addition, according to the equilibrium conditions, surface tension induces a residual stress field in the bulk nano structures in the absence of external loadings[17,18,32]. The residual stress in the bulk can be calculated by the equilibrium conditions and is inversely proportional to the characteristic size of the nano structure[18,32]. When the plate will be very thin, the residual stress can be high. When the surface tension is stress, the bulk of the nano plate will be subjected to compressive stress and there may be self-equilibrium states which correspond to the plate buckling without external loadings. For the bending analyses of nano plates without buckling, the self-equilibrium state (without external loadings) is usually regarded as the reference configuration, from which nano plates deform elastically. However, the classical Hooke's law is often adopted to describe the elastic behavior of the bulk of nano plates[22,23,25–28], where effect of the residual stress field in the bulk is neglected. Therefore, in this paper, the surface tension and the residual stress in the bulk induced by surface tension is considered to study the self-instability and the bending behaviors of nano plates.

The paper is organized as follows. The basic equations for isotropic, homogeneous and linear elastic surfaces and bulk are presented in §II. The self-instability of nano plates under the action of surface tension is investigated in §III; the critical self-instability size of the plate is determined. Based on the variational method of strain energy function, the governing equations and the boundary conditions for bending problems are obtained in §IV. It can be found that the surface effects only modify the coefficients in corresponding equations of the classical Kirchhoff plate theory.

II. BASIC EQUATIONS

Consider a nano plate whose physical properties in the neighborhood of its surfaces are sensibly different from its interior as shown in Fig.1. These surface regions are often modeled as mathematical layers of zero thickness with relevant elastic properties and residual surface tension. In the absence of external loading, the surface tension induces an elastic field in the bulk of the plate. Thus, the existence of surface effects makes nano plates behave elastically from a residual stressed reference configuration (as shown in Fig.2). The basic equations for the surface and the bulk of the nano plates are given in this section.

2.1. Surface Elasticity

After the pioneering work of Shuttleworth[13] on the relations between surface stress and surface strain for small deformations, Gurtin and Murdoch[14,15] established a general theoretical model for the surface elasticity under the classical theory of membrane. Steigmann and Ogden[16] generalized the

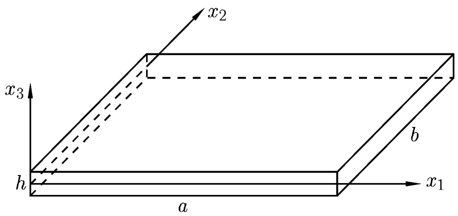

Fig. 1. Simply supported rectangular nano plates with length a, width b and thickness h.

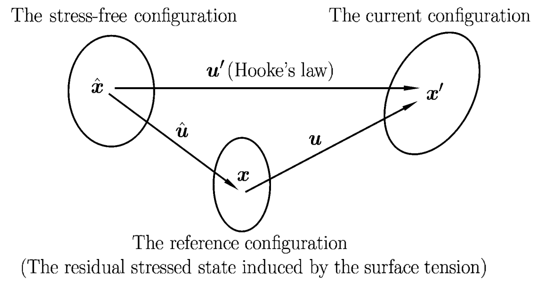

Fig. 2. The choice of the reference configuration: the residual stressed state induced by surface tension.

Gurtin-Murdoch theory to incorporate flexural stiffness of the free surface directly into the constitutive relations. Dingreville and Qu[19] investigated the influence of the Poisson's ratio on the surface properties under general loading conditions. Considering the stationary condition of energy function, Huang et al.[17] proposed a hyperelastic surface model within the framework of finite deformations. Wang et al.[18] investigated the surface elasticity of infinitesimal strains and pointed out that even in the case of small strains the Lagrangian and the Eulerian descriptions should be discriminated. Next, we briefly introduce the surface elasticity.

Assuming the surface to be isotropic, homogeneous and linearly elastic, the constitutive relations of the surface in the Lagrangian description can be written as[18]

$$\boldsymbol{S}_s = \gamma_0^*\boldsymbol{I}_0 + (\gamma_0^* + \gamma_1^*)\,(\mathrm{tr}\boldsymbol{E}_s)\,\boldsymbol{I}_0 - \gamma_0^*(\bar{\nabla}_{0s}\boldsymbol{u}_0) + \gamma_1\boldsymbol{E}_s + \gamma_0^*\boldsymbol{F}_s^{(o)} \tag{1}$$

where $\boldsymbol{S}_s$ is the first kind Piola-Kirchhoff stress of the surface, $\boldsymbol{I}_0$ is the identity tensor on the tangent planes of the surface in the reference configuration; the constants γ_0^*, γ_1^* and γ_1 are the surface tension and the surface Lame moduli; $\boldsymbol{E}_s$, $\bar{\nabla}_{0s}\boldsymbol{u}_0$ and $\boldsymbol{F}_s^{(o)}$ denote, respectively, the surface strain tensor, the in-plane component of the surface displacement gradient and the out-of-plane term of surface deformation gradient. Detailed expressions of these notations are explained in Appendix A. It should be noted that $\boldsymbol{S}_s$ is a 'two-point' tensor with base vectors both on the tangent planes of the surfaces before and after deformations; if it is expressed in the the reference configuration, there should be an out-of-plane term, the contribution of which is often omitted[22,27,30]. The importance of the out-of-plane term was numerically examined[33]. Equation (1) is the complete form of the first kind Piola-Kirchhoff stress of the surface.

In many problems of interest, it is not convenient to work in the current configuration, since the deformed configuration is not known *a priori*. Thus, the surface Cauchy stress $\boldsymbol{\sigma}_s$ can be expressed, in the reference configuration, as in[18]

$$\boldsymbol{\sigma}_s = \gamma_0^*\boldsymbol{I}_0 + \gamma_1^*\,(\mathrm{tr}\boldsymbol{E}_s)\,\boldsymbol{I}_0 + \gamma_1\boldsymbol{E}_s + \gamma_0^*\left(\boldsymbol{F}_s^{(o)} + \boldsymbol{F}_s^{(o)\mathrm{T}}\right) \tag{2}$$

where the last symmetrized term denotes the surface rotation contribution to the surface Cauchy stress. It is shown that since the surface Cauchy stress tensor is defined in the current tangent plane of the surface and is a 2D quantity in current configuration, there are out-of-plane terms when surface Cauchy stress is expressed in the frame of the reference configuration. Hence, even in the case of small strains, it is needed to discriminate the reference and the current configurations of the surface. If identifying the different configurations at small strains, the out-of-plane terms in Eq.(2) are neglected[27,30], which are related to the rotation of the surface.

Comparing Eqs.(1) and (2), we can also find that even in the case of infinitesimal strains, the surface stress tensors $\boldsymbol{S}_s$ and $\boldsymbol{\sigma}_s$ are not the same due to the existence of the residual stress γ_0^*. In the reference configuration, consider a Cartesian coordinate system (x_1, x_2, x_3). The component forms of surface stress are

$$S_{\alpha\beta}^s = \gamma_0^*\delta_{\alpha\beta} + (\gamma_0^* + \gamma_1^*)\,\varepsilon_{\kappa\kappa}^s\delta_{\alpha\beta} - \gamma_0^* u_{\beta,\alpha} + \gamma_1\varepsilon_{\alpha\beta}^s, \quad S_{3\alpha}^s = \gamma_0^* u_{3,\alpha} \tag{3}$$

and

$$\sigma_{\alpha\beta}^s = \gamma_0^*\delta_{\alpha\beta} + \gamma_1^*\varepsilon_{\kappa\kappa}^s\delta_{\alpha\beta} + \gamma_1\varepsilon_{\alpha\beta}^s, \quad \sigma_{3\alpha}^s = \sigma_{\alpha 3}^s = \gamma_0^*\,(u_{3,\alpha} + u_{\alpha,3}) \tag{4}$$

in which α, β and κ range over the integers 1 and 2, summation convention is used, $\delta_{\alpha\beta}$ designates the Kronecker delta, $\varepsilon^s_{\alpha\beta}$ are components of surface strain, and a subscript preceded by a comma indicates differentiation with respect to the corresponding coordinate.

It should be pointed out that besides the surface constitutive relations mentioned-above, the Young-Laplace equations is needed to describe the discontinuity conditions of the traction across the surface, which can be found in Appendix B.

2.2. Constitutive Relations for Bulk

In the absence of external loading, surface tension induces a residual stress field in the bulk nano plates[17,18,32]. Thus, the bulk materials deform elastically from a residual stressed state (as shown in Fig.2). It should be pointed out that, in the prediction of mechanical behaviors of nano plates, the classical Hooke's law is often used to describe the elastic response of the bulk, where the residual stress is neglected. However, the constitutive relations of linearly elastic materials with residual stress are quite different from the classical Hooke's law[10,34].

In view of the importance of the linearization of the general constitutive relations, we present the linear elastic constitutive relations of the bulk with residual stresses as follows[10,34]:

$$\boldsymbol{S} = \hat{\boldsymbol{T}} + \boldsymbol{H} \cdot \hat{\boldsymbol{T}} - \frac{1}{2}\left(\boldsymbol{E} \cdot \hat{\boldsymbol{T}} + \hat{\boldsymbol{T}} \cdot \boldsymbol{E}\right) + \lambda\left(\mathrm{tr}\boldsymbol{E}\right)\boldsymbol{1} + 2\mu\boldsymbol{E} \tag{5}$$

where $\boldsymbol{S}$ is the first Piola-Kirchhoff stress, $\hat{\boldsymbol{T}}$ is the residual stress in the reference configuration, $\boldsymbol{H}$ is the displacement gradient calculated from the reference configuration, $\boldsymbol{E}$ is the infinitesimal strain, $\boldsymbol{1}$ is the identity tensor on 3D Euclidean space, λ and μ are material elastic constants.

Next, we formulate an approximate theory for the thin plate with residual stress. Under the Kirchhoff hypothesis, the linear filaments of the plate in the reference configuration perpendicular to the middle surface remain straight and perpendicular to the deformed middle surface and bear no extensions during bending. Hence, the stress-strain relations in Eq.(5) for the thin plate reduce to

$$\begin{aligned} &S_{\alpha\beta} = \hat{T}_{\alpha\beta} + u_{\alpha,\kappa}\hat{T}_{\kappa\beta} - \frac{1}{2}\left(\varepsilon_{\alpha\kappa}\hat{T}_{\kappa\beta} + \hat{T}_{\alpha\kappa}\varepsilon_{\kappa\beta}\right) + \frac{Y}{1-\nu^2}\left[(1-\nu)\,\varepsilon_{\alpha\beta} + \nu\varepsilon_{\kappa\kappa}\delta_{\alpha\beta}\right] \\ &S_{3\alpha} = u_{3,\kappa}\hat{T}_{\kappa\alpha} \end{aligned} \tag{6}$$

in which $\varepsilon_{\alpha\beta}$ are the strain components, Y and ν designate Young's modulus and the Poisson's ratio of the bulk, respectively. The terms $S_{3\alpha}$ should not be neglected. This is not contrary to Kirchhoff hypothesis, since they are the components of the first kind of Piola-Kirchhoff stress and related to the residual stress in the bulk.

Correspondingly, the displacement components of a point with coordinates (x_1, x_2, x_3) in the reference configuration can be denoted by

$$u_\alpha = u^0_\alpha - x_3 u_{3,\alpha}, \quad u_3 = u^0_3, \tag{7}$$

where $u^0_i\,(x_1, x_2)$ $(i = 1, 2, 3)$ is the displacement components of a point on the middle neutral surface S_m. Then, the strains of von Karman type are

$$\varepsilon_{\alpha\beta} = \varepsilon^0_{\alpha\beta} - x_3 u_{3,\alpha\beta} \tag{8}$$

where

$$\varepsilon^0_{\alpha\beta} = \frac{1}{2}\left(u^0_{\alpha,\beta} + u^0_{\beta,\alpha} + u^0_{3,\alpha}u^0_{3,\beta}\right) \tag{9}$$

are the strain components of the middle surface.

Assume the two surfaces of the plate are identical. Therefore, the membrane forces do not change during the small deflection of nano plates. Hence, we can neglect the stretching in the middle plane during bending, we conclude from Eqs.(8) and (9) that

$$\varepsilon_{\alpha\beta} = -x_3 u_{3,\alpha\beta} \tag{10}$$

$$u^0_{\alpha,\beta} + u^0_{\beta,\alpha} = -u^0_{3,\alpha}u^0_{3,\beta} \tag{11}$$

2.3. Residual Stress in Bulk Induced by Surface Tension

For nano scale structures, according to the equilibrium conditions, the presence of surface tension results in a non-classical boundary condition which gives the traction on the bulk in terms of surface tension. This boundary condition together with the equations of classical elasticity (to be satisfied within the bulk) form a coupled system of field equations to determine the residual stress in the bulk[17,18]. In this paper, the variational method is used to determine the residual stress in the bulk. To obtain the residual stress, we assume a deformation model with respect to the reference configuration. Through the variation of the plate strain energy, we can obtain the equilibrium relations. Let the deformation approach zero. Then, the obtained equilibrium relations reduce to those in the reference configuration, from which we can get the residual stress induced by surface tension in the bulk.

Consider a simply supported rectangular nano plates with two identical surfaces, which is under self-balanced state without buckling (as shown in Fig.1). From the symmetry of nano plates, we can assume that the bulk is under a uniformly distributed stress field $\hat{T}_{x_1} = \hat{T}_{x_2}$ in the reference configuration. For simplicity, we assume that there is a uniform extension λ_1 along x_1 direction from the reference configuration, namely, $u_1 = \lambda_1 x_1,\ u_2 = 0,\ u_3 = 0$. Although this assumed deformation model may be difficult to perform for actual plates, it is effective to theoretically solve the residual stress field in the bulk. Then, from Eq.(6), the corresponding variation of strain energy for the bulk can be expressed as

$$\delta U_B = \frac{1}{2}\iint\limits_{S_m}\left(\int_{-h/2}^{h/2} S_{i\beta}\delta u_{i,\beta}\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2 = Sh\left(\hat{T}_x + \frac{Y}{1-\nu^2}\lambda_1\right)\delta\lambda_1 \tag{12}$$

where S and h designate the surface area in the reference configuration and the thickness of the plate, respectively; the summation convention over repeated indices is implied in the present paper. Then, the surface contribution to the variation of the free energy of the plate is

$$\delta U_S = \iint\limits_{S^+}\left(S^s_{i\beta}\delta u_{i,\beta}\right)\mathrm{d}x_1\mathrm{d}x_2 + \iint\limits_{S^-}\left(S^s_{i\beta}\delta u_{i,\beta}\right)\mathrm{d}x_1\mathrm{d}x_2 = 2S\left[\gamma_0^* + \left(\gamma_1^* + \gamma_1\right)\lambda_1\right]\delta\lambda_1 \tag{13}$$

where S^+ and S^- denote the upper and lower surfaces of the plate, respectively. The principle of virtual work requires that

$$\delta U_s + \delta U_B = 2S\left[\gamma_0^* + \left(\gamma_1^* + \gamma_1\right)\lambda_1\right]\delta\lambda_1 + Sh\left(\hat{T}_x + \frac{Y}{1-\nu^2}\lambda_1\right)\delta\lambda_1 = 0 \tag{14}$$

When $\lambda_1 = 0$, we can obtain residual stress in the bulk under the self-balanced state as

$$\hat{T}_{x_1} = \hat{T}_{x_2} = -\frac{2\gamma_0^*}{h} \tag{15}$$

The above equation shows that residual stresses in the bulk are inversely proportional to the thickness of the nano plate.

III. SELF-INSTABILITIES OF NANO PLATES

Lin et al.[28] considered the effects of the surface on the wrinkling of thin films. However, in the absence of external loading, surface tension induces a compressive residual stress field in the bulk plate and there may be self-equilibrium states which correspond to the plate self-buckling. We shall now formulate a self-buckling problem for the flat nano plates. The residual stress field before buckling in the bulk of the nano plate is treated as initial stresses, which satisfy the equations of equilibrium and boundary conditions[17,18,32].

Since we are interested only in the configuration and critical size for the buckling, we consider the small deflection theory of thin plates in the following discussion of self-instability. The displacement components of an arbitrary point of a plate are measured from the state prior to the onset of buckling.

The governing equations for instabilities of nano plates can be obtained from the principle of virtual work, which is written for the present problem as follows

$$\delta U_B + \delta U_S - \delta W = 0 \tag{16}$$

where

$$\delta W=\oint_C M_n\frac{\partial\delta u_3^0}{\partial n}\mathrm{d}s+\oint_C\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)\delta u_3^0\mathrm{d}s \tag{17}$$

is the virtual work of external forces; s and n are the local coordinates along and normal to the boundary C; Q_n, M_n and M_{ns} denote the corresponding shear force, bending and twisting moments per unit length on the boundary, respectively.

The variation of strain energy for the bulk δU_B is given by

$$\delta U_B=\iint_{S_m}\left(\int_{-h/2}^{h/2}S_{i\alpha}\delta u_{i,\alpha}\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2 \tag{18}$$

By substituting Eqs.(5), (7), (10) and (11) into Eq.(18) and neglecting higher order terms, we obtain

$$\delta U_B=\frac{Yh^3}{12\,(1+\nu)}\iint_{S_m}\left[u^0_{3,\alpha\beta}\delta u^0_{3,\alpha\beta}+\frac{\nu}{(1-\nu)}u^0_{3,\alpha\alpha}\delta u^0_{3,\beta\beta}\right]\mathrm{d}x_1\mathrm{d}x_2 \tag{19}$$

It can be found that due to the no stretching assumption of middle plane, Eq.(19) for the virtual strain energy of the bulk with initial stress is the same as that for the bending of a plate[35, 36].

The variation of surface strain energy is calculated by

$$\delta U_S=\iint_{S^+}S^s_{i\alpha}\delta u_{i,\alpha}\mathrm{d}x_1\mathrm{d}x_2+\iint_{S^-}S^s_{i\alpha}\delta u_{i,\alpha}\mathrm{d}x_1\mathrm{d}x_2 \tag{20}$$

in which $\boldsymbol{S}_s=\gamma_0^*\boldsymbol{I}_0$. From Eqs.(1) and (7), we can get

$$\delta U_S=-2\gamma_0^*\iint_{S_m}u^0_{3,\alpha}\delta u^0_{3,\alpha}\mathrm{d}x_1\mathrm{d}x_2 \tag{21}$$

Substituting Eqs.(19) and (21) into Eq.(16) and using integrations by parts and the Gaussian theorem, we may reduce the principle to the following form

$$\iint_{S_m}\left(Du^0_{3,\alpha\alpha\beta\beta}+2\gamma_0^*u^0_{3,\alpha\alpha}\right)\delta u_3^0\mathrm{d}x_1\mathrm{d}x_2+\oint_C\left[D\nu u^0_{3,\alpha\alpha}+D\,(1-\nu)\,\eta_1+M_n\right]\frac{\partial\left(\delta u_3^0\right)}{\partial n}\mathrm{d}s$$
$$+\oint_C\left[D\,(1-\nu)\frac{\partial\eta_2}{\partial s}-D\eta_3-2\gamma_0^*\eta_4-\left(Q_n+\frac{\partial M_{ns}}{\partial s}\right)\right]\delta u_3^0\mathrm{d}s=0 \tag{22}$$

where $D=Yh^3/\left[12\left(1-\nu^2\right)\right]$ is flexural stiffness and

$$\begin{aligned}
\eta_1&=u^0_{3,11}\cos^2\theta+2u^0_{3,12}\sin\theta\cos\theta+u^0_{3,22}\sin^2\theta\\
\eta_2&=\left(u^0_{3,11}-u^0_{3,22}\right)\sin\theta\cos\theta+u^0_{3,12}\left(\sin^2\theta-\cos^2\theta\right)\\
\eta_3&=\left(u^0_{3,111}+u^0_{3,122}\right)\cos\theta+\left(u^0_{3,222}+u^0_{3,112}\right)\sin\theta\\
\eta_4&=u^0_{3,1}\cos\theta+u^0_{3,2}\sin\theta
\end{aligned} \tag{23}$$

in which θ is the angle between the tangent to periphery and the x_1-axis. Since δu_3^0 are chosen arbitrarily on surface S_m, the corresponding coefficients are required to vanish, and we obtain

$$Du^0_{3,\alpha\alpha\beta\beta}+2\gamma_0^*u^0_{3,\alpha\alpha}=0 \tag{24}$$

For simply supported plates, $\delta u_3^0=0$ and $M_n=0$ on the boundary C. Hence, the third contour integral in Eq.(22) vanishes. But, $\partial\left(\delta u_3^0\right)/\partial n\neq 0$, and its coefficient in Eq.(22) must be zero, so that

$$D\nu u^0_{3,\alpha\alpha}+D\,(1-\nu)\,\eta_1=0. \tag{25}$$

If two edges of the rectangular plate are parallel to x_2-axis and $\theta = 0$, Eq. (25) is reduced to

$$u^0_{3,11} + \nu u^0_{3,22} = 0 \tag{26}$$

This is the boundary condition for bending moment vanishing.

If the edges of plates are free, δu_3^0 and $\partial\left(\delta u_3^0\right)/\partial n$ are chosen arbitrarily on the boundary C. Equation (22) implies that

$$D\nu u^0_{3,\alpha\alpha} + D\left(1-\nu\right)\eta_1 = 0, \tag{27}$$

and

$$D\left(1-\nu\right)\frac{\partial \eta_2}{\partial s} - D\eta_3 - 2\gamma_0^*\eta_4 = 0 \tag{28}$$

If edges of the rectangular plate parallel to x_2-axis and $\theta = 0$, Eqs.(27) and (28) can be simplified as

$$u^0_{3,11} + \nu u^0_{3,22} = 0 \tag{29}$$

and

$$u^0_{3,111} + \left(2-\nu\right)u^0_{3,122} + 2\frac{\gamma_0^*}{D}u^0_{3,1} = 0 \tag{30}$$

To demonstrate the self-buckling of nano plates induced by surface tension, critical size for the instability of simply supported rectangular nano plates (as shown in Fig.1) are given. Since all four edges of the plate are simply supported, the lateral deflection as well as the bending moment vanishes along each edge. The deflection of the buckled plate can be represented by the double series

$$u_3^0 = \sum_{\bar m=1}^{\infty}\sum_{\bar n=1}^{\infty} A_{\bar m\bar n}\sin\frac{\bar m\pi x_1}{a}\sin\frac{\bar n\pi x_2}{b} \qquad (\bar m, \bar n = 1, 2, 3...) \tag{31}$$

which satisfies the boundary conditions. The remaining task is to ensure that it also satisfies the differential equation as in Eq.(24). Substitution of the appropriate derivatives of u_3^0 into Eq.(24) leads to

$$\sum_{\bar m=1}^{\infty}\sum_{\bar n=1}^{\infty} A_{\bar m\bar n}\left\{D\pi^2\left[\left(\frac{\bar m}{a}\right)^2 + \left(\frac{\bar n}{b}\right)^2\right] - 2\gamma_0^*\right\}\sin\frac{\bar m\pi x_1}{a}\sin\frac{\bar n\pi x_2}{b} = 0 \tag{32}$$

This expression can be satisfied in one of two ways, either $A_{\bar m\bar n} = 0$ or the term in the curly brackets vanishes. The first situation corresponds to the case that a and b are very small. This is the equilibrium state of the plate which remains perfectly straight under the action of surface tension. If the values of a and b increase, the expression in the curly brackets may equal to zero. This corresponds to the buckled state of nano plates. Thus

$$b = \pi h\sqrt{\frac{\bar m^2\alpha_A^{-2} + \bar n^2}{24\left(1-\nu^2\right)}}\sqrt{\frac{Yh}{\gamma_0^*}} \tag{33}$$

where $\alpha_A = a/b$ is the aspect ratio. According to Eq.(33), the critical self-instability size b depends on the aspect ratio α_A and the physical properties of the plate and on $\bar m$ and $\bar n$, the numbers of half-waves that plate buckles into. Therefore, the critical size for self-buckling is obtained, by setting $\bar m = \bar n = 1$, as

$$b = \pi h\sqrt{\frac{\alpha_A^{-2} + 1}{24\left(1-\nu^2\right)}}\sqrt{\frac{Yh}{\gamma_0^*}} \tag{34}$$

The relations between the critical self-instability size of Si nano plates and the thickness of the plates are plotted in Fig.3, where material parameters $\nu = 0.3$ and $\gamma_0^*/Y = 0.05$ (Å) are obtained by modular dynamic simulation in Ref.[3]. It can be seen that for the simply supported square plates with the thickness 5 nm (or 8 nm), when the corresponding width is 150 nm (or 304 nm), the surface tensions induce the self-instability even without the external forces. This indicates that thin plates can easily become self-buckled. For nano plates with the same thickness, the change of the aspect ratio has an obvious effect on the critical self-instability size when the aspect ratio is not very large.

It should be pointed out that when the size of the nano plates exceeds the critical self-instability size, the nano plates can bear loadings continually after instabilities and come into the post-buckling state, which is outside the scope of this article.

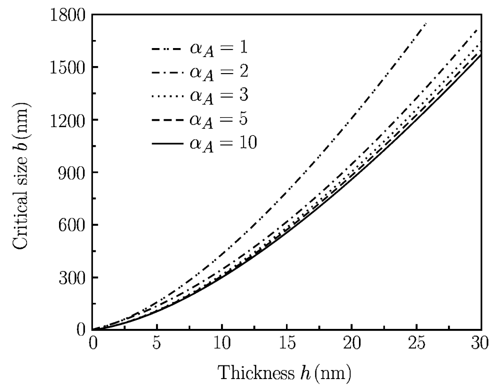

Fig. 3. The relations between self-instability critical size b and the thickness h of Si nano plates for different aspect ratios α_A.

IV. BENDING OF NANO PLATES

The self-instability of the nano plates, due to the existence of the surface tension, is proposed in the previous section. This section considers the bending behaviors of the nano plates without the self-instability. Here, we also adopt the Kirchhoff plate theory, use geometrical descriptions in Eqs.(7)-(11) and chose the self-equilibrium state without external loadings as the reference configuration.

The governing equations for nano plates in bending could be determined by the variations of total potential energy, which can be represented as the same form as in Eq.(16). Since the bulk of nano plates deforms elastically in the same manner of self-buckling, the Eq.(19) is also applicable to the bending problem. But the surfaces of nano plates are in tension in the convex side or compression in the concave side during bending, hence surface elasticity expressed in Eq.(1) should be considered in Eq.(20). Correspondingly, the expression for δU_S is

$$\delta U_S = \frac{h^2}{2} \iint\limits_{S_m} \left[\left(\gamma_1 - \gamma_0^*\right) u^0_{3,\alpha\beta} \delta u^0_{3,\alpha\beta} + \left(\gamma_1^* + \gamma_0^*\right) u^0_{3,\alpha\alpha} \delta u^0_{3,\beta\beta} \right] \mathrm{d}x_1 \mathrm{d}x_2 \tag{35}$$

Furthermore, the lateral load $p(x, y)$ in the direction of the x_3-axis has contributions to the virtual work of external forces. Then, δW becomes

$$\delta W = \iint\limits_{S_m} p \delta u^0_3 \mathrm{d}x_1 \mathrm{d}x_2 + \oint\limits_C M_n \frac{\partial \delta u^0_3}{\partial n} \mathrm{d}s + \oint\limits_C \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \delta u^0_3 \mathrm{d}s \tag{36}$$

Substituting Eqs.(19), (35) and (36) into Eq.(16), we obtain the condition for minimum energy, through the use of integrations by parts and the Gaussian theorem,

$$\begin{aligned} &\iint\limits_{S_m} \left\{ \left[D + \left(\gamma_1^* + \gamma_1\right) \frac{h^2}{2} \right] u^0_{3,\alpha\alpha\beta\beta} - p \right\} \delta u^0_3 \mathrm{d}x_1 \mathrm{d}x_2 \\ &+ \oint\limits_C \left\{ \left[D\nu + \left(\gamma_0^* + \gamma_1^*\right) \frac{h^2}{2} \right] u^0_{3,\alpha\alpha} + \left[D(1-\nu) + \left(\gamma_1 - \gamma_0^*\right) \frac{h^2}{2} \right] \eta_1 + M_n \right\} \frac{\partial \left(\delta u^0_3\right)}{\partial n} \mathrm{d}s \\ &+ \oint\limits_C \left\{ \left[D(1-\nu) + \left(\gamma_1 - \gamma_0^*\right) \frac{h^2}{2} \right] \frac{\partial \eta_2}{\partial s} - \left[D + \left(\gamma_1^* + \gamma_1\right) \frac{h^2}{2} \right] \eta_3 - \left(Q_n + \frac{\partial M_{ns}}{\partial s} \right) \right\} \delta u^0_3 \mathrm{d}s = 0 \end{aligned} \tag{37}$$

Since δu^0_3 are arbitrary on surface S_m, we obtain the equation of equilibrium in bending

$$D_e u^0_{3,\alpha\alpha\beta\beta} - p = 0 \tag{38}$$

in which $D_e = D\left[1 + 6\left(1-\nu^2\right)\dfrac{\gamma_1^* + \gamma_1}{Yh}\right]$ and the boundary conditions which are the second and the third contour integrals in Eq.(37).

For clamped nano plates, the last two contour integrals along the boundary C in Eq.(37) vanish automatically.

For simply supported nano plates, $\delta u_3^0 = 0$ and $M_n = 0$ on the boundary C. Hence, the third contour integral in Eq.(37) vanishes. But, $\partial\left(\delta u_3^0\right)/\partial n \neq 0$, its coefficient in Eq.(37) must be zero. For the cases that two edges of the rectangular plate parallel to x_2-axis and $\theta = 0$, it becomes

$$u_{3,11}^0 + \zeta u_{3,22}^0 = 0 \tag{39}$$

in which $\zeta = \left[\nu + 6\left(1-\nu^2\right)\dfrac{\gamma_0^* + \gamma_1^*}{Yh}\right] \Big/ \left[1 + 6\left(1-\nu^2\right)\dfrac{\gamma_1^* + \gamma_1}{Yh}\right]$. This is the boundary condition for bending moment vanishing.

If the edges of plates are free, $Q_n = M_{ns} = 0$; δu_3^0 and $\partial\left(\delta u_3^0\right)/\partial n$ are chosen arbitrarily on the boundary C. Their coefficients in Eq.(37) must be zero. When two edges of the rectangular plate parallel to x_2-axis and $\theta = 0$, the coefficients can be simplified as

$$u_{3,11}^0 + \zeta u_{3,22}^0 = 0 \tag{40}$$

and

$$u_{3,111}^0 + \xi u_{3,122}^0 = 0, \tag{41}$$

where $\xi = \left[(2-\nu) + 6\left(1-\nu^2\right)\dfrac{\gamma_0^* + \gamma_1^*}{Yh}\right] \Big/ \left[1 + 6\left(1-\nu^2\right)\dfrac{\gamma_1^* + \gamma_1}{Yh}\right]$.

From the governing equations in Eq.(38) and boundary conditions in Eqs.(39)-(41), it can be found that the surface parameters only modify the coefficients in corresponding equations of the classical Kirchhoff plate theory. Hence, the mechanical analysis of nano plates can be obtained through changing the parameters D_e, ζ and ξ in corresponding results of classical plate. When the surface effects are not considered, or the thickness h of plates becomes large enough, Eqs.(38)-(41) reduce to classical plate results. The aforementioned surface elasticity theory assumes that the surface does not slip relative to the body and imposes no further restrictions on the surface parameters. Some researchers[3] assume that the Poisson's ratio of the surface is equal to that of the bulk in the mechanical analyses of nano structures. This assumption is the most accurate for the case that bulk behavior dominates the surface effects, and less accurate as the plate becomes smaller. Based on this simplification, we can obtain that

$$\gamma_1^* + \gamma_1 = \frac{Y_s}{1-\nu^2} \tag{42}$$

and

$$\gamma_0^* + \gamma_1^* = \frac{Y_s \nu}{1-\nu^2} \tag{43}$$

in which Y_s is the surface Young's modulus. Then, the flexural stiffness D_e of nano plates can be represented as

$$D_e = D\left(1 + 6\frac{Y_s}{Yh}\right) \tag{44}$$

which is the same as the results of Ref.[3]. Correspondingly, ζ and ξ may be written as

$$\zeta = \nu, \quad \xi = \left[2 - \nu\left(1 - 6\frac{Y_s}{Yh}\right)\right]\left[1 + 6\frac{Y_s}{Yh}\right]^{-1} \tag{45}$$

The first equation of Eq.(45) indicates that under the same Poisson's ratio assumption, the boundary conditions of simply supported nano plates is the same as those of classical plate theory.

V. CONCLUSIONS AND DISCUSSIONS

The present paper has the following contributions to the nano plate theory:

1. In the absence of external loadings, nano plates may become self-buckled due to the action of surface tension. The critical self-instability size of nano plates is determined.

2. In the bending analysis of nano plate, the residual stress field induced by surface tension is considered. The elastic response of the bulk is modeled as the elasticity with residual stress fields.

Under the assumption that the Poisson's ratio of the surface is equal to that of the bulk, the present expression for the flexural stiffness D_e reduces to the results of Ref.[3].

In addition, the present paper introduces the surface elasticity in the Lagrangian description and points out that even in the case of infinitesimal strains the reference and the current configurations should be discriminated; otherwise the out-of-plane surface displacement gradient, associated with the surface tension, may sometimes be overlooked, particularly for rotated surfaces.

References

[1] Craighead,H.G., Nanoelectromechanical systems. *Science*, 2000, 290: 1532-1535.
[2] Li,M., Tang,H.X. and Roukes,M.L., Ultra-sensitive NEMS-based cantilevers for sensing, scanned probe and very high-frequency applications. *Nature Nanotechnology*, 2007, 2: 114-120.
[3] Miller,R.E. and Shenoy,V.B., Size-dependent elastic properties of nanosized structural elements. *Nanotechnology*, 2000, 11: 139-147.
[4] Guo,J.G. and Zhao,Y.P., The size-dependent elastic properties of nanofilms with surface effects. *Journal of Applied Physics*, 2005, 98: 074306.
[5] Shim,H.W., Zhou,L.G., Huang, H. and Cale, T.S., Nanoplate elasticity under surface reconstruction. *Applied Physics Letters*, 2005, 86: 151912.
[6] Zhang,L. and Huang,H., Young's moduli of ZnO nanoplates: ab initio determinations. *Applied Physics Letters*, 2006, 89: 183111.
[7] Cao,G. and Chen,X., Energy analysis of size-dependent elastic properties of ZnO nanofilms using atomistic simulations. *Physical Review B*, 2007, 76: 165407.
[8] Guo,J.G. and Zhao,Y.P., The size-dependent bending elastic properties of nanobeams with surface effects. *Nanotechnology*, 2007, 18: 295701.
[9] Wang,J., Huang,Q.A. and Yu,H., Young's modulus of silicon nanoplates at finite temperature. *Applied Surface Science*, 2008, 255: 2449-2455.
[10] Sun,C.Q., Thermo-mechanical behavior of low-dimensional systems: The local bond average approach. *Progress in Materials Science*, 2009, 54: 179-307.
[11] Zhou,L.J., Guo, J.G. and Zhao, Y.P., Size and thermal expansion coefficient of a nanofilm temperature dependent. *Chinese Physics Letters*, 2009, 26: 066201.
[12] Pradhan,S.C. and Phadikar,J.K., Nonlocal elasticity theory for vibration of nanoplates. *Journal of Sound and Vibration*, 2009, 325: 206-223.
[13] Shuttleworth,R., The surface tension of solids. *Proceedings of the Physical Society Series A*, 1950, 63: 444-457.
[14] Gurtin,M.E. and Murdoch A.I., A continuum theory of elastic material surfaces. *Archive for Rational Mechanics and Analysis*, 1975, 57: 291-323.
[15] Murdoch,A.I., Some fundamental aspects of surface modeling. *Journal of Elasticity*, 2005, 80: 33-52.
[16] Steigmann,D.J., Elastic surface-substrate interactions. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 1999, 455: 437-474.
[17] Huang,Z.P. and Wang,J., A theory of hyperelasticity of multi-phase media with surface/interface energy effect. *Acta Mechanica*, 2006, 182: 195-210.
[18] Wang,Z.Q., Zhao,Y.P. and Huang, Z.P., The effects of surface tension on the elastic properties of nano structures. *International Journal of Engineering Sciences*, 2009, in press, doi:10.1016/j.ijengsci.2009.07.007.
[19] Dingreville,R. and Qu,J., Interfacial excess energy, excess stress and excess strain in elastic solids: planar interfaces. *Journal of the Mechanics and Physics of Solids*, 2008, 56: 1944-1954.
[20] Cammarata,R.C., Surface and interface stress effects in thin films. *Progress in Surface Science*, 1994, 46: 1-38.
[21] Cammarata,R.C., Trimble,T.M. and Srolovitz,D.J., Surface stress model for intrinsic stresses in thin films. *Journal of Materials Research*, 2000, 15: 2468-2474.
[22] He,L.H., Lim,C.W. and Wu,B.S., A continuum model for size-dependent deformation of elastic films of nano-scale thickness. *International Journal of Solids and Structures*, 2004, 41: 847-857.
[23] Lim,C.W. and He,L.H., Size-dependent nonlinear response of thin elastic films with nano-scale thickness. *International Journal of Mechanical Sciences*, 2004, 46: 1715-1726.
[24] Dingreville,R., Qu,J. and Cherkaoui,M., Surface free energy and its effect on the elastic behavior of nano-sized particles, wires and films. *Journal of the Mechanics and Physics of Solids*, 2005, 53: 1827-1854.
[25] Lu,P., He,L.H., Lee,H.P. and Lu,C., Thin plate theory including surface effects. *International Journal of Solids and Structures*, 2006, 43: 4631-4647.
[26] Huang,R., Stafford,C.M. and Vogt,B.D., Effect of surface properties on wrinkling of ultrathin films. *Journal of Aerospace Engineering*, 2007, 20: 38-44.

[27] Michael,J.L. and John,E.S., Effect of surface stress on the stiffness of cantilever plates. *Physical Review Letters*, 2007, 99: 206102.
[28] Lin,C.C., Yang,F.Q. and Lee,S., Surface wrinkling of an elastic film: effect of residual surface stress. *Langmuir*, 2008, 24: 13627-13631.
[29] Gibbs,J.W., On the equilibrium of heterogeneous substances. In: The Scientific Papers of J. Willard Gibbs. Volume 1: Thermodynamics. New York: Dover, 1961, 55-353.
[30] Zhao,X.J. and Rajapakse,R.K.N.D., Analytical solutions for a surface-loaded isotropic elastic layer with surface energy effects. *International Journal of Engineering Science*, 2009, 47: 1433-1444, doi:10.1016/j.ijengsci.2008.12.013.
[31] Cuenot,S., Frétigny,C., Demoustier-Champagne,S. and Nysten,B., Surface tension effect on the mechanical properties of nanomaterials measured by atomic force microscopy. *Physical Review B*, 2004, 69: 165410.
[32] Gurtin,M.E. and Murdoch,A.I., Surface stress in solids. *International Journal of Solids and Structures*, 1978, 14: 431-440.
[33] Avazmohammadi,R., Yang,F.Q. and Abbasion,S., Effect of interface stresses on the elastic deformation of an elastic half-plane containing an elastic inclusion. *International Journal of Solids and Structures*, 2009, 46: 2897-2906.
[34] Hoger,A., On the determination of residual stress in an elastic body. *Journal of Elasticity*, 1986, 16: 303-324.
[35] Washizu,K., Variational Methods in Elasticity and Plasticity, Oxford: Pergamon, 1982.
[36] Timoshenko,S. and Gere,J.M., Theory of Elastic Stability. New York: McGraw-Hill, 1961.

APPENDIX A: SURFACE ELASTICITY FOR INFINITESIMAL STRAINS

The surface constitutive relations for infinitesimal strains are derived from the finite deformation approximations.

Consider a smooth surface A_0 in the reference configuration, which is described by the position function $\boldsymbol{x} = \boldsymbol{x}\left(\theta^1, \theta^2\right)$. After deformation, this surface becomes A, expressed by the parametric function $\boldsymbol{x}' = \boldsymbol{x}\left(\theta^1, \theta^2\right) + \boldsymbol{u}\left(\theta^1, \theta^2\right)$, where $\boldsymbol{u}$ is the displacement (as shown in Fig.4). The covariant base vectors at A_0 and A are $\boldsymbol{A}_\alpha = \boldsymbol{x}_{,\alpha}$ $(\alpha = 1, 2)$ and $\boldsymbol{a}_\alpha = \boldsymbol{x}'_{,\alpha}$ $(\alpha = 1, 2)$, respectively. Base vectors $\boldsymbol{A}_\alpha$ (or $\boldsymbol{a}_\alpha$) span the tangent plane of the surface at $\boldsymbol{x}$ (or $\boldsymbol{x}'$) in the reference (or current) configuration. Vectors $\boldsymbol{N}$ and $\boldsymbol{n}$ are unit normal vectors of the surface before and after deformations, respectively. The displacement vector $\boldsymbol{u}$ of a point on the surface can be written either in the reference or current configurations

$$\boldsymbol{u} = \boldsymbol{x}' - \boldsymbol{x} = u_0^\alpha \boldsymbol{A}_\alpha + u_0^n \boldsymbol{N} = u^\alpha \boldsymbol{a}_\alpha + u^n \boldsymbol{n} \tag{46}$$

Then, we have the following relations between $\boldsymbol{A}_\alpha$ and $\boldsymbol{a}_\alpha$

$$\begin{aligned} \boldsymbol{a}_\alpha &= \boldsymbol{A}_\alpha + \left[\left(u_0^\lambda\big|_\alpha - u_0^n b_{0\alpha}^\lambda\right)\boldsymbol{A}_\lambda + \left(u_0^\beta b_{0\alpha\beta} + u_{0,\alpha}^n\right)\boldsymbol{N}\right] \\ \boldsymbol{A}_\alpha &= \boldsymbol{a}_\alpha - \left[\left(u^\lambda\big\|_\alpha - u^n b_\alpha^\lambda\right)\boldsymbol{a}_\lambda + \left(u^\beta b_{\alpha\beta} + u_{,\alpha}^n\right)\boldsymbol{n}\right] \end{aligned} \tag{47}$$

where $u_0^\lambda\big|_\alpha$ and $u^\lambda\big\|_\alpha$ are defined by

$$\begin{aligned} u_0^\lambda\big|_\alpha &= u_{0,\alpha}^\lambda + u_0^\beta \bar{\Gamma}_{0\alpha\beta}^\lambda \\ u^\lambda\big\|_\alpha &= u_{,\alpha}^\lambda + u^\beta \bar{\Gamma}_{\alpha\beta}^\lambda \end{aligned} \tag{48}$$

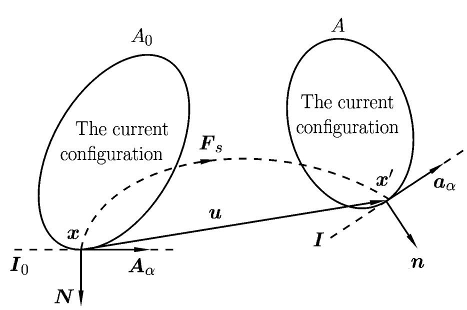

Fig. 4. The deformation of a surface: from the reference configuration A_0 to the current configuration A.

$\bar{\Gamma}^{\lambda}_{0\alpha\beta}$ and $\bar{\Gamma}^{\lambda}_{\alpha\beta}$ are the Christoffel symbols of the second kind of the surface before and after deformations; and the surface curvature tensors in the reference and the current configurations can be expressed as

$$\begin{aligned} \boldsymbol{b}_0 &= b^{\lambda}_{0\alpha}\boldsymbol{A}_{\lambda}\otimes\boldsymbol{A}^{\alpha} = b_{0\alpha\beta}\boldsymbol{A}^{\alpha}\otimes\boldsymbol{A}^{\beta} \\ \boldsymbol{b} &= b^{\lambda}_{\alpha}\boldsymbol{a}_{\lambda}\otimes\boldsymbol{a}^{\alpha} = b_{\alpha\beta}\boldsymbol{a}^{\alpha}\otimes\boldsymbol{a}^{\beta} \end{aligned} \tag{49}$$

where $\boldsymbol{A}^{\alpha}$ and $\boldsymbol{a}^{\alpha}$ are the contra-variant base vectors of $\boldsymbol{A}_{\alpha}$ and $\boldsymbol{a}_{\alpha}$, respectively. From Eq.(47), the surface deformation gradient $\boldsymbol{F}_s$ in the reference configuration and its inverse $\boldsymbol{F}_s^{-1}$ in the current configuration can be written as

$$\begin{aligned} \boldsymbol{F}_s &= \boldsymbol{a}_{\alpha}\otimes\boldsymbol{A}^{\alpha} = \boldsymbol{I}_0 + \boldsymbol{u}_0\bar{\nabla}_{0s} + \boldsymbol{F}_s^{(o)} \\ \boldsymbol{F}_s^{-1} &= \boldsymbol{A}_{\alpha}\otimes\boldsymbol{a}^{\alpha} = \boldsymbol{I} - \left[\boldsymbol{u}\bar{\nabla}_s + \boldsymbol{F}_s^{-1(o)}\right] \end{aligned} \tag{50}$$

where

$$\boldsymbol{u}_0\bar{\nabla}_{0s} = \left(u_0^{\lambda}\big|_{\alpha} - u_0^n b^{\lambda}_{0\alpha}\right)\boldsymbol{A}_{\lambda}\otimes\boldsymbol{A}^{\alpha},\ \ \boldsymbol{F}_s^{(o)} = \left(u_0^{\beta} b_{0\alpha\beta} + u^n_{0,\alpha}\right)\boldsymbol{N}\otimes\boldsymbol{A}^{\alpha} \tag{51}$$

$$\boldsymbol{u}\bar{\nabla}_s = \left(u^{\lambda}\big\|_{\alpha} - u^n b^{\lambda}_{\alpha}\right)\boldsymbol{a}_{\lambda}\otimes\boldsymbol{a}^{\alpha}, \qquad \boldsymbol{F}_s^{-1(o)} = \left(u^{\beta} b_{\alpha\beta} + u^n_{,\alpha}\right)\boldsymbol{n}\otimes\boldsymbol{a}^{\alpha} \tag{52}$$

$\boldsymbol{I}$ is the identity tensor on the tangent planes of the surface after deformations; $\boldsymbol{F}_s^{(o)}$ denotes the out-of-plane term of $\boldsymbol{F}_s$ in the reference configuration; $\boldsymbol{F}_s^{-1(o)}$ is the corresponding out-of-plane term of $\boldsymbol{F}_s^{-1}$ in the current configuration.

In the case of small strain, the surface strain can be approximately expressed by[18]

$$\boldsymbol{C}_s = \boldsymbol{F}_s^{\mathrm{T}}\cdot\boldsymbol{F}_s \approx \boldsymbol{I}_0 + 2\boldsymbol{E}_s, \qquad \boldsymbol{B}_s^{-1} = \boldsymbol{F}_s^{-T}\cdot\boldsymbol{F}_s^{-1} \approx \boldsymbol{I} - 2\boldsymbol{\varepsilon}_s \tag{53}$$

in which surface Green and Cauchy strains tensors are

$$\boldsymbol{E}_s = \frac{1}{2}\left(\bar{\nabla}_{0s}\boldsymbol{u}_0 + \boldsymbol{u}_0\bar{\nabla}_{0s}\right), \qquad \boldsymbol{\varepsilon}_s = \frac{1}{2}\left(\bar{\nabla}_s\boldsymbol{u} + \boldsymbol{u}\bar{\nabla}_s\right) \tag{54}$$

Consequently, $\boldsymbol{F}_s^{-1}$ can be expressed in the reference configuration as

$$\boldsymbol{F}_s^{-1} = \boldsymbol{C}_s^{-1}\cdot\boldsymbol{F}_s^{\mathrm{T}} = (\boldsymbol{I}_0 - 2\boldsymbol{E}_s)\left(\boldsymbol{I}_0 + \bar{\nabla}_{0s}\boldsymbol{u}_0 + \boldsymbol{F}_s^{(o)T}\right) = \boldsymbol{I}_0 - \boldsymbol{u}_0\bar{\nabla}_{0s} + \boldsymbol{F}_s^{(o)\mathrm{T}} \tag{55}$$

Hence, the identity tensor $\boldsymbol{I}$ in the tangent plane of the current configuration can be given by

$$\boldsymbol{I} = \boldsymbol{F}_s\cdot\boldsymbol{F}_s^{-1} = \boldsymbol{I}_0 + \boldsymbol{F}_s^{(o)\mathrm{T}} + \boldsymbol{F}_s^{(o)} \tag{56}$$

Further, the relations of different surface strain tensors for infinitesimal strains can be obtained as

$$\boldsymbol{E}_s = \boldsymbol{\varepsilon}_s \tag{57}$$

For an isotropic and elastic surface, the surface energy per unit area of A_0 in the reference configuration, denoted by $J_2\gamma$, can be assumed to be a function of of J_1 and J_2, where

$$J_1 = 2 + \mathrm{tr}\boldsymbol{E}_s, \qquad J_2 = 1 + \mathrm{tr}\boldsymbol{E}_s + \det\boldsymbol{E}_s \tag{58}$$

Thus, the first kind Piola-Kirchhoff stress of the surface can be written as

$$\boldsymbol{S}_s = 2\boldsymbol{F}_s\cdot\frac{\partial\left(J_2\gamma\right)}{\partial\boldsymbol{C}_s} = \bar{\gamma}\boldsymbol{i}_0 + J_2\frac{\partial\gamma}{\partial J_1}\boldsymbol{E}_s - \bar{\gamma}(\bar{\nabla}_{0s}\boldsymbol{u}) + \bar{\gamma}\boldsymbol{F}_s^{(o)} \tag{59}$$

where $\bar{\gamma} = J_2\left(\partial\gamma/\partial J_1 + J_2\partial\gamma/\partial J_2 + \gamma\right)$. The above equation can be further simplified if γ is expressed as a series expansion with the higher-order terms truncated

$$\gamma = \gamma_0 + \gamma_1(J_1-2) + \gamma_2(J_2-1) + \frac{1}{2}\gamma_{11}(J_1-2)^2 + \gamma_{12}(J_1-2)(J_2-1) + \frac{1}{2}\gamma_{22}(J_2-1)^2 \tag{60}$$

Hence, by neglecting high-order small quantities, we have[18]

$$\boldsymbol{S}_s = \gamma_0^* \boldsymbol{i}_0 + (\gamma_0^* + \gamma_1^*)(\mathrm{tr}\boldsymbol{E}_s)\,\boldsymbol{i}_0 - \gamma_0^* (\bar{\nabla}_{0s}\boldsymbol{u}_0) + \gamma_1 \boldsymbol{E}_s + \gamma_0^* \boldsymbol{F}_s^{(o)} \tag{61}$$

where $\gamma_0^* = \gamma_0 + \gamma_1 + \gamma_2$, $\gamma_1^* = \gamma_1 + 2\gamma_2 + \gamma_{11} + 2\gamma_{12} + \gamma_{22}$.

In the Eulerian description, the stress-strain relations of the surface can be written as

$$\boldsymbol{\sigma}_s = \frac{1}{J_2}\boldsymbol{S}_s \cdot \boldsymbol{F}_s^{\mathrm{T}} = \gamma_0^* \boldsymbol{I} + \gamma_1^* (\mathrm{tr}\boldsymbol{\varepsilon}_s)\,\boldsymbol{I} + \gamma_1 \boldsymbol{\varepsilon}_s \tag{62}$$

By using Eqs.(56) and (57), the above model can also be expressed, in the undeformed configuration, as

$$\boldsymbol{\sigma}_s = \gamma_0^* \boldsymbol{I}_0 + \gamma_1^* (\mathrm{tr}\boldsymbol{E}_s)\,\boldsymbol{I}_0 + \gamma_1 \boldsymbol{E}_s + \gamma_0^* \left(\boldsymbol{F}_s^{(o)} + \boldsymbol{F}_s^{(o)\mathrm{T}}\right) \tag{63}$$

This means that there are out-of-plane terms which are related to the rotation of the surface when surface Cauchy stress is expressed in the reference configuration.

APPENDIX B: GENERALIZED YOUNG-LAPLACE EQUATIONS

The Young-Laplace equations are used to describe the equilibrium conditions of a surface, which can be derived from the stationary condition of the energy function proposed in Refs.[17,18]. The Lagrangian description of the Young-Laplace equations of the surface can be written as

$$\begin{aligned} \boldsymbol{N} \cdot [\![\boldsymbol{S}]\!] \cdot \boldsymbol{N} &= -\left(\boldsymbol{S}_s^{(i)}\right) : \boldsymbol{b}_0 - \left[\boldsymbol{N} \cdot \left(\boldsymbol{S}_s^{(o)}\right)\right] \cdot \nabla_{0s} \\ \boldsymbol{P}_0 \cdot [\![\boldsymbol{S}]\!] \cdot \boldsymbol{N} &= -\left(\boldsymbol{S}_s^{(i)}\right) \cdot \nabla_{0s} + \left[\boldsymbol{N} \cdot \left(\boldsymbol{S}_s^{(o)}\right) \cdot \boldsymbol{b}_0\right] \end{aligned} \tag{64}$$

where $[\![\boldsymbol{S}]\!]$ denotes the discontinuity of $\boldsymbol{S}$ across the surface A_0, $\boldsymbol{P}_0 = \boldsymbol{1} - \boldsymbol{N} \otimes \boldsymbol{N}$ is the perpendicular projection of the space of all vectors upon the space of tangential vectors in the reference configuration, $\boldsymbol{S}_s^{(i)}$ and $\boldsymbol{S}_s^{(o)}$ are the in-plane term and the out-of-plane term of $\boldsymbol{S}_s$, respectively; the operator ∇_{0s} is defined as $\boldsymbol{V}_0 \cdot \nabla_{0s} = V_0^{\alpha}\big|_{\alpha}$ for a vector $\boldsymbol{V}_0 = V_0^{\alpha} \boldsymbol{A}_{\alpha}$.

The corresponding equations under the Eulerian description are[17]

$$\boldsymbol{n} \cdot [\![\boldsymbol{\sigma}]\!] \cdot \boldsymbol{n} = -\boldsymbol{\sigma}_s : \boldsymbol{b}, \quad \boldsymbol{P} \cdot [\![\boldsymbol{\sigma}]\!] \cdot \boldsymbol{n} = -\boldsymbol{\sigma}_s \cdot \nabla_s \tag{65}$$

in which $\boldsymbol{P} = \boldsymbol{1} - \boldsymbol{n} \otimes \boldsymbol{n}$ is the corresponding perpendicular projection in the current configuration, $\boldsymbol{\sigma}$ is the Cauchy stress of the bulk, $[\![\boldsymbol{\sigma}]\!]$ denotes the discontinuity of $\boldsymbol{\sigma}$ across the surface A, the operator ∇_s is defined as $\boldsymbol{v} \cdot \nabla_s = v^{\alpha\beta}\big\|_{\beta}$ for a tensor $\boldsymbol{v} = v^{\alpha\beta} \boldsymbol{a}_{\alpha} \otimes \boldsymbol{a}_{\beta}$.