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CONVERGENCE OF ATTRACTORS

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Abstract

The system of coupled oscillators and its time-discretization (with constant stepsize h) are considered in this paper. Under some conditions, it is showed that the discrete systems have one-dimensional global attractors l_h converging to l which is the global attractor of continuous system.

Key words coupled oscillators system, attractor, cone, horizontal curve

I. Introduction

Consider system of coupled oscillators

$$\dot{x}=Ax+f(x) \tag{1.1}$$

in which $x=(x^1, x^2, \cdots, x^m)\in \mathbf{R}^m$ $(m\geqslant 3)$, A is $m\times m$ matrix, f is a nonlinear map from $\mathbf{R}^m$ to $\mathbf{R}^m$

$$A=m^2\begin{pmatrix} -1 & 1 & & & & 0 & \\ 1 & -2 & 1 & & & & \\ & 1 & -2 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & 0 & & 1 & -2 & 1 & \\ & & & & 1 & -1 & \end{pmatrix},\quad f(x)=\begin{pmatrix} -\sin(x^1)+b \\ -\sin(x^2) \\ \vdots \\ -\sin(x^{m-1}) \\ -\sin(x^m) \end{pmatrix}$$

with constant $b>0$.

(1.1) is discretized by one-step method:

$$x_{n+1}=x_n+h(Ax_n+f(x_n)) \tag{1.2}$$

with constant stepsize $h>0$.

Let F_h denote the map from $\mathbf{R}^m$ to $\mathbf{R}^m$:

$$F_hx=x+h(Ax+f(x)) \tag{1.3}$$

The main purpose of this paper is to investigate the global attractors of discrete systems $\{F_h^n\}_{n=1}^{+\infty}$ (with sufficiently small h) and continuous system (1.1) and to discuss the convergence of attractors of discrete systems.

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II. Global Attractor of Continuous System

Consider the initial value problem of (1.1)

$$\dot{x}=Ax+f(x),\ x(0)=x_0 \tag{2.1}$$

which determines a continuous-time semigroup $\{S(t)\}_{t\geqslant 0}$:

$$S(t):x_0\mapsto x(t)$$

in which $x(t)$ is the solution to initial value problem (2.1).

Let λ_i $(i=0,\ 1,\ \cdots,\ m-1)$ denote the eigenvalues of A,

$$\lambda_i=-4m^2\sin^2(i\pi/2m),\qquad (i=0,\ 1,\ \cdots,\ m-1)$$

The eigenvector corresponding to eigenvalue $\lambda_0=0$ is denoted by $\eta_0=(1,\ 1,\ \cdots,\ 1)^T\in\mathbf{R}^m$. Let $E_1=\mathrm{span}\{\eta_0\}$ and $E_2=E_1^\perp$. The projections from $\mathbf{R}^m$ to E_1 and E_2 are denoted by P and Q respectively. Note that $\lambda_1=-4m^2\sin^2(\pi/2m)<-4$ $(m>3)$. In the sequel, the Euclidean inner product and norm in $\mathbf{R}^m$ is denoted by $\langle\cdot,\ \cdot\rangle$ and $|\cdot|$ respectively.

Definition 2.1 The set

$$Z=\{p+q\in\mathbf{R}^m\,|\,p\in E_1,\ q\in E_2,\ |q|\leqslant|p|\}$$

is said to be a cone in $\mathbf{R}^m$.

Lemma 2.2 For any $y_0,\ x_0\in\mathbf{R}^m$

i) if $y_0-x_0\in Z$, then

$$S(t)y_0-S(t)x_0\in Z,\quad \text{for } t\geqslant 0 \tag{2.2}$$

ii) if for some $t_0>0,\ S(t_0)y_0-S(t_0)x_0\notin Z$, then

$$|Q(S(t)y_0-S(t)x_0)|\leqslant\exp[\lambda_1 t/2]\,|Q(y_0-x_0)|\quad \text{for } 0\leqslant t\leqslant t_0 \tag{2.3}$$

Proof i) Let $x(t)=S(t)x_0,\ y(t)=S(t)y_0,\quad p(t)=P(y(t)-x(t)),\ q(t)=Q(y(t)-x(t))$, then $p(t)$ and $q(t)$ satisfy respectively

$$\dot{p}=P(f(y(t))-f(x(t))),\ p(0)=P(y_0-x_0) \tag{2.4}$$

and

$$\dot{q}=Aq+Q(f(y(t))-f(x(t))),\ q(0)=Q(y_0-x_0) \tag{2.5}$$

By (2.4) and the fact $|f(y)-f(x)|\leqslant|y-x|$, we have

$$\begin{aligned}\frac{d}{dt}|p(t)|^2&=2\langle Pf(y)-Pf(x),\ p\rangle\\ &\geqslant-2|f(y(t))-f(x(t))|\cdot|p(t)|\\ &\geqslant-2|y(t)-x(t)|\cdot|p(t)|\\ &=-2|p(t)+q(t)|\cdot|p(t)|\\ &\geqslant-2(|p(t)|^2+|p(t)|\cdot|q(t)|)\end{aligned} \tag{2.6}$$

Similarly, from (2.5) it follows

$$\frac{d}{dt}|q(t)|^2\leqslant 2\lambda_1|q(t)|^2+2(|q(t)|^2+|p(t)|\cdot|q(t)|) \tag{2.7}$$

Therefore

$$\frac{d}{dt}(|q|^2-|p|^2)\leqslant 2\lambda_1|q|^2+2(|p|^2+|q|^2)+4|p|\cdot|q|$$

and hence

$$\frac{d}{dt}(|q|^2-|p|^2)\leqslant(2\lambda_1+8)|q|^2\leqslant 0 \quad \text{for } |q|=|p|$$

This shows that $y(t)-x(t)\in Z$, $\forall t>0$ if $y_0-x_0\in\partial Z$ (the boundary of Z).

ii) Assume there exists $t_0>0$ such that $y(t_0)-x(t_0)\notin Z$, then $y(t)-x(t)\notin Z$ for $0\leqslant t\leqslant t_0$ by i), namely $|q(t)|>|p(t)|$, $0\leqslant t\leqslant t_0$. By (2.7) we have

$$\frac{d}{dt}|q(t)|^2\leqslant(2\lambda_1+4)|q(t)|^2\leqslant\lambda_1|q(t)|^2$$

and

$$|q(t)|^2\leqslant\exp[\lambda_1 t]|q(0)|^2$$

Consequently,

$$|Q(y(t)-x(t))|\leqslant\exp[\lambda_1 t/2]|Q(y_0-x_0)|, \qquad 0\leqslant t\leqslant t_0$$

Definition 2.3 Suppose Φ is a Lipschitz map from E_1 into E_2. The continuous curve

$$l=\{p+\Phi(p)\,|\,p\in E_1\}$$

is said to be a horizontal curve if

$$|\Phi(p_1)-\Phi(p_2)|\leqslant|p_1-p_2| \quad \text{for any } p_1,\ p_2\in E_1$$

Additionally, if Φ satisfies $\Phi(p+2\pi\eta_0)=\Phi(p)$ for any $p\in E_1$, then l is called a restricted horizontal curve.

Corollary 2.4 If l is a horizontal curve (restricted horizontal curve), then $S(t)l$ ($\forall t>0$) is still a horizontal curve (restricted horizontal curve).

Proof For any y, $x\in S(t)l$, there exist $\exists\, y_0,\ x_0\in l$, such that $S(t)y_0=y$, $S(t)x_0=x$. Since x_0, $y_0\in l$, $y_0-x_0\in Z$. From Lemma 2.2 it follows $S(t)y_0-S(t)x_0=y-x\in Z$, hence $S(t)l$ is a horizontal curve.

Note the fact $f(x+2\pi\eta_0)=f(x)$, then $S(t)l$ is a restricted horizontal curve if l is a restricted horizontal one.

Lemma 2.5 There is a constant $c>0$, such that, for any $x\in\mathbf{R}^m$, there exists $t_0>0$, $|QS(t)x|\leqslant c/4$ for $t>t_0$. If $|Qx|\leqslant c/4$, then $|QS(t)x|\leqslant c/4$ for all $t>0$.

Proof Obviously, f is a bounded map from $\mathbf{R}^m$ to $\mathbf{R}^m$, that is, there is a constant $c>0$ such that $|f(x)|\leqslant c$ for any $x\in\mathbf{R}^m$. Suppose $x\in\mathbf{R}^m$, then $S(t)x$ satisfies

$$S(t)x=e^{At}x+\int_0^t e^{A(t-\tau)}f(S(\tau)x)d\tau \tag{2.8}$$

$$\begin{aligned}\Rightarrow\quad |QS(t)x| &\leqslant\|e^{At}Q\|\,|Qx| \\ &\quad+\int_0^t\|e^{(At-\tau)}Q\|\cdot|f(S(\tau)x)|d\tau \\ &\leqslant\exp[\lambda_1 t]|Qx|+c\int_0^t\exp[\lambda_1(t-\tau)]d\tau \\ &=\exp[\lambda_1 t]|Qx|+(c/|\lambda_1|)(1-\exp[\lambda_1 t])\end{aligned}$$

This immediately leads to lemma 2.5.

Theorem 2.6 For any $\tau>0$, the map $S(\tau)$ has an invariant restricted horizontal curve l: $S(\tau)l=l$.

Proof Let $H=[0, 2\pi]\cdot\eta_0\subset E_1$, $\mathscr{B}=\{x\in \mathbf{R}^m | Px\in H\}$

$$\mathscr{M}=\{l\,|\,l \text{ is a restricted horizontal curve }\}$$

$$\hat{\mathscr{M}}=\{\hat{l}=l\cap\mathscr{B}\,|\,l\in\mathscr{M}\}$$

let Π denote the map from $\mathscr{M}$ to $\hat{\mathscr{M}}$:

$$\Pi l=\hat{l}=l\cap\mathscr{B}$$

For any $\hat{l}\in\hat{\mathscr{M}}$, $\hat{l}$ corresponds to a unique Lipschtz map from H to E_2. Let M denote the set of Lipschitz maps corresponding to all the elements of $\hat{\mathscr{M}}$. The operations in M are defined as usual and the norm is defined as follows:

$$\|g\|=\max_{p\in H}|g(p)|, \qquad g\in M$$

Then M is a Banach space.

Let

$$\tilde{M}=\{g\,|\,g\in M,\ \|g\|\leqslant c/4\}$$

In the sequel, the element in M and the corresponding one in $\hat{\mathscr{M}}$ are regarded as the same. $\tilde{M}$ is compact in M by Arzela-Ascoli theorem, since $\tilde{M}$ is uniformly bounded and equicontinuous. We define a map F from M to M such that $\Pi\circ S(\tau)=F\circ\Pi$, namely, $F=\Pi\circ S(\tau)\circ\Pi^{-1}$. By lemma 2.2 and 2.5 we have $F\tilde{M}\subset\tilde{M}$. Since F is continuous from M into M, F has at least one fixed point $\hat{l}$ in $\tilde{M}$ by Schauder fixed point theorem. Let $l=\Pi^{-1}\hat{l}$, then

$$S(\tau)l=S(\tau)\Pi^{-1}\hat{l}=\Pi^{-1}F\hat{l}=\Pi^{-1}\hat{l}=l$$

thus l is an invariant restricted horizontal curve for $S(\tau)$.

Theorem 2.7 System (1.1) has one-dimensional global attractor l. l is a restricted horizontal curve in $\mathbf{R}^m$.

Proof For $\tau>0$, by theorem 2.6, there exists $l\in\mathscr{M}$, such that $S(\tau)l=l$. We shall prove: 1° l is invariant for the semigroup $\{S(t)\}_{t\geqslant 0}$, i. e. $S(t)l=l$, $\forall t>0$. 2° for any $x\in\mathbf{R}^m$, $d(S(t)x, l)\to 0$ as $t\to+\infty$, therefore, l is the global attractor of system (1.1).

Proof of 1° For any $x\in\mathbf{R}^m$, we show $d(S(n\tau)x, l)\to 0$, as $n\to+\infty$, here $d(x,l)$ denotes the distance between x and l. Without loss of generality, we assume $|Qx|\leqslant c/4$. For any integer n, there exists $\hat{y}\in l$ such that $PS(n\tau)x=P\hat{y}$. Since $S(\tau)l=l$, there is a $y\in l$ such that $S(n\tau)y=\hat{y}$ which implies $S(n\tau)y-S(n\tau)x\notin Z$. By lemma 2.2 we have

$$\begin{aligned} d(S(n\tau)x, l)&\leqslant|Q(S(n\tau)x-S(n\tau)y)|\\ &\leqslant\exp[\lambda_1 n/2]\,|Qx-Qy|\\ &\leqslant\exp[\lambda_1 n\tau]\cdot(c/2)\end{aligned}$$

which leads to $d(S(n\tau)x, l)\to 0$ as $n\to+\infty$. Hence $S(\tau)$ has a unique invariant restricted horizontal curve.

Suppose t is sufficiently small such that $\Pi S(t)l\in\tilde{M}$.

$$F\circ\Pi S(t)l=\Pi\circ S(\tau)S(t)l=\Pi\circ S(t)S(\tau)l=\Pi\circ S(t)l$$

This shows $\Pi S(t)l$ is a fixed point of F in $\tilde{M}$, and hence $S(t)l$ is an invariant restricted horizontal curve for $S(\tau)$. From the above discussion, it follows $S(t)l=l$, $\forall t>0$.

Proof of 2° For any $x\in\mathbf{R}^m$, it follow from (2.3) $d(S(t)x, l)\to 0$ as $t\to+\infty$. Therefore, l is global attractor of semigroup $\{S(t)\}_{t\geq 0}$, (or system (1.1)).

III. Attractors of Discrete Systems

In this section, the notations are the same as in the previous section.

Let

$$F_h x=Bx+hf(x), \qquad x\in\mathbf{R}^m$$

in which $B=I+hA$. I is the identity in $\mathbf{R}^m$. In the sequel we suppose $h<1/4m^2$ such that the eigenvalues of B $\mu_i=1+h\lambda_i$ satisfy $0\leqslant\mu_i\leqslant 1$, $i=0, 1, \cdots, m-1$.

Lemma 3.1 For any y, $x\in\mathbf{R}^m$,

i) if $y-x\in Z$, then

$$F_h^n y-F_h^n x\in Z, \qquad n=1, 2, \cdots \tag{3.1}$$

ii) if $F_h^k y-F_h^k x\notin Z$, then

$$|Q(F_h^k y-F_h^k x)|\leqslant(1-2h)^k|Q(y-x)| \tag{3.2}$$

The proof is similar to that of lemma 2.2.

Lemma 3.2 $\forall x\in\mathbf{R}^m$, $\exists N>0$, when $n>N$, $|QF_h^n x|\leqslant c/4$ (here c is the same as in lemma 2.5). Furthmore, if $|Qx|\leqslant c/4$, then $|QF_h x|\leqslant c/4$.

The proof is similar to that of lemma 2.5.

Theorem 3.3 F_h has an invariant restricted horizontal curve l_h : $F_h l_h=l_h$.

Theorem 3.4 l_h is the global attractor of F_h.

The proofs of theorem 3.3 and 3.4 are similar to those of theorem 2.6 and 2.7 and hence omitted here.

IV. Convergence of Attractors

From the discussions of section II and section III, we know in phase space $\mathbf{R}^m$ $(m\geqslant 3)$ the discrete system $\{F_h^n\}_{n=1}^{+\infty}(h<1/4m^2)$ and the continuous system have one-dimensional attractors l_h and l respectively, which are restricted horizontal curves in $\mathbf{R}^m$. Suppose l_h and l correspond respectively to the elements Φ_h and Φ in Banach space M.

For the solution $x(t)$ to the initial value problem of (1.1)

$$\dot{x}=Ax+f(x), \quad x(0)=x_0$$

the approximation value at time T calculated by one-step method is denoted by $F_h^N x_0$ where the stepsize $h=T/N$, N is an integer. By the convergent property of one-step method we know $F_h^N x_0\to x(T)$ as $h\to 0$. For the whole system, what can be said about the relation between l_h, the attractor of the discrete system, and l which is the attractor of the continuous system? General results seem to be unknown. In this section we show that for the system of coupled oscillators (1.1), l_h approaches l as $h\to 0$, by which we mean that the corresponding elements Φ_h converge to Φ as $h\to 0$ in Banach space M.

Theorem 4.1 $\lim\limits_{h\to 0}\|\Phi_h-\Phi\|=0$

Proof Note that $\{\Phi_h\}(0<h<1/4m^2)$ is uniformly bounded and equi-continuous, hence $\{\Phi_h\}$ is precompact in M. We prove theorem by contradiction. Suppose $\Phi_h\not\to\Phi$ as $h\to 0$, that is, $\exists\,\varepsilon_0>0$ and series $\{\Phi_{h_i}\}$ such that $\|\Phi_{h_i}-\Phi\|\geqslant 2\varepsilon_0$ $(i=1, 2, ...)$ while $h_i\to 0$ as $i\to+\infty$. $\{\Phi_{h_i}\}$ has a convergent subsequence since $\{\Phi_h\}$ is precompact in M. For simplicity of presentation, we assume $\Phi_{h_i}\to\Psi\in M$, and $\|\Psi\|\leqslant c/4$. Consequently, $\|\Psi-\Phi\|>\varepsilon_0$. According to the definition of the norm in M, there exists a $p\in H$ such that $|\Psi(p)-\Phi(p)|>\varepsilon_0$; on the other hand, $|\Phi_{h_i}(p)-\Psi(p)|\to 0$ as $i\to+\infty$, then there is an integer number N_1 such that $|\Phi_{h_i}(p)-\Phi(p)|>\varepsilon_0$ for $i>N_1$. Let $x=p+\Phi(p)$, then we have

$$d(x,\ l_{h_i})>2\varepsilon_1 \tag{4.1}$$

if $i>N_1$, here ε_1 is some positive number. Take $T_i=h_in_i$ such that T_i is less than some $T>0$ and $T_i>T_0=[-\ln(2\varepsilon_1/c)/2]$. Since l is invariant for $S(T_i)$, there exist x_i' such that $S(T_i)x_i'=x$, $i=1, 2\cdots$. On the interval $[0, T_i]$, calculating the initial value problem

$$\dot{x}=Ax+f(x),\ x(0)=x_i'.$$

by one-step method, the error tends to 0 as $h_i\to 0$.
Consequently $F_{h_i}^{n_i}x_i'\to x$ as $i\to+\infty$.
Let us suppose that

$$|F_{h_i}^{n_i}x_i'-x|<\varepsilon_1 \quad \text{if } i>N_2 \tag{4.2}$$

From (3.2) it follows

$$d(F_{h_i}^{n_i}x_i',\ l_{h_i})\leqslant(1-2h_i)^{n_i}\cdot(c/2)\leqslant(1-2h_i)^{T_0/h_i}\cdot(c/2)$$

Since

$$\lim_{h_i\to 0}(1-2h_i)^{T_0/h_i}=\exp[-2T_0]$$

there exists $N_3>0$, such that

$$d(F_{h_i}^{n_i}x_i',l_{h_i})\leqslant\exp[-2T_0](c/2)<\varepsilon_1 \quad \text{for } i>N_3 \tag{4.3}$$

Taking $N=\max\{N_1,\ N_2,\ N_3\}$, we have $d(x,\ l_{h_i})<2\varepsilon_1$ as $i>N$ by (4.2) and (4.3). This is in contradiction with (4.1).

From Theorem 4.1 it follows immediately that l_h converges to l as $h\to 0$

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