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CRACK PROPAGATION IN THE POWER-LAW NONLINEAR VISCOELASTIC MATERIAL*

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Abstract

An analysis on crack creep propagation problem of power-law nonlinear visco-elastic materials is presented. The creep incompressilility assumption is used. To simulate fracture behavior of craze region, it is assumed that in the fracture process zone near the crack tip, the cohesive stress σ_f acts upon the crack surfaces and resists crack opening. Through a perturbation method, i. e., by superposing the Mode-I applied force onto a referential uniform stress state, which has a trivial solution and gives no effect on the solution of the original problem, the nonlinear viscoelastic problem is reduced to linear problem. For weak nonlinear materials, for which the power-law index $n \cong 1$, the expressions of stress and crack surface displacement are derived. Then, the fracture process zone local energy criterion is proposed and based on which the formulas of cracking incubation time t^* and crack slow propagation velocity $\dot{a}$ are derived.

Key words nonlinear viscoelasticity, creep incompressibility, perturbation method, crack propagation, crack incubation time

I. Introduction

Crazing damage is a common phenomenon of fracture of polymeric materials. The formation of craze zone is a mid-state in the fracture process of the materials from perfect state to failure. Microscopically, in this region there exists some fibrils linking the two crack surfaces and resisting the crack opening. This zone is usually called process zone. Therefore, this kind of fracture problem is more complex.

Viscoelastic materials have constitutive relationship of hereditary integration type, which renders much more difficulties than the elastic ones. The delayed fracture-the Griffith problem for linear viscoelastic materials was studied by Knauss[1] for the first time. The research on crack initiation and growth in viscoelastic medium was carried out by Schapery[2], he used the assumption that the process zone is simply the crack extension in which the cohesive stress σ_f acts upon the two crack surfaces and resists crack opening. He and et al[3], based on the plastic increment theory, analyzed the fracture problem of power-law hardening incompressible medium by using the perturbation concept and asserted that the analysis method proposed can be used to analyze creep problem. However, since the time dependence of modulus μ of the

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material has not been taken into consideration, strictly speaking that method is not applicable for creep (viscoelasticity) fracture problem.

The present paper attempts to deal with the fracture (cracking) problem of the power-law nonlinear viscoelastic medium. The assumption of creep incompressibility was made. With uniform stress and strain fields as referential state and using perturbation approach suggested in [3], the nonlinear viscoelastic problem was reduced to pseudo-isotropic linear problem. The expressions of asymptotic solution of stress and crack opening displacement of the material subjected to the combined action of Mode-I fracture force at remote boundary q and the cohesive stress σ_f in the craze zone were derived. Finally, according to the local energy criterion suggested by the present authors, the formulas of cracking incubation time and crack slow propagation velocity were proposed.

II. Proposition of the Boundary Value Problem

2.1 Configuration of the cracked medium

The central cracked infinite viscoelastic medium is schematically shown in Fig. 1. The applied load is perpendicular and symmetric to the crack surface. The crack surface is parallel to x coordinate and lies in the x–z coordinate plane, and $y=0$. The crack length is $2a(t)$. At the vicinity of the crack tip there exists a process zone, its enlarged diagram is as shown in Fig. 2, the origin of local coordinate locates at the crack tip. The coordinate of the crack surface is $\xi=a(t)-x$; and the coordinate ahead crack is $\xi_1=x-a(t)$. Within $0\leqslant\xi\leqslant\alpha$ is the process zone. In the craze band cohesive stress $\sigma_f(\xi, t)$ is acting on the crack surfaces. If σ_f equals to yield stress and is constant, this is the well-known Dugdale model.

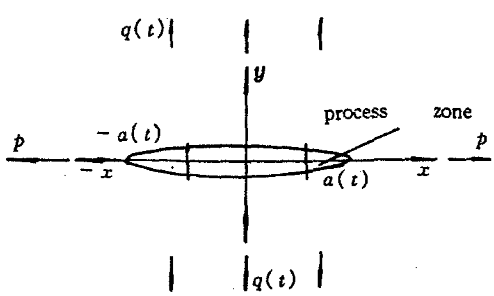

Fig. 1 Plane strain cracked medium and load configuration

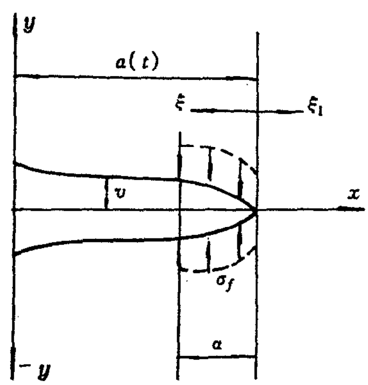

Fig. 2 Schematic drawing of fracture process zone

2.2 Constitutive relationships

Making use of the assumption of small deformation and creep incompressibility, the creep typed power-law nonlinear viscoelastic constitutive relation is expressed as:

$$\varepsilon_{ij}=\frac{3}{2}\int_0^t D(t-\tau)\frac{\partial}{\partial\tau}\left[\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\cdot S_{ij}\right]d\tau \tag{2.1}$$

where $n\geqslant 1$ is a material power-law constant. S_{ij} is deviatoric stress; σ_0 is a power-law parameter and has stress unit, say it can be yielding stress; $D(t-\tau)$ is creep compliance function (or hereditary integration kernel function) and can be determined with creep tests;

σ_e is equivalent stress:

$$\sigma_e=\sqrt{\frac{3}{2}S_{ij}S_{ij}} \tag{2.2}$$

where indexes i, j can be 1, 2, 3 and obey the summation convention of tensor operation.

2.3 Boundary conditions

- Loading condition: At remote boundary i. e., $x^2+y^2\to\infty$,

$$\sigma_{xx}=-T\leqslant 0 \tag{2.3a}$$

$$\sigma_{yy}=q(t)\geqslant 0 \tag{2.3b}$$

- Crack surface condition:

$$\sigma_{yy}=\sigma_{xy}=0, \qquad \text{when } y=0 \text{ and } |x|<a(t)-\alpha(t) \tag{2.3c}$$

$$\sigma_{yy}=\sigma_f(\xi,\ t),\ \sigma_{xy}=0,\ \text{when } y=0 \text{ and } a(t)-\alpha(t)\leqslant|x|\leqslant a(t) \tag{2.3d}$$

Equation (2.3a) is the uniform referential stress state. It should be pointed out that the load (2.3a) will cause a uniform deformation and the crack will be closed; this state is called referential state and will be denoted by the asterisk " * ". Loading (2.3b) is the Mode-I fracture load and (2.3d) is the cohesive stress in the process zone. It is (2.3b) and (2.3d) that constitute the perturbation upon the referential state.

2.4 Basic equations

The mechanical fields caused by T, $q(t)$ and $\sigma_f(\xi,\ t)$ are:

- Displacement field: $\bar{u}_i=u_i^*+u_i$ (2.4)
- Strain field: $\bar{\varepsilon}_{ij}=\varepsilon_{ij}^*+\varepsilon_{ij}$ (2.5)
- Strain ~ displacement relations: $\varepsilon_{ij}^*=\frac{1}{2}(u_{i,j}^*+u_{j,i}^*)$, and $\varepsilon_{ij}=\frac{1}{2}(u_{i,j}+u_{j,i})$ (2.6)
- Incompressibility condition: $\bar{\varepsilon}_{ii}=\varepsilon_{ii}^*=\varepsilon_{ii}=0$, and $\bar{u}_{i,i}=u_{i,i}^*=u_{i,i}=0$ (2.7)
- Stress field: $\bar{\sigma}_{ij}=\sigma_{ij}^*+\sigma_{ij}$ or $\bar{S}_{ij}=S_{ij}^*+S_{ij}$ (2.8)
- Equilibrium equation: $\sigma_{ij,j}^*=0$ and $\sigma_{ij,j}=0$ (2.9)

The equations of (2.3), (2.6), (2.7), and (2.9) plus equation (2.1) constitute the complete proposition of the boundary valued problem.

III. Perturbation Analysis

3.1 Establishment of referential state

Applying uniform compression in x direction $P=P(t)=-TH(t)$ at $t=0$, and then keeping it constant, $H(t)$ is Heaviside function. Crack is closed and the stress fields caused inside the body are:

$$\sigma^*(t)=\sigma^*H(t),\ S_{ij}^*(t)=\underline{S}_{ij}^*H(t) \text{ and } \sigma_e^*(t)=\underline{\sigma}_e^*H(t) \tag{3.1}$$

where the quantity with lower bar "$-$" is independent on time. Thereby the strain field is also uniform:

$$\varepsilon_{ij}^*(t)=\frac{3}{2}\int_0^t D(t-\tau)\frac{\partial}{\partial\tau}\left[\left(\frac{\sigma_e^*}{\sigma_0}\right)^{n-1}\cdot\underline{S}_{ij}^*H(\tau)\right]d\tau=\frac{3}{2}\left(\frac{\sigma_e^*}{\sigma_0}\right)^{n-1}\cdot\underline{S}_{ij}^*D(t) \tag{3.2}$$

Let

$$\varepsilon_e=\left[\frac{2}{3}\varepsilon_{ij}\varepsilon_{ij}\right]^{\frac{1}{2}},\quad \text{thus}\ \varepsilon_e^*(t)=D(t)\left[\frac{\sigma_e^*}{\sigma_0}\right]^{n-1}\sigma_e^* \tag{3.3}$$

So that expression of $\varepsilon^*_{ij}\varepsilon^*_{kl}/(\sigma^*_e)^2$ is independent on time.

For plane strain problems $\varepsilon^*_{zz}=0$, thus, $\sigma^*_{zz}=\frac{1}{2}(\sigma^*_{xx}+\sigma^*_{yy})$. So we have

$$\sigma^*_{xx}=-TH(t),\quad \sigma^*_{yy}=0,\ \text{and}\ \sigma^*_{zz}=\frac{T}{2}H(t) \tag{3.4}$$

Therefore

$$\sigma^*_e=\frac{\sqrt{3}}{2}TH(t)\qquad (T\geqslant 0) \tag{3.5}$$

The uniform strain field is as follows

$$\varepsilon^*_{yy}=-\varepsilon^*_{xx}=\frac{\sqrt{3}}{2}\varepsilon^*_e,\quad \varepsilon^*_{zz}=0,\ \text{and}\ \varepsilon^*_e=\sigma_0\left(\frac{\sqrt{3}}{2}\frac{T}{\sigma_0}\right)^n D(t) \tag{3.6}$$

$D(t)$ is creep compliance function.

3.2 The stress and strain increment fields of perturbation

Upon the uniform referential state caused by $P=-TH(t)$ superposing the perturbation forces σ_f and $q(t)$, the incremental stress σ_{ij} and strain ε_{ij} are produced. Referring to the increment plastic theory[3], the relation between stress increment and strain increment for the perturbation state was obtained:

$$\frac{3}{2}\left(\frac{\sigma^*_e}{\sigma_0}\right)^{n-1}\int_0^t D(t-\tau)\frac{\partial S_{ij}}{\partial\tau}d\tau=\Big[\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})-\frac{1}{3}\delta_{ij}\delta_{kl}$$
$$+\frac{2}{3}\frac{1-n}{n}\varepsilon^*_{ij}\varepsilon^*_{kl}\cdot(\varepsilon^*_e)^{-2}\Big]\varepsilon_{kl} \tag{3.7}$$

where δ_{mn} is Kronecker notation. This expression is similar to the relation of incremental stress versus incremental strain for power-law hardening plastic material given in Ref. [3]. The first two terms on the right side of equation (3.7) is elastic strain, whereas the third term is plastic strain. Performing tensor manipulation of equation (3.7) the constitutive relation for the perturbation state:

$$\varepsilon_{xx}=nc\int_0^t D(t-\tau)\frac{\partial S_{xx}}{\partial\tau}d\tau \tag{3.8a}$$

$$\varepsilon_{yy}=nc\int_0^t D(t-\tau)\frac{\partial S_{yy}}{\partial\tau}d\tau \tag{3.8b}$$

$$\varepsilon_{xy}=c\int_0^t D(t-\tau)\frac{\partial S_{xy}}{\partial\tau}d\tau \tag{3.8c}$$

where

$$c=\frac{3}{2}\left(\frac{\sigma^*_e}{\sigma_0}\right)^{n-1}=\frac{3}{2}\left[\frac{\sqrt{3}}{2}\frac{T}{\sigma_0}\right]^{n-1} \tag{3.9}$$

It can be seen that equation (3.8) is not isotropic viscoelastic constitutive relation, it is still quite difficult. For weak nonlinear material, $n \leqq 1$ and introducing notation $\beta = c\sqrt{n}$, equations (3.8a), (3.8b) and (3.8c) can be written as (3.10), which is a good approximation for some polymeric materials having weak nonlinearity.

$$\varepsilon_{ij} = \int_0^t \beta \times D(t-\tau) \frac{\partial S_{ij}}{\partial \tau} d\tau \tag{3.10}$$

Thus the nonlinear viscoelastic problem was reduced to pseudo-isotropic linear viscoelastic one; and the solving method depicted in Ref. [2] can be employed here.

3.3 Correspondence principle

According to the classical correspondence principle, the viscoelastic solution for the standing crack (fixed boundary problem) can be easily obtained from its corresponding elastic solution, and then through the Laplace inversely transformation the solution of the original problem can be obtained. For the growing crack problem, the thing is not so straightforward. In order to deal this problem, if the following three conditions are satisfied the extended correspondence principle proposed by Graham[4] can be used:

- The crack is growing monotonically, no closure occurs;
- Stress σ_{yy} is independent on material properties;
- The crack opening displacement can be decomposed as: $v = v_a(E, \nu) v_b(\xi_1, t)$.

The case dealt with here meets the above conditions perfectly. Therefore the extended correspondence principle was employed. In the following, the superscript letter R is used to indicate the solution of reference elastic problem. Under the action of $q(t)$ and $\sigma_f(\xi, t)$, the stress distribution of σ_{yy}^R for the corresponding elastic problem is as follows[2]:

$$\sigma_{yy}^R = \left\{ \frac{q\sqrt{\pi a}}{\sqrt{2\pi\xi_1}} - \frac{1}{\pi\sqrt{\xi_1}} \int_0^a \frac{\sqrt{\xi}\,\sigma_f(\xi)}{(\xi+\xi_1)} d\xi \right\} \cdot H(\xi_1) \tag{3.11}$$

where $H(\xi_1)$ is Heaviside function, $H(\xi_1)=1$ when $\xi_1 \geqslant 0$; and $H(\xi_1)=0$ when $\xi_1 < 0$ (see Fig. 2).

According to the condition of σ_{yy}^R being finite at crack tip, we have

$$q\sqrt{\pi a} = \sqrt{\frac{2}{\pi}} \int_0^a \frac{\sigma_f(\xi) d\xi}{\sqrt{\xi}} \tag{3.12}$$

Therefore (3.11) becomes to:

$$\sigma_{yy}^R = \frac{\sqrt{\xi_1}}{\pi} \int_0^a \frac{\sigma_f(\xi)}{\sqrt{\xi}\,(\xi+\xi_1)} d\xi \tag{3.13}$$

The crack surface displacement normal to the crack for the corresponding elastic problem is as:

$$v^R = H(\xi) \left(\frac{D_0 \beta}{2\pi} \right) \int_0^a \sigma_f(\xi') \left\{ 2\left(\frac{\xi}{\xi'}\right)^{\frac{1}{2}} - \ln\left[\left| \frac{\xi'^{\frac{1}{2}} + \xi^{\frac{1}{2}}}{\xi'^{\frac{1}{2}} - \xi^{\frac{1}{2}}} \right| \right] \right\} d\xi' \tag{3.14}$$

where $D_0 = D(t)|_{t=0}$ is the initial compliance of the material.

According to the correspondence principle, we have for the original problem,

$$\sigma_{yy} = \sigma_{yy}^R(\xi_1) H(t) \tag{3.15}$$

$$v(\xi,\ t)=\int_0^t D(t-\tau)\frac{\partial}{\partial\tau}\left(\frac{v^R}{D_0}\right)d\tau \tag{3.16}$$

For sake of concise, introducing the notation $I(\xi)$,

$$I(\xi)=\int_0^a \sigma_f(\xi')g(\xi,\ \xi')d\xi';\ g(\xi,\ \xi')=2\sqrt{\frac{\xi}{\xi'}}-\ln\left|\frac{\sqrt{\xi'}+\sqrt{\xi}}{\sqrt{\xi'}-\sqrt{\xi}}\right|;$$

Equation (3.16) can be written as:

$$v(\xi,\ t)=H(\xi)\times\frac{\beta}{2\pi}\int_0^t D(t-\tau)\frac{\partial}{\partial\tau}(I(\xi))d\tau \tag{3.17}$$

The last equation is similar to the modified superposition principle (MSP)[5, 6].

IV. Cracking Incubation Time and Subcritical Growth Rate

4.1 Local energy criterion

When the cracked viscoelastic body is loaded at time $t=0$, the crack does not grow immediately; there exists a incubation time t^*. In the time interval $0\leqslant t<t^*$, the crack is stationary; crack length is constant, whereas the crack opening displacement is enhancing. When the crack incubates ripe, i. e. the mechanical state meet certain critical condition, the crack begins to grow. This time is called incubation time t^*.

Here local energy criterion is used to predict t^*. It is assumed that during the cracking incubation time t^*, the cohesive stress σ_f resists crack opening and absorbs energy: $W=\int_0^a \frac{1}{2}\sigma_f v d\xi$. When the work W equals to the surface energy of the crack Γ, the crack begins to grow. Thus

$$\Gamma=\frac{W}{2a}=\frac{1}{a}\int_0^a v\sigma_f d\xi \tag{4.1}$$

where Γ is fracture surface energy which is material constant.

4.2 Cracking incubation time

Substituting (3.17) into (4.1), and introducing the following notation

$$J(t)=\int_0^a \sigma_f(\xi,t)I(\xi)d\xi \tag{4.2}$$

we have,

$$\Gamma=\frac{\beta}{4\pi a}\int_0^{t^*} D(t-\tau)\frac{\partial}{\partial\tau}[J(t)]d\tau \tag{4.3}$$

Thus, the cracking incubation time t^* was determined through the above equation. This criterion is similar to the classical fracture energy release rate criterion.

4.3 Crack slow propagation rate

Upon onset of cracking, it is assumed that the size of process zone α and the distribution and value of σ_f are invariable and moving forward as time. This is similar to the Dugdale model of the fracture process zone for the plane stress fracture. When the crack extends Δa the fracture energy $2\Delta a\Gamma$ is consumed, which can only be supplied by the applied force. At Δt, the work done by the load $q(t)$ acting at the remote boundary equals to $2q(t)\bar{v}^{\infty}\Delta t$, $\bar{v}^{\infty}$, is average displacement of the load boundary.

$$\Delta a\Gamma = q(t)\dot{v}^{\infty}\Delta t \tag{4.4}$$

Thus, we have

$$\dot{a} = \frac{q(t)\dot{v}^{\infty}}{\Gamma} \tag{4.5}$$

$\dot{a}$ is the crack slow propagation velocity.

V. Concluding Remarks

An analysis on the plane strain fracture problem of power-law nonlinear viscoelastic material was attempted. Creep deformation was assumed to be incompressible. In the vicinity of crack tip, (i. e. fracture process zone) cohesive stress was assumed to act upon crack surfaces and resist crack opening. Through perturbation approach the expressions of stress and displacement increments near crack tip were given. The local energy criterion has been proposed, the formulas predicting crack incubation time and subcritical crack growth rate were obtained. Owing to the large difficulty of the original problem, an approximation has to be resorted. The results are applicable to the weak nonlinear medium with $n \cong 1$. In addition, the cohesive stress σ_f acting in the process zone can be analyzed according to Dugdale model. To verify the validity of this model, experiment is necessary, and is complicated.

References

[1] W. G. Knauss, Delayed fracture-the Griffith problem for linearly viscoelastic materials, *Int. J. Fract.*, **6**, 1 (1970), 7～20.

[2] R. A. Schapery, A theory of crack initiation and growth in viscoelastic medium: I. theoretical development, *Inter. J. Fract.*, **11**, 1 (1975), 141～159.

[3] M. Y. He and J. W. Hutchinson, The penny-shaped crack and the plane strain crack in an infinite body of power-law materials, *Trans. J. Appl. Mech.*, **48**, 4 (1981), 830～840.

[4] G. A. C. Graham, The correspondence principle of linear viscoelastic theory for mixed mode boundary value problems involving time-dependent boundary regions, *Quart. Appl. Math.*, **26**, 2 (1968), 167～174.

[5] W. N. Findley, et al, *Creep and Relaxation of Nonlinear Viscoelastic Theory for Materials.*, North-Holland Pub. Co. (1976).

[6] A. C. Pipkin and T. G. Rogers, A nonlinear integral representation for viscoelastic behaviour, *J. Mech. Phy. Solids.*, **16**, 1 (1968), 59～72.