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# p Dissipative Operator 

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#### Abstract

In this paper the authors prove that the generalized positive $p$ selfadjoint $(\mathrm{GPpS})$ operators in Banach space satisfy the generalized Schwarz inequality, solve the maximal dissipative extension representation of $p$ dissipative operators in Banach space by using the inequality and introducing the generalized indefinite inner product (GIIP) space, and apply the result to a certain type of Schrödinger operator.


## 1. Introduction

This research studies a sort of $p$ dissipative operator in Banach space by means of the generalized semi-inner product (GSIP) space and the generalized indefinite inner product (GIIP) space. The research on its maximal dissipative extension representation on an infinite dimensional dynamic system in Banach space is of great importance. Moreover, this paper will give some applications in quantum mechanics and the Schrödinger operator. Now, research on the Schrödinger operator is one of the key problems to study soliton wave and quantum mechanics. On the basis of the above discussion the authors make further research on the behavior of the nonlinear Schrödinger equation and scattering of the corresponding particle collision.

The dissipative operator in Hilbert space comes from the Cauchy problem on the hyperbolic partial differential equation with $L^{2}$ measure. Maximal dissipative operators occur in many applications, for instance they are the infinitesimal generators of strongly continuous semigroups of a contraction operator. Now, with the intensive research on the infinite dimensional dynamic system and such problems as the soliton wave, the scattering of particle collisions in quantum mechanics, great attention was focused on the initial value problems of partial differential equations with the measure of Banach space. For example, the natural measure of the heat equation is the supreme of temperature; the measure of the diffusion equation is in $L^{1}$; the measure of dealing with the scattering of a particle collision is in $L^{p}(\mathrm{p}=2$ or $p \neq 2)$ (see [7, 12, 13, 16, 23, 29, 30]). It is well
known that the dissipative operator in Hilbert space is one with wider applications (see [12, 16, 19, 22, 29, 30, 42, 43, 44, 50]. Especially in [12] R. S. Phillips and G. Lumer researched the operator $L_{0}=A-S$ in Hilbert space, where $A$ is a skew-symmetric operator and $S$ is a positive operator. And there is a one to one correspondence between the maximal dissipative expansion on $L_{0}$ and the maximal negative subspace in indefinite inner product space. In this paper we extend the result to the Banach space by using the GSIP space and introducing the GIIP space. Because Banach space does not have the bilinear character as the inner product, this makes it difficult to study the operators in Banach space.

The research on GSIP and operator theory in GSIP space originated in Lumer's semiinner product (SIP) in Banach space (see [31]) and Nath's GSIP space in Banach space. Many researchers (see [1, 4, 9, 31-35, 37-40, 51]) studied the geometric properties of GSIP space or SIP space which include orthogonal projection, isometry and the Riesz Representation Theorem, etc. And also researchers (see [2, 3, 4, 8, 13, 15, 24, $27,28,32,33,38,39,41])$ studied the operator theory in SIP or GSIP space: the adjoint operator, adjoint Abelian operator, generalized $p$ selfadjoint operator, generalized $p$ normal operator, etc. They also established the function models of adjoint Abelian operators in $L^{p}(\Omega)$ and $C(K)$ (see [15]). These researches added more to the theory of the SIP and GSIP spaces. G. Lumer and R. S. Phillips [13] researched the characters of the dissipative operator in Banach space and paper [10] obtained the properties of the $J$ dissipative operator in indefinite inner product space. Based on these results, the authors of the paper [7, 8, 25] obtained some important results by dealing with the Schrödinger operator on Banach space $L^{p}[0,2 \pi]$. It is important whether the results obtained in [7] can be extended to general Banach space. What we will do in our paper is to further extend in general the Banach space. The most difficult point is to determine if the generalized Schwarz inequality holds for the generalized positive $p$ selfadjoint operator (GPpS operator).

The research on indefinite inner product space comes from quantum field theory. Until now, the operator theory in the space is fruitful, for example $[9,10,11,14,42,43$, 44]. It is of real importance in physics to solve the difficulty of divergence by establishing scattering theory with an indefinite inner product space. However, for general Banach space which can't be changed into Hilbert space we may not use the indefinite inner product to deal with the scattering of particle the collision with the measure $L^{p}(p \neq 2)$, as there is a bilinear Hermite function in the indefinite inner product space. Therefore, we introduce GIIP space into the study of the $p$ dissipative operator in Banach space. This extension is of particular significance not only in mathematics but also in a real physical system, which we may find when the maximal dissipative extension of a sort of Schrödinger operator is dealt with in the paper.

The research on the linear operator with character of chaos and the dynamic behavior of the linear operators in infinite dimension Banach space gives the foundation to look into the complexity and dynamic behavior of infinite dimensional dynamic systems with the metric of the Banach space (see [25, 26, 46, 47, 48]). Based on it we develop the maximal dissipative extension of the $p$ dissipative operator in Banach space and we apply the result to a certain kind of Schrödinger operator.

The paper includes six sections. In Sect. 1 we give the introduction. In the next section we set up the generalized Schwarz inequality of the generalized positive $p$ selfadjoint operator in GSIP space. In Sect. 3 the generalized indefinite inner product space and generalized Krein space are introduced and some properties are obtained. In Sect. 4 we construct the natural boundary space of $p$ dissipative operator in Banach space. In Sect. 5 we give the maximal dissipative extension representation of a sort of $p$ dissipative
operator in Banach space by the natural boundary space. Finally a kind of Schrödinger operator is studied by using the results obtained in Sects. 4 and 5 and we solve the maximal dissipative extension representation of the operator.

The main results in the paper are described as follows.
Theorem 2.1. Let $X$ be a GSIP space. $T \in L(X)$, if $T$ is a generalized positive $p$ selfadjoint operator in $X$, then the generalized Schwarz inequality of $T$ is right.

Theorem 5.1. Let $L_{0}=A-S$, where $A$ is $p$ skewsymmetric, $\operatorname{Re}[A u, u]_{p}=0$ and $S$ is a reversible generalized positive p selfadjoint operator. Suppose that the maximal dissipative extension of $L_{0}$ is $L$. Then, there is an one to one correspondence between the maximal dissipative extension $L$ of $L_{0}$ and the maximal negative subspace $\tilde{N}$ of GIIP space $\tilde{H}$, and

$$
\begin{gathered}
L u=L_{1} u+S^{1 / 2} \varphi(\hat{u}), \quad L_{1}=A^{*}-S \\
D(L)=\left\{u \in D\left(L_{1}\right) \mid \hat{u} \in \widehat{N}, \widehat{N} \text { is the projection of } \tilde{N} \text { from } \widetilde{H} \text { to } \widehat{H}\right\}
\end{gathered}
$$

Theorem 6.3. In $X=L^{p^{\prime}}[0,2 \pi], 1<p^{\prime}<\infty$, suppose the Schrödinger operator

$$
\begin{gathered}
L_{0}=i f^{\prime \prime}-f \\
D\left(L_{0}\right)=\left\{f \mid f, f^{\prime \prime} \in X, \quad f(0)=f(2 \pi), \quad f^{\prime}(0)=f^{\prime}(2 \pi)\right\} .
\end{gathered}
$$

If the maximal dissipative extension of $L_{0}$ is $L$, then there is an one to one correspondence between the operator $L$ and the maximal negative subspace $\tilde{N}$ of $\widetilde{H}$ and

$$
\begin{gathered}
L u=i u^{\prime \prime}-u+u^{\prime}(0) u(0) f \\
D(L)=\left\{u\left|u, u^{\prime}, u^{\prime \prime} \in X, \alpha u^{\prime}(2 \pi) u(2 \pi)+\beta u^{\prime}(0) u(0)=0,|\beta| \leq|\alpha|\right\}\right.
\end{gathered}
$$

where $f \in X$ satisfy the inequality

$$
-2(\beta / \alpha+1) u^{\prime}(0) u(0)+\left|u^{\prime}(0) u(0)\right|^{p}\|f\|^{p} \leq 0
$$

## 2. GSIP Space

In order to carry over Hilbert space arguments to the theory of Banach space, Lumer [31] introduced the concept of SIP space which has a more general axiom system than that of Hilbert space. Furthermore, Nath [1] introduced the GSIP space. From [1], a complex Banach space $X$ is called a complex generalized semi-inner product (GSIP) space if corresponding to an arbitrary pair of elements $x, y \in X$, there exists a complex number $[x, y]_{p}$ in $X \times X$ which satisfies the following properties for any $x, y, z \in X$, and $\lambda \in C$ ( $C$ denotes the complex field):
(1) $[\alpha x+\beta y, z]_{p}=\alpha[x, z]_{p}+\beta[y, z]_{p}$,
(2) $[x, x]_{p}>0$, for $x \neq 0 ; x=0$ iff $[x, x]_{p}=0$,
(3) $\left|[x, y]_{p}\right| \leq[x, x]_{p}^{1 / p}[y, y]_{p}^{1 / q}, \quad 1<p, q<+\infty, \quad 1 / p+1 / q=1$.

A GSIP $[x, y]_{p}, 1<p<+\infty$, generates the norm $\|\cdot\|$ that for $x \in X,\|x\|=[x, x]_{p}^{1 / p}$. If $p=2$, the GSIP space is the SIP space. Then we denote the SIP to $[\cdot, \cdot]$.

From [2, 3], if $X$ is a complex Banach space with norm $\|\cdot\|$, for each $p \in(1,+\infty)$, then there exists a GSIP $[x, y]_{p}$ which generates the norm $\|\cdot\|$, and in this case we have

$$
\begin{aligned}
{[x, \lambda y]_{p} } & =|\lambda|^{p-2} \bar{\lambda}[x, y]_{p}, \text { for any } x, y \in X, \lambda \in C \\
{[t x, y]_{p} } & =\left[x,|t|^{(2-p) /(p-1)} \bar{t} y\right]_{p}, \text { for } t \in C
\end{aligned}
$$

Moreover if $p \neq p^{\prime}, p, p^{\prime} \in(1,+\infty)$ and $[\cdot, \cdot]_{p},[\cdot, \cdot]_{p^{\prime}}$ are respectively the corresponding GSIP which generalized the norm $\|\cdot\|$ in Banach space $X$, then for all $x, y \in X, y \neq 0$ :

$$
[x, y]_{p}=\|y\|^{p-p^{\prime}}[x, y]_{p^{\prime}}
$$

Suppose $p \in(1,+\infty), T \in L(X)(L(X)$ denotes all bounded linear operators). Papers [2,3] proved that if $X$ is a smooth strictly convex and reflexive Banach space, then there is a unique GSIP $[\cdot, \cdot]_{p}$ which generates the norm and for each $f \in X^{*}$ there is a unique $y \in X$ such that $f(x)=[x, y]_{p}$ for all $x \in X$, and in this case we have $\|f\|=\|y\|^{p-1}$.

From [3] we have that for each $f \in X^{*}$ and $p \in(1,+\infty)$ there is a unique $y^{\prime} \in X$ such that $f(x)=\left[x, y^{\prime}\right]_{p}$, for all $x \in X$, where GSIP $[\cdot, \cdot]_{p}$ generates the norm. Throughout the paper, we shall always assume that $X$ is a Banach space which is smooth strictly convex and reflexive.

Definition 2.1 (see $[2,3])$. Suppose $p \in(1,+\infty)$, $T \in L(X)$, and $y \in X$, by $[T x, y]_{p}=$ $\left[x, y^{*}\right]_{p}$, we obtain $T_{p}^{*}$ satisfying $T_{p}^{*} y=y^{*}$, defines a mapping which maps $X$ into $X$, $T_{p}^{*}$ is called a generalized $p$ adjoint operator. If $T_{p}^{*}=T, T$ is called a generalized $p$ selfadjoint operator. If $p=2$, the generalized 2 selfadjoint operator also is called generalized selfadjoint operator. If $T$ satisfies $[T x, x]_{p} \geq 0, \forall x \in X, T$ is called a generalized positive operator. If for $T,[T x, x]_{p}$ is real, call it generalized Hermite operator.

Of course the generalized adjoint operator and the generalized $p$ selfadjoint operator depend on $p$. Generalized positive operators under GSIP $[\cdot, \cdot]_{p^{\prime}}, 1<p^{\prime}<\infty$, are generalized positive operators under GSIP, $[\cdot, \cdot]_{p}, p \neq p^{\prime}, 1<p<\infty$. If $T \in L(X)$, is both a generalized $p$ selfadjoint operator and generalized positive operator, then we call it generalized positive $p$ selfadjoint operator (GPpS operator).
Example 2.1. There exists an operator, which is a generalized positive $p$ selfadjoint $(\mathrm{GPpS})$ operator in Banach space, but isn't both a generalized selfadjoint operator in SIP space and a selfadjoint operator in Hilbert space.

Suppose $X=l^{p}, 1<p<\infty, p \neq 2$. Define the unique GSIP $[\cdot, \cdot]_{p}$ in $X$ following that:

$$
[x, y]_{p}=\sum_{i=1}^{\infty} x_{i}\left|y_{i}\right|^{p-2} \overline{y_{i},} \quad \text { where } x=\left\{x_{i}\right\}, \quad y=\left\{y_{i}\right\} \in l^{p}
$$

Define the operator $T: l^{p} \rightarrow l^{p}$, such that $T\left\{x_{i}\right\}=\left\{x_{i}^{\prime}\right\}$, where $x_{1}^{\prime}=x_{1}, x_{i}^{\prime}=0, i \neq 1$. Then $[T x, x]_{p}=\left|x_{1}\right|^{p} \geq 0 . T$ is a generalized positive operator. Since

$$
[T x, y]_{p}=x_{1} \overline{y_{1}}\left|y_{1}\right|^{p-2}=[x, T y]_{p}
$$

we also have that $T$ is a generalized $p$ selfadjoint operator. Hence $T$ is a GPpS operator in $X$. Notice that the unique $\operatorname{SIP}[\cdot, \cdot]$ in $X$ is

$$
[x, y]=\|y\|^{2-p} \sum_{i=1}^{\infty} x_{i}\left|y_{i}\right|^{p-2} \overline{y_{i}}, \quad \text { where } x=\left\{x_{i}\right\}, \quad y=\left\{y_{i}\right\} \in X
$$

Then

$$
[T x, y]=x_{1} \overline{y_{1}}\left|y_{1}\right|^{p-2} /\|y\|^{p-2}, \quad[x, T y]=x_{1} \overline{y_{1}}\left|y_{1}\right|^{p-2} /\|T y\|^{p-2}
$$

where $x=\left\{x_{i}\right\}, y=\left\{y_{i}\right\} \in X$. Hence $T$ isn't a generalized selfadjoint operator in SIP space. Since $l^{p}, p \neq 2$ isn't Hilbert space, $T$ also isn't a selfadjoint operator in Hilbert space.

Example 2.2. There exists an operator $T$ in Banach space such that $T$ is a generalized $p$ selfadjoint operator but isn't both a generalized Hermite operator and a generalized positive operator. Hence the generalized $p$ selfadjoint operator in Banach space differs from the selfadjoint operator in Hilbert space.

Let $X=Y \oplus Y, \oplus$ is $l_{3}$-sum, $Y$ is a two dimensional Hilbert space, inner product $(\cdot, \cdot)$ in $Y$. Define the GSIP following that, for $1<p<\infty$,

$$
\left[\left\langle y_{1}, y_{2}\right\rangle,\left\langle y_{1}^{\prime}, y_{2}^{\prime}\right\rangle\right]_{p}=\left\|\left\langle y_{1}^{\prime}, y_{2}^{\prime}\right\rangle\right\|^{p-3}\left\{\left(y_{1}, y_{1}^{\prime}\right)\left\|y_{1}^{\prime}\right\|+\left(y_{2}, y_{2}^{\prime}\right)\left\|y_{2}^{\prime}\right\|\right\}
$$

Define the operator $T=\left[\begin{array}{cccc}0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0\end{array}\right]$ in $X$. We easily prove that

$$
[T x, y]_{p}=[x, T y]_{p}, \quad x, y \in X
$$

Hence $T$ is a generalized $p$ selfadjoint operator. But, for $x=\left\langle y_{1}, y_{2}\right\rangle \in X, y_{1}=$ $\ll x_{11}, x_{12} \gg, y_{2}=\ll x_{21}, x_{22} \gg \in Y$, has

$$
\begin{aligned}
{[T x, x]_{p}=} & \left\|\left\langle y_{1}, y_{2}\right\rangle\right\|^{p-3}\left\{\left(-i x_{21} \bar{x}_{11}+i x_{22} \bar{x}_{12}\right)\left\|y_{1}\right\|\right. \\
& \left.+\left(-i x_{11} \bar{x}_{21}+i x_{12} \bar{x}_{22}\right)\left\|y_{2}\right\|\right\} .
\end{aligned}
$$

This is a complex number. So $T$ is not a generalized Hermite operator and generalized positive operator.

Example 2.3. There exists an operator $T$ such that $T$ is a generalized Hermitz operator but isn't a generalized $p$ selfadjoint operator in Banach space.

Let $X=l^{p^{\prime}}, 1<p^{\prime}<, \infty, p^{\prime} \neq 2$. There exists a unique GSIP $[\cdot, \cdot]_{p}, 1<p<\infty$, $p \neq p^{\prime}$, following that

$$
[x, y]_{p}=\|y\|^{p-p^{\prime}} \sum_{i=1}^{\infty} x_{i}\left|y_{i}\right|^{p^{\prime}-2} \bar{y}_{i}, \quad x=\left\{x_{i}\right\}, y=\left\{y_{i}\right\} \in X
$$

The same operator $T$ is defined as in Example 2.1. Then $[T x, x]_{p}$ is a real number and $T$ is a generalized Hermite operator. As $p \neq p^{\prime}$, we easily prove that $[T x, y]_{p} \neq[x, T y]_{p}$. Hence $T$ isn't a generalized $p$ selfadjoint operator.

Definition 2.2. Let $T \in L(X)$, and satisfy the following the inequality:

$$
\left|[T x, y]_{p}\right| \leq\left|[T x, x]_{p}\right|^{1 / p}\left|[T y, y]_{p}\right|^{1 / q}, \quad 1<p, \quad q<\infty, \quad 1 / p+1 / q=1
$$

then we say that $T$ satisfies the generalized Schwarz inequality.

Proposition 2.1 (see [4] Lemma 1). If $T \in L(X), T^{n}=I$ ( $I$ is a unit operator in $X$ ) $n$ is a integral. Then $T$ is a scalar type operator and $T=\sum_{i=1}^{n} \omega_{i} E_{i}$, where $E_{i} E_{j}=$ $\delta_{i j} E_{i}, i=1,2, \cdots, n, \sum_{i=1}^{n} E_{i}=I, \omega_{i}$ is a root of unity, $i=1,2, \ldots, n$.

Because GSIP isn't bilinear in Banach space, the proof of the generalized Schwarz inequality of GPpS operator of Banach space is more difficult than for the selfadjoint operator of Hilbert space (see [6]).

Theorem 2.1. If $T \in L(X)$ and $T$ is a GPpS operator, then $T$ satisfies the generalized Schwarz inequality.

Proof. We prove the theorem in four steps.
(1) Step one. Let $1 / 2 \leq\|T\|<1, N(T)=\{0\}$, where $N(T)$ is the kernel space of $T$.

First we prove that there exists $A \in L(X)$ such that $A^{2}=T$ and $A$ is a GPpS operator.

If $x \in X,\|x\|=[x, x]_{p}^{1 / p}=1$, then $[(I-T) x, x]_{p}=1-[T x, x]_{p} \geq 0$ by using $\|T\|<1$. Then $I-T$ is a generalized positive operator.

As $(I-T)^{*} T=(T(I-T))^{*}=((I-T) T)^{*}=T(I-T)^{*}$, from [3] then $|\sigma(I-T)|=\|I-T\|$, where $\sigma(I-T)$ is the spectrum of $I-T$. Because of $|\sigma(I-T)| \leq$ $|W(I-T)| \leq\|I-T\|$, where the set of numbers $W(I-T)=\left\{[(I-T) x, x]_{p}\right.$ : $\|x\|=1\}$ is called the numerical range of the operator $I-T$, and $|\sigma(I-T)|$ denote the spectral radius, $|W(I-T)|=\sup \{|\lambda|, \lambda \in W(I-T)\}$ is called the numerical range's radius respectively, then $|\sigma(I-T)|=|W(I-T)|=\|I-T\|$ (see [3]). As $\sup _{\|x\|=1}[(I-T) x, x]_{p}=1-\inf _{\|x\|=1}[T x, x]_{p} \leq 1 / 2$, then $\|I-T\| \leq 1 / 2$.

Let $A_{0}=I, 2 A_{n+1}=T+2 A_{n}-A_{n}^{2}$ or $A_{n+1}=\left(T-A_{n}^{2}\right) / 2+A_{n}$. We can easily prove that $\left\{A_{n}\right\}$ is the sequence of bounded linear operators. As

$$
\begin{gathered}
I-A_{0}=0,\left\|I-A_{1}\right\|=\|I-T\| / 2 \leq 1 / 4 \\
\left\|I-A_{2}\right\| \leq\left\|(I-T)+\left(I-A_{1}\right)^{2}\right\| / 2 \leq\left(1 / 2+(1 / 4)^{2}\right) / 2 \leq 1 / 2, \cdots, \\
\left\|I-A_{n}\right\| \leq 1 / 2
\end{gathered}
$$

then $\left\|\left(I-A_{n+1}\right)+\left(I-A_{n}\right)\right\| \leq 1$. Hence

$$
\begin{aligned}
\left\|A_{n+1}-A_{n}\right\| & \leq\left\|\left(I-A_{n}\right)+\left(I-A_{n-1}\right)\right\|\left\|A_{n}-A_{n-1}\right\| / 2 \\
\leq\left\|A_{n}-A_{n-1}\right\| / 2 & \leq \cdots \leq\left\|A_{1}-A_{0}\right\| / 2^{n} \leq 1 / 2^{n+2} \rightarrow 0, \text { as } n \rightarrow \infty
\end{aligned}
$$

It follows that $\left\{A_{n}\right\}$ is a Cauchy sequence. Hence there exists $A \in L(X)$ such that $\left\{A_{n}\right\}$ converge to $A$ by using the completion of $L(X)$. Therefore $A=\left(T-A^{2}\right) / 2+A, T=A^{2}$.

Next, we prove that A is a generalized positive operator. As $\left\|I-A_{n}\right\| \leq 1 / 2$ it follows that $\|I-A\| \leq 1 / 2$. Because $T$ is a GPpS operator, then $\left[T^{i} x, \bar{x}\right]_{p}, i \in N$ is a real number. From the construction of $A_{n}$, we have $\left[A_{n} x, x\right]_{p}$ is a real number. Hence $[A x, x]_{p}$ is a real number. As $\|x\|=1,-1 / 2 \leq-[A x, x]_{p}+[x, x]_{p} \leq$ $1 / 2$, then $[A x, x]_{p}=[x, x]_{p}-\left\{[x, x]_{p}-[A x, x]_{p}\right\} \geq 0$. If $x \in X,[A x, x]_{p}=$ $\|x\|^{p-1}\left[A \frac{x}{\|x\|}, \frac{x}{\|x\|}\right]_{p} \geq 0$. Then $A$ is a generalized positive operator.

Third, we prove that $A$ is a generalized $p$ selfadjoint operator.

As $T=A^{2}, T=T^{*}$, then $\left(A^{*}\right)^{2}=A^{2}$. From the hypothesis $N(T)=\{0\}$ of the first step, we have $N(A)=N\left(A^{*}\right)=\{0\}$ and there exists the inverse operator of $A$ and $T$ exchanges to $A$ and $A^{*}$.

Because of $A_{0}=I, A_{1} A^{*}=\frac{1}{2}(I+T) A^{*}=A^{*} A_{1}, A_{2} A^{*}=\left\{\frac{1}{2}\left(T-A_{1}^{2}\right)+A_{1}\right\} A^{*}=$ $A^{*} A_{2}, \cdots, A_{n} A^{*}=A^{*} A_{n}, \cdots, n \in N$, then $A A^{*}=A^{*} A$. We have $A^{*}=A^{-1} A^{*} A$ , $A^{* 2}=A^{2}=A^{*} A^{-1} A^{*} A,\left(A-A^{*} A^{-1} A^{*}\right) A=0$ or $\left(A-A^{*} A^{-1} A^{*}\right) T=0$. Hence $T\left(A-A^{*} A^{-1} A^{*}\right)=0$. From $N(A)=\{0\}$ and $A^{2}=A^{* 2}$, we have $\left(A^{-1} A^{*}\right)^{2}=I$.
Using Proposition 2.1, then

$$
A^{-1} A^{*}=E_{1}-E_{2}, \text { where } E_{i} E_{j}=\delta_{i j} E_{i}, \quad i, j=1,2, \quad E_{1}+E_{2}=I
$$

Thus

$$
\begin{aligned}
A^{*} & =A\left(E_{1}-E_{2}\right), \quad A^{*} E_{2}=-A E_{2}, \\
T E_{2} x & =A^{* 2} E_{2} x=A^{*}\left(-A E_{2}\right) x=-A A^{*} E_{2} x 0 \leq\left[T E_{2} x, E_{2} x\right]_{p} \\
& =-\left[A A^{*} E_{2} x, E_{2} x\right]_{p}=-\left[A^{*} E_{2} x, A^{*} E_{2} x\right]_{p} \leq 0,
\end{aligned}
$$

for arbitrary $x \in X$.
$\left[A^{*} E_{2} x\right]=0$, for arbitrary $x \in X . A^{*} E_{2} x=0$. Since $N(A)=N\left(A^{*}\right)=\{0\}$, we have $E_{2} x=0$, for arbitrary $x \in X$. Thus $E_{2}=0 . A^{-1} A^{*}=E_{1}=I . A=A^{*}$. Hence $A$ is a GPpS operator. Then

$$
\begin{aligned}
\left|[T x, y]_{p}\right| & =\left|\left[A^{2} x, y\right]_{p}\right|=\left|[A x, A y]_{p}\right| \leq|[A x, A x]|^{1 / p}[A y, A y]_{p}^{1 / q} \\
& =\left|[T x, x]_{p}\right|^{1 / p}\left|[T y, y]_{p}\right|^{1 / q}, \text { for arbitrary } x, y \in X, \quad 1 / p+1 / q=1 .
\end{aligned}
$$

(2) Step two. Let $1 / 2 \leq\|T\|<1, N(T) \neq\{0\}$.

Now we denote the quotient space $Y=X / N(T)$.
According to [2] Theorem 2.22, for the generalized $p$ selfadjoint operator $T, X=$ $\overline{N(T) \oplus R(T)}=N(T) \oplus \overline{R(T)}$, and for $\forall x \in N(T), \forall y \in \overline{R(T)}$, we have $[y, x]_{p}=0$. Hence the space $Y$ exists.

Let $\bar{x} \in Y$, then $\bar{x}=x_{0}+x_{1}$, where $x_{1} \in N(T)$.If $x_{o} \in N(T)$, we have $\bar{x} \in N(T)$ and $\bar{x}$ is zero element of $Y$; if $x_{0} \notin N(T)$, then $T \bar{x}=T x_{0} \neq 0$ and the kernel space of $T$ on $Y$ is $\{0\}$.

According to the projection properties of GSIP space in [5], there exist the projection operator $P: X \rightarrow N(T)$ and $Q: X \rightarrow \overline{R(T)}$.

Define GSIP in $Y$ the following:

$$
[\bar{x}, \bar{y}]_{Y}=[Q x, Q y]_{p}, \forall \bar{x}, \bar{y} \in Y .
$$

As $Q$ is a linear operator and $[\cdot, \cdot]_{p}$ is a GSIP, the definition of $[\cdot, \cdot]_{Y}$ has the significance and it is easy to prove that $[\cdot, \cdot]_{Y}$ is a GSIP in $Y$. As $P T x \in N(T)$, then $[P T x, Q y]_{p}=0$, we have

$$
\begin{aligned}
& {[Q T x, Q y]_{p}=[Q T x+P T x, Q y]_{p}=[T x, Q y]_{p}} \\
& \quad=[T(Q x+P x), Q y]_{p}=[T Q x, Q y]_{p},
\end{aligned}
$$

for $x, y \in X$. Then

$$
\begin{equation*}
\left[(Q T-T Q) x_{2}, z\right]_{p}=0, \forall x_{2}, z \in \overline{R(T)} \tag{2.1}
\end{equation*}
$$

According to the Riesz Representation Theorem in GSIP space in [2-5] and the formula (2.1), it follows that

$$
(Q T-T Q) x_{2}=0, \forall x_{2} \in \overline{R(T)} \quad \text { or } Q T x_{2}=T Q x_{2}, \quad \forall x_{2} \in \overline{R(T)} .
$$

As $x_{1} \in N(T)$, then $T x_{1}=Q x_{1}=0,(Q T-T Q) x_{1}=0 . X=N(T) \oplus \overline{R(T)}$ for the generalized $p$ selfadjoint operator $T([2])$. When $x \in X, x=x_{1}+x_{2}, x_{1} \in N(T), x_{2} \in$ $\overline{R(T)}$, then

$$
Q T x=Q T x_{1}+Q T x_{2}=T Q x_{1}+T Q x_{2}=T Q x .
$$

So

$$
[T \bar{x}, \bar{y}]_{Y}=[Q T x, Q y]_{p}=[Q x, T Q y]_{p}=[Q x, Q T y]_{p}=[\bar{x}, T \bar{y}]_{Y} .
$$

As a result $T$ is a generalized $p$ selfadjoint operator in $Y$. Because $[T \bar{x}, \bar{x}]_{Y}=$ $[T Q x, Q x]_{p}=0, T$ is a generalized positive operator in $Y$. From the condition $1 / 2 \leq\|T\|<1$, we can obtain that $T$ satisfies the generalized Schwarz inequality in the space $\left(Y,[\cdot, \cdot]_{Y}\right)$ by Step (1):

$$
\left|[T \bar{x}, \bar{y}]_{Y}\right| \leq\left|[T \bar{x}, \bar{x}]_{Y}\right|^{1 / p}\left|[T \bar{y}, \bar{y}]_{Y}\right|^{1 / q} .
$$

Thus, we have the following:

$$
\begin{aligned}
{[T \bar{x}, \bar{y}]_{Y} } & =[Q T x, Q y]_{p}=[T x, Q y]_{p}=[x, T Q y]_{p} \\
& =[x, T Q y+T P y]_{p}=[x, T y]_{p}=[T x, y]_{p}
\end{aligned}
$$

for $\forall x \in \bar{x}, y \in \bar{y}, \bar{x}, \bar{y} \in Y$,

$$
\begin{aligned}
\left|[T x, y]_{p}\right| & =\left|[T \bar{x}, \bar{y}]_{Y}\right| \leq\left|[T \bar{x}, \bar{x}]_{Y}\right|^{1 / p}\left|[T \bar{y}, \bar{y}]_{Y}\right|^{1 / q} \\
& =\left|[T x, x]_{p}\right|^{1 / p}\left|[T y, y]_{p}\right|^{1 / q} .
\end{aligned}
$$

(3) Step three. Let $0<\|T\|<1 / 2$.

Suppose $K=\alpha T, \alpha>0$. Choose $\alpha$ such that $1 / 2 \leq\|K\| \leq 1$. From Step (1), there is the generalized positive operator $A, K=A^{2}$. By using $[\alpha T x, y]_{p}=\left[x, \alpha^{1 /(p-1)} T y\right]_{p}$, we have

$$
\begin{equation*}
\alpha T=K=A^{2},(\alpha T)^{*}=K^{*}=\alpha^{1 /(p-1)} T . \tag{2.2}
\end{equation*}
$$

Then

$$
\left(A^{*}\right)^{2}=K^{*}=\alpha^{(2-p) /(p-1)} A^{2} \text { or } A^{2}=\alpha^{(p-2) /(p-1)}\left(A^{*}\right)^{2} .
$$

Analogously to the discussion in Steps (1) and (2), when $N(T)=\{0\}$ then we have

$$
\alpha^{(p-2) / 2(p-1)} A^{-1} A^{*}=E_{1}-E_{2} .
$$

As in the discussion in Step (1), we have $E_{2}=0$. Then $\alpha^{(p-2) / 2(p-1)} A^{*}=A$. Thus

$$
\begin{aligned}
\left|[T x, y]_{p}\right| & =\left|\alpha^{-1}[\alpha T x, y]_{p}\right|=\left|\alpha^{-1}[K x, y]_{p}\right| \\
& =\alpha^{-1}\left|\left[A^{2} x, y\right]_{p}\right|=\alpha^{-1}\left|\left[A x, A^{*} y\right]_{p}\right| \\
& =\alpha^{-1}\left|\left[A x, \alpha^{(p-2) / 2(p-1)} A y\right]_{p}\right|=\alpha^{-1}\left(\alpha^{(p-2) / 2(p-1)}\right)^{p-2}\left|\left[A x, A^{*} y\right]_{p}\right| \\
& \leq \alpha^{-1}\left(\alpha^{(p-2) / 2}\right)^{1 / p}\left(\alpha^{(p-2) / 2}\right)^{1 / q}\left|[A x, A x]_{p}\right|^{1 / p}\left|[A y, A y]_{p}\right|^{1 / q} \\
& =\alpha^{-1}\left|\left[A x, A^{*} x\right]_{p}\right|^{1 / p}\left|\left[A y, A^{*} y\right]_{p}\right|^{1 / q} \\
& =\alpha^{-1}\left|\left[A^{2} x, x\right]_{p}\right|^{1 / p}\left|\left[A^{2} y, y\right]_{p}\right|^{1 / q} \\
& =\alpha^{-1}\left|[K x, x]_{p}\right|^{1 / p}\left|[K y, y]_{p}\right|^{1 / q}=\alpha^{-1}\left|[\alpha T x, x]_{p}\right|^{1 / p}\left|[\alpha T y, y]_{p}\right|^{1 / q} \\
& =\left|[T x, x]_{p}\right|^{1 / p}\left|[T y, y]_{p}\right|^{1 / q} .
\end{aligned}
$$

When $N(T) \neq\{0\}$, the same result can be obtained by the same method as in Step (2). Hence the generalized Schwarz inequality of $T$ is satisfied.
(4) Step four. Let $\|T\| \geq 1$.

We suppose $K=\alpha T, \alpha>0$, choosing $\alpha$ such that $0<\|K\|<1$. By using the following formula:

$$
(I-\alpha T)^{*}(\alpha T)=(\alpha T)(I-\alpha T)^{*}
$$

and similarly to Steps (1), (2) and (3) we easily prove that $T$ satisfies the generalized Schwarz inequality.

From Step (1)-(4), we prove that the GPpS operator satisfies the generalized Schwarz inequality.

Corollary 2.1. When $T$ is a GPpS operator and $T^{*}=\alpha T, \alpha>0$, then the generalized Schwarz inequality of $T$ is satisfied.

Theorem 2.2. When $T$ is a GPpS operator, then there exists a GPpS operator $P$ such that $T=P^{2}$ and $P$ is unique. $P$ is called a positive square root of $T$.

Proof. When $1 / 2 \leq\|T\| \leq 1$, by the result of Steps (1), (2) in Theorem 2.1, there exists a GPpS operator such that $P^{2}=T$.

When $0<\|T\|<1 / 2$, suppose $K=(2\|T\|)^{-1} T$, then $\|K\|=1 / 2$. By Steps (1), (2) in Theorem 2.1, there exists the generalized positive operator $A$, such that $K=A^{2}$, but $A$ is not a generalized $p$ selfadjoint operator. Suppose $P=\sqrt{2\|T\|} A$. Because

$$
\begin{aligned}
{[P x, y]_{p} } & =[\sqrt{2\|T\|} A x, y]_{p}=\left[\sqrt{2\|T\|} x, A^{*} y\right]_{p}=\left[x,(2\|T\|)^{1 / 2(p-1)} A^{*} y\right]_{p} \\
& =\left[x,(2\|T\|)^{1 / 2(p-1)}(2\|T\|)^{(p-2) / 2(p-1)} A y\right]_{p}=[x, \sqrt{2\|T\|} A y]_{p} \\
& =[x, P y]_{p} .
\end{aligned}
$$

Thus $P$ is a GPpS operator and $T=P^{2}$.
When $\|T\|>1$, the same result for $P$ is obtained by similar reasoning.
Now we prove $P$ is unique.

If $P, Q$ are the GPpS operators and $T=P^{2}=Q^{2}$, we prove $P=Q$.
By the above discussion of this theorem, for the GPpS operators $P, Q$, there exist respectively the GPpS operators $P^{\prime}$ and $Q^{\prime}$ such that $P^{\prime 2}=P, Q^{\prime 2}=Q$. Let $y=(P-Q) x$, then

$$
\begin{aligned}
& \left\|P^{\prime} y\right\|^{p}+\left\|Q^{\prime} y\right\|^{p}=\left[P^{\prime} y, P^{\prime} y\right]_{p}+\left[Q^{\prime} y, Q^{\prime} y\right]_{p} \\
& =\left[P^{\prime 2} y, y\right]_{p}+\left[Q^{\prime 2} y, y\right]_{p}=[P y, y]_{p}+[Q y, y]_{p}=[(P+Q) y, y]_{p} \\
& =[(P+Q)(P-Q) x, y]_{p}=\left[\left(P^{2}+Q P-P Q-Q^{2}\right) x, y\right]_{p} .
\end{aligned}
$$

From the construction of $P$ and $Q, P T=T P=P^{3}, Q T=T Q=Q^{3}$, and it easily follows that $Q P=P Q$. Thus

$$
\begin{gathered}
\left\|P^{\prime} y\right\|_{p}+\left\|Q^{\prime} y\right\|_{p}=\left[\left(P^{2}-Q^{2}\right) x, y\right]_{p}=0, \text { and } P^{\prime} y=Q^{\prime} y=0 ; \quad P y=Q y=0 . \\
\text { Then }\left\|(P-Q)^{2} x\right\|^{p}=\left[(P-Q)^{2} x,(P-Q)^{2} x\right]_{p}=\left[(P-Q) y,(P-Q)^{2} x\right]_{p}= \\
{\left[P y-Q y,(P-Q)^{2} x\right]_{p}=0, \forall x \in X .(P-Q)^{2} x=0, \forall x \in X, \text { or }(P-Q)^{2}=0 .}
\end{gathered}
$$

Hence

$$
T-P Q=0, \quad P(P-Q)=0, \quad Q(P-Q)=0
$$

Because

$$
\begin{aligned}
& \|(P-Q) x\|^{p}=[(P-Q) x,(P-Q) x]_{p} \\
= & {[x, P(P-Q) x]_{p}-[x, Q(P-Q) x]_{p}=0, }
\end{aligned}
$$

then $(P-Q) x=0, \forall x \in X$. Therefore $P=Q$. The theorem is proved.
Corollary 2.2. If $T$ is a generalized positive operator and $T^{*}=\alpha T, \alpha>0$, then there exists a unique generalized positive operator $P$ such that $T=P^{2}$ and $P^{*}=\alpha^{1 / 2} P$.

Definition 2.3. If $U, U^{*} \in L(X)$, and $U U^{*}=U^{*} U=I, U$ is called a generalized $p$ unitary operator (see [3]).

It is easy to prove that the generalized $p$ unitary operator is an isometric operator. From Theorem 2.1, 2.2 and simulating to the result of Hilbert space, the following theorem can be obtained.

Theorem 2.3. (1) $I f\left(T^{*}\right)^{*}=T, T \in L(X)$, there exist $U, P \in L(X)$ such that $T=U P$, where $U$ is a generalized $p$ unitary operator and $P$ is a GPpS operator.
(2) If $\left(T^{*}\right)^{*}=M_{p} T, M_{p}>0, T \in L(X)$ (the definition of the operator is seen in [3]), there exist $U, P \in L(X), T=U P$ and $U^{*} U=U U^{*}=M_{p}^{-(p-1) / 2} I, P^{*}=M_{p}^{1 / 2} P$ on $\overline{P X}$. Then $T=U P$ is called a polar decomposition of $T$.

## 3. Generalized Indefinite Inner Product (GIIP) Space

In order to investigate the $p$ dissipative operator in Banach space, we set up the GIIP space in this section. Using the new space we can give the maximal dissipative extension representation of the $p$ dissipative operator by the negative subspace in the space. The GIIP space comes from the indefinite inner product space but differs from it. Many results on indefinite inner product space have been published, for example, see $[9,10$, $11,14]$. It attracted great attention because the indefinite inner product space caused some important applications in quantum field theory [9], scattering theory [10] and control theory [14]. Here we will set up the GIIP space by means of GSIP.

Definition 3.1. Let $R$ be a complex (or real) linear space, $y, z \in R$, define a complex (or real) number $\langle y, z\rangle$ :
(1) $\langle x+y, z\rangle=\langle x, z\rangle+\langle y, z\rangle,\langle\lambda x, y\rangle=\lambda\langle x, y\rangle$;
(2) If $\langle x, y\rangle=0$ for arbitrary $y$ then $x=0$.

The space $(R,\langle\cdot, \cdot\rangle)$ satisfying (1), (2) is called a generalized indefinite inner product (GIIP) space, $\langle\cdot, \cdot\rangle$ is called the generalized indefinite inner product (GIIP) in $R$.

Definition 3.2. Let $(R,\langle\cdot, \cdot\rangle)$ be GIIP space. If it includes two subspaces $H_{+}$and $H_{-}$ with the properties below:
(1) $R=H_{+} \oplus H_{-}$where $\oplus$ is an orthogonal direct sum for $\langle\cdot, \cdot\rangle$, that is, for arbitrary $x \in H_{+}, y \in H_{-},\langle x, y\rangle=0$, where $H_{+}, H_{-}$are linear subspaces with the following:

$$
H_{+}=\{x \in R \mid\langle x, x\rangle \geq 0, x \neq 0\} ; \quad H_{-}=\{x \in R \mid\langle x, x\rangle \leq 0, \quad x \neq 0\} .
$$

(2) For $1<p<\infty$, spaces $\left(H_{+},\langle\cdot, \cdot\rangle\right)$ and $\left(H_{-},\langle\cdot, \cdot\rangle\right)$ are GSIP spaces. Then we call $(R,\langle\cdot, \cdot\rangle)$ a generalized Krein space. Here $\langle\cdot, \cdot\rangle$ is a GIIP.

If $\left(H_{+},\langle\cdot, \cdot\rangle\right),\left(H_{-},-\langle\cdot, \cdot\rangle\right)$ become Banach spaces and $\|\cdot\|$ is the norm of the spaces $H_{+}, H_{-}$, we call the space $(R,\langle\cdot, \cdot\rangle)$ a complete generalized Krein space and $H_{+}, H_{-}$ are called a regular decomposition of $(R,\langle\cdot, \cdot\rangle)$. Considering generality, we suppose $H_{ \pm} \neq\{0\}$.

Theorem 3.1. In the generalized Krein space $(R,(\cdot, \cdot))$, for $x, y \in R, x=x_{+}+x_{-}$, $y=y_{+}+y_{-}$and $x_{+}, y_{+} \in H_{+}, x_{-}, y_{-} \in H_{-}$denote

$$
[x, y]_{p}=\left\langle x_{+}, y_{+}\right\rangle-\left\langle x_{-}, y_{-}\right\rangle, \text {for } 1<p<+\infty
$$

Then $\left(R,[\cdot, \cdot]_{p}\right)$ is a GSIP space.
Proof. We only need to verify that $[\cdot, \cdot]_{p}$ meets with the following inequality:

$$
\begin{equation*}
\left|[x, y]_{p}\right| \leq[x, x]_{p}^{1 / p}[y, y]_{p}^{1 / q}, 1 / p+1 / q=1 \tag{3.1}
\end{equation*}
$$

or

$$
\begin{aligned}
& \left|\left\langle x_{+}, y_{+}\right\rangle-\left\langle x_{-}, y_{-}\right\rangle\right|^{p} \\
& \leq\left|\left\langle x_{+}, x_{+}\right\rangle-\left\langle x_{-}, x_{-}\right\rangle\right| \cdot\left|\left\langle y_{+}, y_{+}\right\rangle-\left\langle y_{-}, y_{-}\right\rangle\right|^{p-1} \\
& =\left\{\left\|x_{+}\right\|^{p}+\left\|x_{-}\right\|^{p}\right\} \cdot\left\{\left\|y_{+}\right\|^{p}+\left\|y_{-}\right\|^{p}\right\}^{p-1} .
\end{aligned}
$$

Noticing the basic Young's inequality, for $p>1, a>0, b>0$, we have

$$
\begin{equation*}
a b \leq \frac{1}{p} a^{p}+\frac{1}{q} b^{q}, \quad 1 / p+1 / q=1 . \tag{3.2}
\end{equation*}
$$

Let $x_{+} \neq 0, y_{+} \neq 0$ (otherwise formula (3.1) can easily be proved). Thus

$$
\begin{align*}
\mid\left\langle x_{+}, y_{+}\right\rangle & -\left.\left\langle x_{-}, y_{-}\right\rangle\right|^{p} \leq\left(\left|\left\langle x_{+}, y_{+}\right\rangle\right|+\left|\left\langle x_{-}, y_{-}\right\rangle\right|\right)^{p} \\
& \leq\left(\left\|x_{+}\right\|\left\|y_{+}\right\|^{p-1}+\left\|x_{-}\right\|\left\|y_{-}\right\|^{p-1}\right)^{p} \\
& =\left\|x_{+}\right\|^{p}\left\|y_{+}\right\|^{p(p-1)}\left(1+\frac{\left\|x_{-}\right\|}{\left\|x_{+}\right\|}\left(\frac{\left\|y_{-}\right\|}{\left\|y_{+}\right\|}\right)^{p-1}\right)^{p} . \tag{3.3}
\end{align*}
$$

Let $k=\left\|x_{-}\right\| /\left\|x_{+}\right\|, m=\left\|y_{-}\right\| /\left\|y_{+}\right\|$. We might as well assume $k \geq 1$, otherwise we only let $k=\left\|x_{+}\right\| /\left\|x_{-}\right\|$. Then the right side of (3.3) $=\left\|x_{+}\right\|^{p}\left\|y_{+}\right\|^{p(p-1)}\left(1+k m^{p-1}\right)^{p}$. Thus

$$
\begin{aligned}
{[x, x]_{p}^{1 / p}[y, y]_{p}^{1 / q} } & =\left(\left\|x_{+}\right\|^{p}+\left\|x_{-}\right\|^{p}\right)^{1 / p}\left(\left\|y_{+}\right\|^{p}+\left\|y_{-}\right\|^{p}\right)^{1 / q} \\
& =\left\|x_{+}\right\|\left(1+\left(\frac{\left\|x_{-}\right\|}{\left\|x_{+}\right\|}\right)^{p}\right)^{1 / p}\left\|y_{+}\right\|^{p-1}\left(1+\left(\frac{\left\|y_{+}\right\|}{\left\|y_{-}\right\|}\right)^{p}\right)^{1 / q} \\
& =\left\|x_{+}\right\|\left\|y_{+}\right\|^{p-1}\left(1+k^{p}\right)^{1 / p}\left(1+m^{p}\right)^{1 / q}
\end{aligned}
$$

Therefore, proving formula (3.1) is equal to proving the following:

$$
\left(1+k m^{p-1}\right)^{p} \leq\left(1+k^{p}\right)\left(1+m^{p}\right)^{p-1} .
$$

Considering the inequality (3.2), then we have

$$
a b^{p-1} \leq a^{p} / p+b^{p}(p-1) / p, \quad a>0, \quad b>0
$$

Thus

$$
\begin{aligned}
k^{n} m^{n(p-1)} & \leq \frac{1}{p} k^{n p}+\frac{p-1}{p} m^{n p}, \quad 1<p<+\infty, \\
p \frac{k^{n} m^{n(p-1)}}{n} & \leq \frac{k^{n p}}{n}+(p-1) \frac{m^{n p}}{n}, \quad n=1,2,3, \cdots .
\end{aligned}
$$

By summing the above formula, we can obtain

$$
p \sum_{i=1}^{\infty} \frac{k^{n} m^{n(p-1)}}{n} \leq \sum_{i=1}^{\infty} \frac{k^{n p}}{n}+(p-1) \sum_{i=1}^{\infty} \frac{m^{n p}}{n}
$$

or

$$
\begin{gathered}
p \ln \left(1+k m^{p-1}\right) \leq \ln \left(1+k^{p}\right)+(p-1) \ln \left(1+m^{p}\right), \\
\left(1+k m^{p-1}\right)^{p} \leq\left(1+k^{p}\right)\left(1+m^{p}\right)^{p-1} .
\end{gathered}
$$

This theorem has been proved.
Example 3.1. Let $H=H_{+} \oplus H_{-}$and $H_{ \pm}$respectively are GSIP spaces in which the GSIP are $[\cdot, \cdot]_{ \pm}$respectively. We construct the GIIP as following that

$$
\left\langle x_{+}+x_{-}, y_{+}+y_{-}\right\rangle=\left[x_{+}, y_{+}\right]_{+}-\left[x_{-}, y_{-}\right]_{-},
$$

where $x=x_{+}+x_{-}, y=y_{+}+y_{-} \in H$. Easily, we can prove that $(H,\langle\cdot, \cdot\rangle)$ is a generalized Krein space.

Example 3.2. Let $(X,[\cdot, \cdot])$ be a GSIP space (for example $X=C(K)$ or $L^{p^{\prime}}, 1<$ $p^{\prime}<\infty$ ). Suppose $T$ is a generalized $p$ selfadjoint operator and $N(T) \neq\{0\}$ (such $T$ exists in general, for example, the paper [15] gives the functional models of the adjoint Abelian operators in $C(K)$ and $L^{p^{\prime}}\left(1<p^{\prime}<\infty\right)$ ). From [2] Theorem 2.22, $X=\overline{R_{p}(T) \oplus N_{p}(T)}=\overline{R_{p}(T)} \oplus N_{p}(T)$, where $R_{p}(T), N_{p}(T)$ are the numerical range and kernel space, respectively. Then $\overline{R_{p}(T)}, N_{p}(T)$ are closed subspaces in $X$ and $\left(\overline{R_{p}(T)},[\cdot, \cdot]\right),\left(N_{p}(T),[\cdot, \cdot]\right)$ are GSIP spaces. Let the GIIP in $X$ be the following

$$
\left\langle x_{1}+x_{2}, y_{1}+y_{2}\right\rangle=\left[x_{1}, y_{1}\right]-\left[x_{2}, y_{2}\right], \quad \text { for } x_{1}, y_{1} \in \overline{R_{p}(T)}, \quad x_{2}, y_{2} \in N_{p}(T) .
$$

Being proved easily, $(X,\langle\cdot, \cdot\rangle)$ is a generalized Krein space and $\overline{R_{p}(T)}, N_{p}(T)$ is a regular decomposition of $X$. For $1<p<+\infty$, we can obtain different regular decompositions of $X$ because there exist a lot of different generalized $p$ selfadjoint operators in GSIP space $X$. Hence the regular decomposition of $X$ is not unique.
Definition 3.3. Define projection operators $P_{ \pm}: \Pi=(R,\langle\cdot, \cdot\rangle) \rightarrow H_{ \pm}$such that $x=x_{+}+x_{-} \in \Pi \rightarrow x_{ \pm} \in H_{ \pm}$. Denote $J=P_{+}-P_{-}$.
Theorem 3.2. (1) $J^{2}=I, J=J^{*}$, where $J^{*}$ is a generalized $p$ adjoint operator in generalized Krein space $\left(R,[\cdot, \cdot]_{p}\right)$.
(2) $\langle x, y\rangle=[J x, y]_{p},[x, y]_{p}=\langle J x, y\rangle$.

Proof. We only prove $J=J^{*}$, the others may be obtained easily:

$$
\begin{aligned}
{[J x, y]_{p} } & =\left[P_{+} x-P_{-} x, y\right]_{p}=\left[P_{+} x, y\right]_{p}-\left[P_{-} x, y\right]_{p} \\
& =\left\langle P_{+} x, y_{+}\right\rangle+\left\langle 0, y_{-}\right\rangle-\left\{\left\langle 0, y_{+}\right\rangle+\left\langle P_{-} x, y_{-}\right\rangle\right\} \\
& =\left\langle P_{+} x, y_{+}\right\rangle-\left\langle P_{-} x, y_{-}\right\rangle \\
& =\left\langle P_{+} x, y_{+}\right\rangle-\left[P_{-} x, y_{-}\right]_{p}=\left\langle P_{+} x, y_{+}\right\rangle+\left[P_{-} x,-y_{-}\right]_{p} \\
& =\left\langle P_{+} x, y_{+}\right\rangle+\left\langle P_{-} x,-y_{-}\right\rangle=\left[P_{+} x+P_{-} x, y_{+}-y_{-}\right]_{p}=[x, J y]_{p} .
\end{aligned}
$$

Hence $J$ is a generalized $p$ selfadjoint operator in $\left(R,[\cdot, \cdot]_{p}\right)$.
We remark that the GSIP in Theorem 3.1 depends on the regular decomposition of $R$. In general the GSIP isn't unique.
Definition 3.4. Let $(R,\langle\cdot, \cdot\rangle)$ be a GIIP space, and $x \in R$. If $\langle x, x\rangle \geq 0$ (or $\langle x, x\rangle \leq 0$ ), $x$ is called a semipositive (or seminegative) vector in $R$. If $\langle x, x\rangle>0$ (or $\langle x, x\rangle<0$ ), $x$ is called a positive (or negative) vector in $R$. If $\langle x, x\rangle=0, x$ is called a neutral vector or isotropic vector.

Definition 3.5. Lis called a positive (or negative, semi-positive, semi-negative, neutral, respectively) subspace if all vectors in linear subspace $L$ in $R$ are positive (or negative, semi-positive, semi-negative, neutral, respectively). Suppose that $L$ is a positive (or negative, semi-positive, semi-negative, neutral) subspace, and there is not any positive (or negative, semi-positive, semi-negative, neutral) subspace $L$ such that $L$ is a proper subspace of $L^{\prime}$. Then $L$ is called a maximal positive (or maximal negative, maximal semi-positive, maximal semi-negative, maximal neutral) subspace in $(R,\langle\cdot, \cdot\rangle)$.

According to the related results on indefinite inner product space (see [9-12]) we can easily obtain: Any positive (or negative, semi-positive, semi-negative, neutral) subspace of GIIP space $R$ can be extended as a maximal positive (or negative, semi-positive, semi-negative, neutral) subspace but the extension isn't unique.
Theorem 3.3. Suppose that $L$ is a semi-positive subspace in $\Pi=(R,\langle\cdot, \cdot\rangle)$. Then there exist an orthogonal projection $P_{+}^{L}$ in $H_{+}$and a contraction linear operator $T$ : $P_{+}^{L} H_{+} \rightarrow$ $H_{-}$such that

$$
\begin{equation*}
L=\left\{x_{+}+T x_{+} \mid x_{+} \in P_{+}^{L} \Pi\right\} \quad(\text { or }\|T\|<1) \tag{3.3}
\end{equation*}
$$

where the orthogonal projection $P_{+}^{L}: L \rightarrow H_{+}$means that $x \rightarrow x_{+}$, for arbitrary $x=x_{+}+x_{-} \in L,\left\langle x_{+}, x_{-}\right\rangle=0$. L satisfying Eq. (3.3) is a semi-positive (or positive) subspace in $\Pi$ if and only if $\left\|T x_{+}\right\| \leq\left\|x_{+}\right\|$, for arbitrary $x \in P_{+} \Pi$ ( or $\|T\|<1$ ).

Proof. For arbitrary $x \in L,-\left\|P_{-} x\right\|^{p}+\left\|P_{+} x\right\|^{p}=\langle x, x\rangle \geq 0$, then $\left\|P_{-} x\right\| \leq\left\|P_{+} x\right\|$. $\left\|P_{-} x\right\|^{p} \leq\|x\|^{p}=[x, x]_{p}=\left[P_{+} x, P_{+} x\right]_{p}+\left[P_{-} x, P_{-} x\right]_{p}=\left\|P_{+} x\right\|^{p}+\left\|P_{-} x\right\|^{p} \leq$ $2\left\|P_{+} x\right\|^{p}$. Thus $\left.P_{+}\right|_{L}$ is reversible. Let the reversibility be $E x_{+}=x, x \in L$. Then $P_{+} x=x_{+}$. Denote $P_{+}^{L}: H_{+} \rightarrow P_{+} L$ a projection operator. Writing $T: P_{+}^{L} H_{+} \rightarrow$ $H_{-}: P_{+} x=P_{+}^{L} x_{+} \rightarrow P_{-} E P_{+} x$, for arbitrary $x \in L$, then we easily get that $P_{+}^{L}$ is an orthogonal projection operator. From the semi-positive of $L$ we obtain that $T$ is a contraction linear operator, or, $\langle T x, x\rangle \leq\langle x, x\rangle, x \in D(T)$. Then $L=\left\{x_{+}+T x_{+}: x_{+} \in\right.$ $\left.P_{+}^{L} \Pi\right\}$.

The other conclusions in the theorem are omitted because their proofs are easy.
Corollary 3.1. The semi-positive subspace $L$ is a maximal semi-positive subspace if and only if $P_{+} L=H_{+}$; any semi-positive subspaces are contained in one maximal semi-positive subspace.
Corollary 3.2. All maximal semi-positive (or semi-negative) subspaces in generalized Krein space have identical dimension.

Theorem 3.4. In Banach space $\left(R,[\cdot, \cdot]_{p}\right)$, if the GSIP $[\cdot, \cdot]_{p}$ is continuous to the first variable, any norms in $R$ are equivalent to each other.
Proof. Suppose $\|\cdot\|,\|\cdot\|^{\prime}$ are two norms in $R$ and denote $\|x\|^{\prime \prime}=\|x\|+\|x\|^{\prime}$. Now we prove $\|x\|^{\prime \prime}$ is also the norm of $R$. In fact we only need to prove the completeness in $R$. For arbitrary $\left\{x_{n}\right\} \in R$, if $\left\|x_{n}-x_{m}\right\|^{\prime \prime} \rightarrow 0, m, n \rightarrow \infty$, then there exist $x_{0}, x_{0}^{\prime} \in R$ such that $\left\|x_{n}-x_{0}\right\| \rightarrow 0,\left\|x_{n}-x_{0}^{\prime}\right\| \rightarrow 0$, as $n \rightarrow \infty$. Hence $\left[x_{n}-x_{0}^{\prime}, y\right]_{p} \rightarrow 0$ (from the continuous condition), $\left[x_{n}-x_{0}^{\prime}, y\right]_{p} \rightarrow 0$, for arbitrary $y \in R$. Therefore for arbitrary $y \in R$,

$$
\left|\left[x_{0}-x_{0}^{\prime}, y\right]_{p}\right| \leq\left|\left[x_{n}-x_{0}^{\prime}, y\right]_{p}\right|+\left|\left[x_{n}-x_{0}, y\right]_{p}\right| \rightarrow 0, \quad \text { as } n \rightarrow \infty
$$

Because $y$ is arbitrary then $x_{0}=x_{0}^{\prime}$ and

$$
\left\|x_{n}-x_{0}\right\|^{\prime \prime}=\left\|x_{n}-x_{0}\right\|+\left\|x_{n}-x_{0}\right\|^{\prime} \rightarrow 0, \quad \text { as } n \rightarrow \infty .
$$

If $\|\cdot\|^{\prime \prime}$ is a norm in $\left(R,[\cdot, \cdot]_{p}\right)$, the space is still complete. We get $\|\cdot\|^{\prime \prime}$ is also a norm in $R$ and $\|x\| \leq\|x\|^{\prime \prime}$. From the Banach Inverse Theorem's Corollary (see [50]) in Banach space, we have that $\|\cdot\|$ is equivalent to $\|\cdot\|^{\prime \prime}$. For the same reason, $\|\cdot\|$ is equivalent to $\|\cdot\|^{\prime}$. Therefore $\|\cdot\|$ is equivalent to $\|\cdot\|^{\prime \prime}$. The proof is complete.

## 4. The Construction of Natural Boundary Space

In the section we set up the natural boundary space of $p$ dissipative operators in Banach space. The natural boundary space is a GIIP space. In the next section we will give the natural boundary space's application. For similar results in Hilbert space and indefinite inner product space see M. G. Crandall and R. S. Phillips [12]. Because there doesn't exist an inner product in Banach space, it is very difficult to extend the results. Using GSIP space and GIIP space, we solve this difficulty.
(i) Hypotheses of spaces $H_{0}, H_{1}, H_{2} . H_{0}$ : Let $\left(H_{0},[\cdot, \cdot]_{p}\right), 1<p<\infty$ be a GSIP space and the norm of $H_{0}$ is $\|x\|=[x, x]_{p}^{1 / p}$. Suppose that $S$ is a GPpS operator in $H_{0}(S$ may be an unbounded operator). And suppose $F$ is a linear operator with the following conditions:
(1) domain $D(F)=D(S)$,
(2) $\overline{R(F)}=H_{0}$,
(3) $F$ is a generalized $p$ selfadjoint operator,
(4) $[F u, u]_{p} \geq[u, u]_{p}$, for arbitrary $u \in D(S)$ or denote $F-I \geq 0$.

The operator $F$ exists. For example, in SIP space $X$, let $S$ be the GPpS operator then $X=\overline{R(S)} \oplus N(S)$ (see [2] Theorem 2.22). We assume that $F x=\alpha S x, x \in \overline{R(S)}$ $; F x=\beta x, x \in N(S)$ and $F$ is a linear operator in $X$. If $S$ is a bounded operator, choose $\alpha$ and $\beta$ such that $\alpha \geq \frac{1}{\|S\|}, \beta \geq 1$; if $S$ is an unbounded operator, choose $\alpha$ and $\beta$ such that $\alpha=1, \beta \geq 1$. Then we easily prove that $F$ is a GPpS operator in $X$. From [2] Theorem 2.22, we have

$$
X=\overline{R(F) \oplus N(F)}
$$

Because of $N(F)=0$, then $X=\overline{R(F)}$. Therefore $F$ satisfies the above (1)-(4).
Remark 4.1. It is evident that $\|F\| \geq 1,\left\|F^{-1}\right\| \leq 1$. From Theorem $2.2 S, F$ have unique positive square roots respectively: we denote $S^{1 / 2}, F^{1 / 2}$ and $S^{1 / 2}, F^{1 / 2}$ are GPpS operators and $D\left(S^{1 / 2}\right)=D\left(F^{1 / 2}\right)$.

$$
\begin{aligned}
& H_{1} \text { : Let }\left(H_{1},[\cdot, \cdot]_{1}\right) \text { be a GSIP space with GSIP }[\cdot, \cdot]_{1} \text { such that } \\
& \qquad[u, v]_{1}=\left[F^{1 / 2}, F^{1 / 2}\right]_{p}, \quad u, v \in D\left(F^{1 / 2}\right)
\end{aligned}
$$

and the norm in $H_{1}$ is denoted $\|x\|_{1}=[x, x]_{1}^{1 / p}, x \in H_{1}$. It is easy to see that $H_{1}$ is a dense set of $H_{0}$.

## $H_{2}$ : Let $\left(H_{2},[\cdot, \cdot]_{2}\right)$ be a GSIP space and the GSIP is:

$$
[u, v]_{2}=\left[F^{-1 / 2} u, F^{-1 / 2} v\right]_{p}=\left[F^{-1} u, v\right]_{p}, \quad u, v \in H_{0}
$$

and the norm in $H_{2}$ is denoted $\|x\|_{2}=[x, x]_{2}^{1 / p}$.
Remark 4.2. (1) For arbitrary $u \in H_{0},\|u\| \leq\|u\|_{1}$. In fact

$$
\|u\|=[u, u]_{p}^{1 / p} \leq\left([F u, u]_{p}\right)^{1 / p}=\left(\left[F^{1 / 2} u, F^{1 / 2} u\right]_{p}\right)^{1 / p}=\|u\|_{1}
$$

(2) For arbitrary $u \in H_{0}$, generates a continuous functional $l_{u}(v)$ on $H_{1}$ according to the formula:

$$
l_{u}(v)=[v, u]_{p}, \text { for arbitrary } v \in H_{1} .
$$

Its continuity follows from the estimate:

$$
\begin{aligned}
\left|l_{u}(v)\right| & =\left|[v, u]_{p}\right| \leq\left|[v, v]_{p}\right|^{1 / p}\left|[u, u]_{p}\right|^{1 / q} \\
& =\|v\|\|u\|^{p-1} \leq\|v\|_{1}\|u\|^{p-1} .
\end{aligned}
$$

(3) Let $u_{0} \in H_{0}$, then $\|u\|_{2}=\left\|l_{u}\right\|^{1 /(p-1)}$ and $\|u\|_{2} \leq\|u\|$. In fact

$$
\begin{aligned}
\left\|l_{u}\right\|^{1 /(p-1)} & =\left(\sup _{v \in H_{1}} \frac{\left|[v, u]_{p}\right|}{\|v\|_{1}}\right)^{1 /(p-1)}=\left(\sup _{v \in H_{1}} \frac{\left[F^{-1 / 2} v, F^{-1 / 2} u\right]_{p}}{\left\|F^{-1 / 2} v\right\|}\right)^{1 /(p-1)} \\
& =\left\|F^{-1 / 2} u\right\|=\|u\|_{2}
\end{aligned}
$$

From Remark $4.1\left\|F^{-1}\right\| \leq 1$, then

$$
\|u\|_{2}^{p}=\left[F^{-1} u, u\right]_{p} \leq\left\|F^{-1} u\right\|\|u\|^{p-1} \leq\|u\|^{p}, \quad\|u\|_{2} \leq\|u\| .
$$

Hence $\left\{l_{u}, u \in H_{0}\right\}$ is a dense subset of $H_{1}^{*}$, where $H_{1}^{*}$ is a dual space of $H_{1}$.
Remark 4.3. By using Remarks 4.1 and 4.2 , then

$$
\|u\|_{1} \leq\|u\| \leq\|u\|_{2}, \quad u \in H_{0} ; \quad H_{1} \subset H_{0} \subset H_{2}
$$

in the topological sense. If $u \in H_{1}, v \in H_{2}$, we define $[\cdot, \cdot]_{p}$ in $H_{1} \times H_{2}$ satisfying the following formula:

$$
[u, v]_{p}=\left[F^{1 / 2} u, F^{-1 / 2} v\right]_{p}=[F u, v]_{2}=\left[u, F^{-1} v\right]_{1}, u \in H_{1}, v \in H_{2}
$$

If $u \in H_{2}, v \in H_{1}$, define $[u, v]_{p}=\left[F^{-1 / 2} u, F^{1 / 2} v\right]_{p}$.
Example 4.1. For $1<p<\infty, p \neq 2$, if $H_{0}=L^{p}(R)$, denote the unique SIP $[\cdot, \cdot]$ in $H_{0}$ :

$$
[f, g]=\|g\| \int_{R} f\left(\frac{|g|}{\|g\|}\right)^{p-1} \operatorname{sgn} g d x, \quad f, g \in L^{p}(R)
$$

Let $F=\alpha I, \alpha>1$. Then $H_{1}=L^{p}(R, \alpha d x), H_{2}=L^{q}\left(R, \alpha^{-1} d x\right), q^{-1}+p^{-1}=1$.
Proposition 4.1. $\left(H_{1},[\cdot, \cdot]_{1}\right),\left(H_{2},[\cdot, \cdot]_{2}\right)$ are GSIP spaces.
Proposition 4.2. $H_{2}=\left(H_{1}\right)^{\prime}$, where $\left(H_{1}\right)^{\prime}$ is a dual space of $H_{1}$.
Proof. It is easy to see that $H_{2} \subset\left(H_{1}\right)^{\prime}$. For arbitrary $l \in\left(H_{1}\right)^{\prime}$, there exists $a \in H_{1}$ such that $l(u)=[u, a]_{1}$ from the Riesz Representation Theorem in GSIP space. Then

$$
l(u)=[u, a]_{1}=[u, F a]_{p}, \quad u \in H_{1} .
$$

Let $\alpha=F a \in H_{2}$. Thus $l(u)=[u, \alpha]_{p}, u \in H_{1}$. From the Riesz Representation Theorem in GSIP space, $l \in H_{2}$. Then $\left(H_{1}\right)^{\prime} \subset H_{2}$. Hence $\left(H_{1}\right)^{\prime}=H_{2}$.

Proposition 4.3. (1) If $u \in H_{1}, v \in H_{2}$, then

$$
\left|[u, v]_{p}\right| \leq\left\|F^{1 / 2} u\right\|\left\|F^{-1 / 2} v\right\|^{p-1} \leq\|u\|_{1}\|v\|_{2}^{p-1}
$$

(2) $u \in H_{2}, v \in H_{1}$, then $\left|[u, v]_{p}\right| \leq\|u\|_{2}\|v\|_{1}^{p-1}$.
(3) $u, v \in H_{1}$, then $[S u, v]_{p}=\left[S^{1 / 2} u, S^{1 / 2} v\right]_{p}=[u, S v]_{p}$, and $S^{1 / 2}$ defines a bounded mappings on $H_{1}$ to $H_{2}$ and on $H$ to $H_{2}$. $S$ defines a bounded mapping on $H_{1}$ to $H_{2}$.
(ii) The definition of $p$ dissipative operator.

Definition 4.1. Let L be a linear operator on $H_{1}$ to $H_{2}$ with domain $D(L)$ dense in $H_{1}$. We define $L^{*}$, the generalized padjoint operator to $L$, as the operator on $H_{1}$ to $H_{2}$ given by: $v$ in $D\left(L^{*}\right)$ and $L^{*} v=f$ if $[L u, v]_{p}=[u, f]_{p}$ for all $u$ in $D(L)$.
Definition 4.2. Let $L$ be a densely defined linear operator on $H_{1}$ to $H_{2}$. Then $L$ is (1) p symmetric if $L^{*} \supset L$; (2)p skew-symmetric if $L^{*} \supset-L$; (3) generalized $p$ selfadjoint if $L^{*}=L$.

Definition 4.3. Let $L$ be a densely defined linear operator on $H_{1}$ to $H_{2}$. $L$ is called dissipative operator if $\operatorname{Re}[L u, u]_{p} \leq 0, u \in D(L)$. $L$ is maximal dissipative if it is dissipative and not a proper restriction of a p dissipative operator.

Remark 4.4. For $1<p<+\infty$, if the space $H_{0}$ has GSIP $[\cdot, \cdot]_{p}$ and $L$ is a $p$ dissipative operator from $H_{1}$ to $H_{2}$, then $L$ is a $p^{\prime}$ dissipative operator where $p \neq p^{\prime}, 1<p^{\prime}<+\infty$ by using the formula

$$
\|y\|^{p-p^{\prime}}[x, y]_{p^{\prime}}=[x, y]_{p}, \quad 1<p, \quad p^{\prime}<+\infty
$$

and $[x, y]_{p^{\prime}}$ is a GSIP of $H_{0}$. Hence, for the $p$ dissipative operator in Definition 4.3, $p$ means that the space $H_{0}$ has GSIP $[\cdot, \cdot]_{p}, 1<p<+\infty$.

Let $L_{0}=A-S$ on $H_{1}$ to $H_{2}$, where $A$ is a $p$ skew-symmetric operator, $D(A)$ is dense in $H_{1}, \operatorname{Re}[A u, u]_{p}=0$ for any $u \in D\left(L_{0}\right)$ and $S$ is a GPpS operator. Then $L_{0}$ is a $p$ dissipative operator because $\operatorname{Re}\left[L_{0} u, u\right]_{p}=-[S u, u] \leq 0$. In this and the next section we investigate the very important operator $L_{0}$ in Banach space.
(iii) The construction of natural boundary space $\hat{H}$ of $L_{0}$. We introduce the product space $H_{12}=H_{1} \times H_{2}$ with element $\bar{u}=\left\{u^{1}, u^{2}\right\}$ and $Q(\cdot, \cdot)$ :

$$
Q(\bar{u}, \bar{v})=\operatorname{Re}\left[u^{2}, v^{1}\right]_{p}+\operatorname{Re}\left[S u^{1}, v^{1}\right]_{p}
$$

for any $\bar{u}=\left\{u^{1}, u^{2}\right\}, \bar{v}=\left\{v^{1}, v^{2}\right\} \in H_{1} \times H_{2}$. Let the graph of $L_{0}$ be

$$
G\left(L_{0}\right)=\left\{\left\{u, L_{0} u\right\} \mid u \in D\left(L_{0}\right)\right\} .
$$

As $\bar{u}=\left\{u, L_{0} u\right\} \in G\left(L_{0}\right)$, then

$$
Q(\bar{u}, \bar{u})=\operatorname{Re}\left[L_{0} u, u\right]_{p}+\operatorname{Re}[S u, u]_{p}=0 .
$$

Let $L_{1}=A^{*}-S$; we have the set $\overline{\text { span } G\left(L_{1}\right)}$, where $G\left(L_{1}\right)$ is a graph of $L_{1}$. If $\bar{u}, \bar{v} \in G\left(L_{1}\right)$ then

$$
Q(\bar{u}, \bar{v})=\operatorname{Re}\left[L_{1} u, v\right]_{p}+\operatorname{Re}[S u, v]_{p}=\operatorname{Re}\left[A^{*} u, v\right]_{p} .
$$

The sets $H_{+}, H_{-}$are defined by

$$
\begin{aligned}
H_{+} & =\left\{\bar{u}: Q(\bar{u}, \bar{u}) \geq 0, \bar{u} \in G\left(L_{1}\right)\right\} \subset G\left(L_{1}\right), \\
H_{-} & =\left\{\bar{u}: Q(\bar{u}, \bar{u}) \leq 0, \bar{u} \in G\left(L_{1}\right)\right\} \subset G\left(L_{1}\right) .
\end{aligned}
$$

Because $H_{+}, H_{-}$may not be linear spaces, we define $\bar{H}_{+}, \bar{H}_{-}$as follows:

$$
\bar{H}_{+}=\overline{\operatorname{span} H_{+}}, \bar{H}_{-}=\overline{\operatorname{span} H_{-}}
$$

Then $\bar{H}_{+}, \bar{H}_{-}$are closed linear subspaces of $\overline{\operatorname{span} G\left(L_{1}\right)}$ and $G\left(L_{0}\right) \subset \bar{H}_{+} \cap \bar{H}_{-}$.
Define $\bar{Q}_{+}(\cdot, \cdot)$ in $\bar{H}_{+}$:

$$
\bar{Q}_{+}(\bar{u}, \bar{v})=Q(\bar{u}, \bar{v}) \operatorname{sgn} Q(\bar{v}, \bar{v}), \bar{u}, \bar{v} \in \bar{H}_{+} .
$$

Define $\bar{Q}_{-}(\cdot, \cdot)$ in $\bar{H}_{-}$:

$$
\bar{Q}_{-}(\bar{u}, \bar{v})=Q(\bar{u}, \bar{v})(-\operatorname{sgn} Q(\bar{v}, \bar{v})), \bar{u}, \bar{v} \in \bar{H}_{-} .
$$

Let $\overline{\bar{H}}$ be the direct sum space $\overline{\bar{H}}=\bar{H}_{+} \oplus \bar{H}_{-} ; \oplus$ means direct sum. Define the form $\overline{\bar{Q}}(\cdot, \cdot)$ in $\overline{\bar{H}}$, for $\overline{\bar{u}}=\bar{u}_{+}+\bar{u}_{-}, \overline{\bar{v}}=\bar{v}_{+}+\bar{v}_{-} \in \overline{\bar{H}}$ :

$$
\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}})=\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right) .
$$

Let $\widehat{H}=\bar{H} / G\left(L_{0}\right)=\bar{H}_{+} / G\left(L_{0}\right) \oplus \bar{H}_{-} / G\left(L_{0}\right)$. Introduce the form $\widehat{Q}(\widehat{u}, \widehat{v}), \widehat{u}$, $\widehat{v} \in \widehat{H}$, such that

$$
\widehat{Q}(\widehat{u}, \widehat{v})=\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}}) \text {, where } \overline{\bar{u}}, \overline{\bar{v}} \text { belong to the coset } \widehat{u}, \widehat{v} \text { in } \widehat{H} .
$$

Theorem 4.1. $\left(\bar{H}_{+}, \bar{Q}_{+}(\cdot, \cdot)\right),\left(\bar{H}_{-},-\bar{Q}_{-}(\cdot, \cdot)\right)$ are GSIP spaces. Let $\bar{H}_{+} / G\left(L_{0}\right)$, $\bar{H}_{-} / G\left(L_{0}\right)$ be quotient spaces, then

$$
\left(\bar{H}_{+} / G\left(L_{0}\right), \bar{Q}_{+}\right),\left(\bar{H}_{-} / G\left(L_{0}\right),-\bar{Q}_{-}\right)
$$

are GSIP spaces.
Theorem 4.2. $\widehat{Q}(\widehat{u}, \widehat{v})=\overline{\bar{Q}}(\overline{\bar{u}}, \bar{v})$, for any $\overline{\bar{u}} \in \widehat{u}, \overline{\bar{v}} \in \widehat{v}$, where $\overline{\bar{u}}, \overline{\bar{v}} \in \overline{\bar{H}}, \widehat{u}, \widehat{v} \in \widehat{H}$.
Theorem 4.3. $(\widehat{H}, \widehat{Q}(\cdot, \cdot))$ is a GIIP space.
Let $\widetilde{H}=\widehat{H} \oplus \check{H}, \oplus$ is a direct sum, where $\check{H}$ is a GSIP space with GSIP $[\cdot, \cdot]_{p}$. For any $\widetilde{u}=\{\hat{u} \check{u}\} \in \widetilde{H}$, define $\widetilde{Q}$ :

$$
\widetilde{Q}(\widetilde{u}, \widetilde{v})=\widehat{Q}(\widehat{u}, \widehat{v})+[\check{u}, \check{v}]_{p}
$$

It is easy to see that $(\widetilde{H}, \widetilde{Q})$ is a GIIP space.
Let $\widetilde{N}$ be a negative subspace on $(\widetilde{H}, \widetilde{Q})$. Suppose $\widehat{N}$ is a projection subspace of $\widetilde{N}$ in $\widehat{H}$. From the definition of $\widetilde{Q}$, then $\widehat{N}$ is a negative subspace on $(\widehat{H}, \widehat{Q})$ and

$$
\|\check{u}\|^{p} \leq-\widehat{Q}(\widehat{u}, \widehat{u})=-\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{u}}) \leq C\|\widehat{u}\|^{p}, \text { for any }\{\widehat{u}, \check{u}\} \in \widetilde{N} \text {. }
$$

Theorem 4.4. If $\widetilde{N}$ is a maximal negative subspace on $(\widetilde{H}, \widetilde{Q})$, then $\widehat{N}$ is a maximal negative subspace on $(\widehat{H}, \widehat{Q})$.
Theorem 4.5. If $\widetilde{N}$ is a maximal negative subspace on $\widetilde{H}$, then $\|\check{u}\|^{p} \leq-\widehat{Q}(\widehat{u}, \widehat{u})$, for $\widetilde{u}=\{\widehat{u}, \breve{u}\} \in \widetilde{N}$. Define a transformation $\varphi: \widehat{u} \rightarrow \check{u}, \widetilde{u}=\{\widehat{u}, \breve{u}\} \in \widetilde{N}$. Then it is a linear contraction transformation with respect to the form $-\widehat{Q}$ on the maximal negative subspace $\widetilde{N}$ to $\check{H}$. Conversely, if the transformation $\varphi$ is a contraction (in this sense) on the maximal negative subspace of $\widetilde{H}$ to $\breve{H}$, then the graph of $\varphi$ is a maximal negative subspace of $\widetilde{H}$.
(iv) The proof of Theorem 4.1-Theorem 4.5.

Proof of Theorem 4.1. Because of $\bar{Q}_{ \pm}(\bar{u}, \bar{u})=\bar{Q}(\bar{u}, \bar{u})( \pm \operatorname{sgn} Q(\bar{u}, \bar{u}))=|Q(\bar{u}, \bar{u})|>0$ for $\bar{u} \neq 0$. Now to prove $\left(\bar{H}_{+}, \bar{Q}_{+}\right)$is a GSIP space, we only need to prove:

$$
\begin{equation*}
\left|\bar{Q}_{+}(\bar{u}, \bar{v})\right| \leq\left|\bar{Q}_{+}(\bar{u}, \bar{u})\right|^{1 / p}\left|\bar{Q}_{+}(\bar{v}, \bar{v})\right|^{(p-1) / p}, \quad \text { for any } \bar{u}, \bar{v} \in \bar{H}_{+} \tag{4.1}
\end{equation*}
$$

In fact,

$$
\begin{aligned}
\left|\bar{Q}_{+}(\bar{u}, \bar{v})\right| & =|Q(\bar{u}, \bar{v})|=\left|\operatorname{Re}\left[A^{*} u, v\right]_{p}\right| \\
& =0.5\left|\operatorname{Re}\left[A^{*} u, v\right]_{p}+\operatorname{Re}\left[A^{*} u, v\right]_{p}\right| \\
& =0.5\left|\operatorname{Re}\left[A^{*} u, v\right]_{p}+\operatorname{Re}\left[u, A^{* *} v\right]_{p}\right| .
\end{aligned}
$$

As $A^{*} \supset-A,-A^{* *} \subset A^{*}$, then

$$
\left|\bar{Q}_{+}(\bar{u}, \bar{v})\right|=0.5\left|\operatorname{Re}\left[A^{*} u, v\right]_{p}+\operatorname{Re}\left[u,-A^{*} v\right]_{p}\right| .
$$

Construct a new GSIP space $H_{1} \times H_{2}$ with GSIP $[\cdot, \cdot]_{12}$ as follows:

$$
\begin{aligned}
{[\bar{u}, \bar{v}]_{12} } & =\operatorname{Re}\left[u^{1}, v^{1}\right]_{1}+\operatorname{Re}\left[u^{2}, v^{2}\right]_{2}, \\
& \forall \bar{u}=\left\{u^{1}, u^{2}\right\}, \quad \bar{v}=\left\{v^{1}, v^{2}\right\} \in H_{1} \times H_{2} .
\end{aligned}
$$

Imitating Theorem 3.1, $[\cdot, \cdot]_{12}$ is a GSIP in $H_{1} \times H_{2}$. Let

$$
W \bar{u}=W\left\{u^{1}, u^{2}\right\}=\left\{F^{-1} u^{2}, F u^{1}\right\}, \quad \forall \bar{u} \in H_{1} \times H_{2} .
$$

Then $W^{2}=I$ and $[W \bar{u}, \bar{v}]_{12}=[\bar{u}, W \bar{v}]_{12}$, or $W$ is a generalized $p$ selfadjoint operator in GSIP space $\left(H_{1} \times H_{2},[\cdot, \cdot]_{12}\right)$.

Next we prove that $W$ satisfies the generalized Schwarz inequality in ( $H_{1} \times$ $\left.H_{2},[\cdot, \cdot]_{12}\right)$ :

$$
\begin{equation*}
\left|[W x, y]_{12}\right| \leq\left|[W x, x]_{12}\right|^{1 / p}\left|[W y, y]_{12}\right|^{1 / q}, \quad \text { for } x, y \in H_{1} \times H_{2} . \tag{4.2}
\end{equation*}
$$

As $W^{2}=I$, from Proposition 2.1, there exist linear operators $E_{1}, E_{2}$ satisfying

$$
E_{i} E_{j}=\left\{\begin{array}{l}
0, i \neq j \\
E_{i}, i=j
\end{array} \quad \text { and } E_{1}+E_{2}=I, \quad W=E_{1}-E_{2}\right.
$$

Then, for any

$$
\begin{aligned}
f, g & \in H_{1} \times H_{2}, \quad f=f_{1}+f_{2}, \quad g=g_{1}+g_{2} \\
f_{1}, g_{1} & \in E_{1}\left(H_{1} \times H_{2}\right), \quad f_{2}, g_{2} \in E_{2}\left(H_{1} \times H_{2}\right) . \\
W f & =f_{1}-f_{2}, \quad W g=g_{1}-g_{2}
\end{aligned}
$$

Thus

$$
E_{1}\left(H_{1} \times H_{2}\right), \quad E_{2}\left(H_{1} \times H_{2}\right), \quad W\left(H_{1} \times H_{2}\right)=\left(E_{1}-E_{2}\right)\left(H_{1} \times H_{2}\right)
$$

are linear subspaces in $\mathrm{H}_{1} \times \mathrm{H}_{2}$.
Construct a new GSIP $[\cdot, \cdot]_{E_{1}}$ in the product space $\left(E_{1}\left(H_{1} \times H_{2}\right)\right) \times\left(W\left(H_{1} \times H_{2}\right)\right)$ as follows:

$$
\left[f_{1}, z\right]_{E_{1}}=\left[f_{1}, h-g\right]_{12},
$$

where

$$
\begin{gathered}
f_{1}, h \in E_{1}\left(H_{1} \times H_{2}\right) \subset H_{1} \times H_{2}, \quad g \in E_{2}\left(H_{1} \times H_{2}\right), \\
z=h-g \in\left(E_{1}-E_{2}\right)\left(H_{1} \times H_{2}\right)=W\left(H_{1} \times H_{2}\right) \subset H_{1} \times H_{2} .
\end{gathered}
$$

Construct a new GSIP $[\cdot, \cdot]_{E_{2}}$ in the product space $\left(E_{2}\left(H_{1} \times H_{2}\right)\right) \times\left(W\left(H_{1} \times H_{2}\right)\right)$ as follows:

$$
\left[f_{2}, z\right]_{E_{2}}=\left[f_{2}, h-g\right]_{12},
$$

where

$$
\begin{gathered}
f_{2}, g \in E_{2}\left(H_{1} \times H_{2}\right) \subset H_{1} \times H_{2}, h \in E_{1}\left(H_{1} \times H_{2}\right), \\
z=h-g \in\left(E_{1}-E_{2}\right)\left(H_{1} \times H_{2}\right)=W\left(H_{1} \times H_{2}\right) \subset H_{1} \times H_{2} .
\end{gathered}
$$

Because $[\cdot, \cdot]_{12}$ is a GSIP, then it is easy to prove that $[\cdot, \cdot]_{E_{1}},[\cdot, \cdot]_{E_{2}}$ are GSIPs in spaces $\left(E_{1}\left(H_{1} \times H_{2}\right)\right) \times\left(W\left(H_{1} \times H_{2}\right)\right),\left(E_{2}\left(H_{1} \times H_{2}\right)\right) \times\left(W\left(H_{1} \times H_{2}^{2}\right)\right)$ respectively.

Construct a new GSIP $[\cdot, \cdot]_{E_{3}}$ in the space $\left(H_{1} \times H_{2}\right) \times W\left(H_{1} \times H_{2}\right)$ such that

$$
[\widehat{f}, \widehat{g}]_{E_{3}}=\left[f_{1}, g_{1}-g_{2}\right]_{E_{1}}+\left[f_{2}, g_{1}-g_{2}\right]_{E_{2}}
$$

where

$$
\widehat{f}=f_{1}+f_{2}, \quad \widehat{g}=g_{1}+g_{2} \in H_{1} \times H_{2}, W \widehat{g}=g_{1}-g_{2}
$$

Similar to Theorem 3.1, $[\cdot, \cdot]_{E_{3}}$ is a GSIP in the space $\left(H_{1} \times H_{2}\right) \times\left(H_{1} \times H_{2}\right)$ because $[\cdot, \cdot]_{E_{1}},[\cdot, \cdot \cdot]_{E_{2}}$, are GSIP in the spaces $\left(E_{1}\left(H_{1} \times H_{2}\right)\right) \times\left(W\left(H_{1} \times H_{2}\right)\right),\left(E_{2}\left(H_{1} \times\right.\right.$ $\left.\left.H_{2}\right)\right) \times\left(W\left(H_{1} \times H_{2}\right)\right)$ respectively. Hence we have the generalized Schwarz inequality in $\left(\left(H_{1} \times H_{2}\right) \times\left(H_{1} \times H_{2}\right),[\cdot, \cdot]_{E_{3}}\right)$ :

$$
\left|[\widehat{f}, \widehat{g}]_{E_{3}}\right| \leq\left|[\widehat{f}, \widehat{f}]_{E_{3}}\right|^{1 / p}\left|[\widehat{g}, \widehat{g}]_{E_{3}}\right|^{1 / q}, \widehat{f}, \widehat{g} \in\left(H_{1} \times H_{2}\right) \times\left(H_{1} \times H_{2}\right)
$$

It is enough to remark that since for any $\widehat{f}, \widehat{g}$,

$$
\begin{aligned}
\left|[\widehat{f}, \widehat{g}]_{E_{3}}\right| & =\left|\left[f_{1}, g_{1}-g_{2}\right]_{E_{1}}+\left[f_{2}, g_{1}-g_{2}\right]_{E_{2}}\right| \\
& =\left|\left[f_{1}, g_{1}-g_{2}\right]_{12}+\left[f_{2}, g_{1}-g_{2}\right]_{12}\right| \\
& =\left|[\widehat{f}, W \widehat{g}]_{12}\right|=\left|[W \widehat{f}, \widehat{g}]_{12}\right|
\end{aligned}
$$

the generalized Schwarz inequality follows:

$$
\begin{aligned}
\left|[\widehat{f}, \widehat{g}]_{E_{3}}\right| & \leq\left|\left[f_{1}, f_{1}-f_{2}\right]_{E_{1}}+\left[f_{2}, f_{1}-f_{2}\right]_{E_{2}}\right|^{1 / p}\left|\left[g_{1}, g_{1}-g_{2}\right]_{E_{1}}+\left[g_{1}, g_{1}-g_{2}\right]_{E_{2}}\right|^{1 / q} \\
& =\left|\left[f_{1}, f_{1}-f_{2}\right]_{12}+\left[f_{2}, f_{1}-f_{2}\right]_{12}\right|^{1 / p}\left|\left[g_{1}, g_{1}-g_{2}\right]_{12}+\left[g_{2}, g_{1}-g_{2}\right]_{12}\right|^{1 / q} \\
& =\left|[\widehat{f}, W \widehat{f}]_{12}\right|^{1 / p}\left|[\widehat{g}, W \widehat{g}]_{12}\right|^{1 / q}=\left|[W \widehat{f}, \widehat{f}]_{12}\right|^{1 / p}\left|[W \widehat{g}, \widehat{g}]_{12}\right|^{1 / q} .
\end{aligned}
$$

Hence the formula (4.2) is proved. And we easily prove that

$$
\left|\bar{Q}_{+}(\bar{u}, \bar{v})\right|=0.5\left|\operatorname{Re}\left[A^{*} u, v\right]_{p}+\operatorname{Re}\left[u, A^{* *} v\right]_{p}\right|=0.5\left|\left[W u^{\prime}, v^{\prime}\right]_{12}\right|
$$

where $u^{\prime}=\left\{u, A^{*} u\right\}, v^{\prime}=\left\{v, A^{* *} v\right\} \in H_{1} \times H_{2}$. By using Eq. (4.2),

$$
\begin{aligned}
\left|Q_{+}(\bar{u}, \bar{v})\right| & \leq 0.5\left|\left[W u^{\prime}, u^{\prime}\right]_{12}\right|^{1 / p}\left|\left[W v^{\prime}, v^{\prime}\right]_{12}\right|^{1 / q} \\
& =0.5\left|\operatorname{Re}\left[A^{*} u, u\right]_{p}+\operatorname{Re}\left[u, A^{* *} u\right]_{p}\right|^{1 / p}\left|\operatorname{Re}\left[A^{*} v, v\right]_{p}+\operatorname{Re}\left[v, A^{* *} v\right]_{p}\right|^{1 / q} \\
& =\left|\bar{Q}_{+}(\bar{u}, \bar{u})\right|^{1 / p}\left|\bar{Q}_{+}(\bar{v}, \bar{v})\right|^{1 / q} .
\end{aligned}
$$

Therefore $\left(\bar{H}_{+}, \bar{Q}_{+}\right)$is a GSIP space. By similar reasoning we also conclude that $\left(\bar{H}_{-},-\bar{Q}_{-}\right)$is a GSIP space.

Then the quotient spaces $\bar{H}_{ \pm} / G\left(L_{0}\right)$ exist. The forms $\bar{Q}_{ \pm}$of $\bar{H}_{ \pm}$bring about the forms of the quotient spaces $\bar{H}_{ \pm} / G\left(L_{0}\right)$. It is easy to see that $\left(\bar{H}_{ \pm} / G\left(L_{0}\right), \pm \bar{Q}_{ \pm}\right)$are GSIP spaces. This completes the proof.

Proof of Theorem 4.2. Obviously $\widehat{Q}(\widehat{u}, \widehat{u})=0$ when $\widehat{u} \in G\left(L_{0}\right)$. As $\widehat{u}, \widehat{v} \in \widehat{H}, \overline{\bar{u}}_{0} \in$ $G\left(L_{0}\right)$, first we prove that $\widehat{Q}\left(\widehat{u}+\bar{u}_{0}, \widehat{v}\right)=\widehat{Q}(\widehat{u}, \widehat{v})$,

$$
\begin{align*}
\widehat{Q}\left(\widehat{u}+\overline{\bar{u}}_{0}, \widehat{v}\right) & =\bar{Q}_{+}\left(\bar{u}_{+}+\overline{\bar{u}}_{0}, \bar{v}_{+}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right),  \tag{4.4}\\
\bar{Q}_{+}\left(\bar{u}_{+}+\overline{\bar{u}}_{0}, \bar{v}_{+}\right) & =Q\left(\bar{u}_{+}+\overline{\bar{u}}_{0}, \bar{v}_{+}\right) \operatorname{sgn} Q\left(\bar{v}_{+}, \bar{v}_{+}\right) \\
& =\operatorname{Re}\left[A^{*}\left(u_{+}+u_{0}\right), v_{+}\right]_{p} \operatorname{sgn} Q\left(\bar{v}_{+}, \bar{v}_{+}\right) \\
& =\operatorname{Re}\left[A^{*} u_{+}, v_{+}\right]_{p} \operatorname{sgn} Q\left(\bar{v}_{+}, \bar{v}_{+}\right)+\operatorname{Re}\left[A u_{0}, v_{+}\right]_{p} \operatorname{sgn} Q\left(\bar{v}_{+}, \bar{v}_{+}\right) \\
& =\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}\right)+\operatorname{Re}\left[A u_{0}, v_{+}\right]_{p} \operatorname{sgn} Q\left(\bar{v}_{+}, \bar{v}_{+}\right),
\end{align*}
$$

where $\bar{u}_{+}=\left\{u_{+}, L_{1} u_{+}\right\}, \bar{v}_{+}=\left\{v_{+}, L_{1} v_{+}\right\} \in \bar{H}_{+}, \overline{\bar{u}}_{0}=\left\{u_{0}, L_{0} u_{0}\right\} \in G\left(L_{0}\right)$.
Now we need to prove $\operatorname{Re}\left[A u_{0}, v_{+}\right]_{p}=0$. By using the same notation for $W$ and the form (4.2) of Theorem 4.1 we have

$$
\begin{aligned}
\left|\operatorname{Re}\left[A u_{0}, v_{+}\right]_{p}\right| & =\left|Q\left(\overline{\bar{u}}_{0}, \bar{v}_{+}\right)\right|=0.5\left|\left[W u_{0}^{\prime}, v_{+}^{\prime}\right]_{12}\right| \\
& \leq 0.5\left|\left[W u_{0}^{\prime}, u_{0}^{\prime}\right]_{12}\right|^{1 / p}\left|\left[W v_{+}^{\prime}, v_{+}^{\prime}\right]_{12}\right|^{1 / q} \\
& =0.5\left|\operatorname{Re}\left[A u_{0}, u_{0}\right]_{12}\right|^{1 / p}\left|\left[W v_{+}^{\prime}, v_{+}^{\prime}\right]_{12}\right|^{1 / q}=0
\end{aligned}
$$

where $u^{\prime}=\left\{u, A^{*} u\right\}, \overline{\bar{u}}_{0}=\left\{u, L_{0} u\right\}, \bar{v}_{+}=\left\{v_{+}, L_{1} v_{+}\right\}, v^{\prime}=\left\{v_{+}, A^{* *} v_{+}\right\}$. Hence $\operatorname{Re}\left[A u_{0}, v_{+}\right]_{p}=0$. Then $\bar{Q}_{+}\left(\bar{u}_{+}+\overline{\bar{u}}_{0}, \bar{v}_{+}\right)=\bar{Q}\left(\bar{u}_{+}, \bar{v}_{+}\right)$. The form (4.4) changes into

$$
\widehat{Q}\left(\widehat{u}+\overline{\bar{u}}_{0}, \widehat{v}\right)=\overline{\bar{Q}}\left(\overline{\bar{u}}+\overline{\bar{u}}_{0}, \overline{\bar{v}}\right)=\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right)=\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}}) .
$$

Next we prove

$$
\widehat{Q}\left(\widehat{u}, \widehat{v}+\overline{\bar{v}}_{0}\right)=\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}}), \quad \overline{\bar{u}}, \overline{\bar{v}} \in \bar{H}_{+} \oplus \bar{H}_{-}, \quad \overline{\bar{v}}_{0} \in G\left(L_{0}\right), \quad \overline{\bar{u}}=\bar{u}_{+}+\bar{v}_{-}
$$

It suffices to note that $Q\left(\widehat{u}, \widehat{v}+\bar{v}_{0}\right)=\bar{Q}(\bar{u}, \bar{v})$ by similar reasoning as in the preceding proof. Obviously $\bar{v}_{+}+\overline{\bar{v}}_{0} \in \bar{H}_{+}, \bar{v}_{-}+\overline{\bar{v}}_{0} \in \bar{H}_{-}$. We obtain

$$
\begin{aligned}
\overline{\bar{Q}}\left(\overline{\bar{u}}, \overline{\bar{v}}+\overline{\bar{v}}_{0}\right) & =\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}+\overline{\bar{v}}_{0}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right) \\
& =\overline{\bar{Q}}\left(\bar{u}_{+}+0, \bar{v}_{+}+\bar{v}_{0}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right) \\
& =\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}\right)+\bar{Q}_{-}\left(0, \overline{\bar{v}}_{0}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right) \\
& =\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right)=\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}}) .
\end{aligned}
$$

Then

$$
\widehat{Q}\left(\widehat{u}, \widehat{v}+\overline{\bar{v}}_{0}\right)=\overline{\bar{Q}}\left(\overline{\bar{u}}, \overline{\bar{v}}+\overline{\bar{v}}_{0}\right)=\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}}) .
$$

Therefore

$$
\widehat{Q}(\widehat{u}, \widehat{v})=\overline{\bar{Q}}(\overline{\bar{u}}, \overline{\bar{v}}), \widehat{u}, \widehat{v} \in \widehat{H}, \quad \forall \overline{\bar{u}} \in \widehat{u}, \quad \overline{\bar{v}} \in \widehat{v}
$$

where $\widehat{u}, \bar{v}$ are the cosets of $\overline{\bar{u}}, \overline{\bar{v}}$ respectively. This completes the proof.

Proof of Theorem 4.4. Let $P_{-}: \widehat{H} \rightarrow \bar{H}_{-} / G\left(L_{0}\right)$ be a projection operator such that

$$
\overline{\bar{u}}=\bar{u}_{+}+P_{-} \overline{\bar{u}}, \quad P_{-} \overline{\bar{u}}=\bar{u}_{-}, \quad \overline{\bar{u}} \in \widehat{H}, \quad \bar{u}_{+} \in \bar{H}_{+} / G\left(L_{0}\right), \quad \bar{u}_{-} \in \bar{H} / G\left(L_{0}\right)
$$

First we prove that a necessary and sufficient condition for $\widehat{N}$ to be a maximal negative subspace is that $P_{-} \widehat{N}=\bar{H}_{-} / G\left(L_{0}\right)$. If $\widehat{N}$ is a maximal negative subspace, then $P_{-} \widehat{N} \subset \bar{H}_{-} / G\left(L_{0}\right)$. If $P_{-} \widehat{N}$ does not fill out $\bar{H}_{-} / G\left(L_{0}\right)$ then there exists $\widehat{u} \in\left(\bar{H}_{-} / G\left(L_{0}\right)\right) \backslash\left(P_{-} \widehat{N}\right)$. Hence $\widehat{u} \notin \widehat{N}$ and $\{\widehat{u}\} \cup \widehat{N}$ is a negative subspace in $\widehat{H}$ and properly contain $\widehat{N}$. This is a contradiction. Conversely, if $P_{-} \widehat{N}=\bar{H}_{-} / G\left(L_{0}\right)$ and $\widehat{N}$ is not maximal negative, there exists $\widehat{u} \notin \widehat{N}, \widehat{Q}(\widehat{u}, \widehat{u})<0$. Hence $P_{-} \widehat{u} \in \bar{H}_{-} / G\left(L_{0}\right)$, $P_{-} \widehat{u} \in P_{-} \widehat{N}$. As $\widehat{u}=\left(I-P_{-}\right) \widehat{u}+P_{-} \widehat{u}$, then $\left(I-P_{-}\right) \widehat{u} \in P_{+} \widehat{N}$, where $P_{+}=I-P_{-}$. Then $\widehat{u} \in\left(P_{+}+P_{-}\right) \widehat{N}=\widehat{N}$, but $\widehat{u} \notin \widehat{N}$; this is a contradiction. Hence $\widehat{N}$ is a maximal negative subspace.

Now to prove that if $\widetilde{N}$ is a maximal negative subspace in $\widetilde{H}$, then $\widetilde{N}$ is a maximal negative subspace in $\widehat{H}$. As $\widehat{N}$ is a closed negative subspace, then $P_{-} \widehat{N}$ is a closed subspace in $\bar{H}_{-} / G\left(L_{0}\right)$. If $P_{-} N \neq \bar{T}_{-} / G\left(L_{0}\right)$, then the set $\left.\left\{\left(\bar{H} /{\underset{\sim}{N}}^{( } L_{0}\right)\right) \backslash\left(P_{-} \widehat{N}\right), 0\right\} \cup \widetilde{N}$ is a negative subspace relative to $\widetilde{Q}$ in $\widetilde{N}$ and properly contains $\widetilde{N}$. This is a contradiction. Therefore $\widehat{N}$ is a maximal negative subspace.

This completes the proof.
Proofs of Theorem 4.3 and Theorem 4.5, follow from the above, so we omit the proofs.

## 5. The Maximal Dissipative Extension Representation of $\boldsymbol{p}$ Dissipative Operator

Theorem 5.1. Let $L_{0}=A-S, A$ is a $p$ skew-symmetric operator, and satisfty $\operatorname{Re}[A u, u]_{p}=0, S$ is an reversible GPpS operator. Suppose the maximal dissipative extension of $L_{0}$ is $L$. Then, there is a one to one correspondence between the maximal dissipative extension $L$ of $L_{0}$ and the maximal negative subspace $\widetilde{N}$ of GIIP space $\widetilde{H}$, and

$$
\begin{gather*}
L u=L_{1} u+S^{1 / 2} \varphi(\widehat{u}), L_{1}=A^{*}-S,  \tag{5.1}\\
D(L)=\left\{u \in D\left(L_{1}\right) \mid \widehat{u} \in \widehat{N}, \widehat{N} \text { is the projection of } \widetilde{N} \text { from } \widetilde{H} \text { to } \widehat{H}\right\} \tag{5.2}
\end{gather*}
$$

Proof. Assume that $L$ is the maximal dissipative extension, then

$$
\operatorname{Re}[L u, u]_{p} \leq 0, u \in D(L)
$$

If $u \in D\left(L_{0}\right), v \in D(L)$, then

$$
\begin{aligned}
\operatorname{Re}\left[L_{0} u, v\right]_{p} & =\operatorname{Re}[A u, v]_{p}-\operatorname{Re}[S u, v]_{p}, \\
\operatorname{Re}[A u, v]_{p} & =\operatorname{Re}\left[L_{0} u, v\right]_{p}+\operatorname{Re}[S u, v]_{p}, \\
\left|\operatorname{Re}[A u, v]_{p}\right| & \leq\left|\operatorname{Re}\left[L_{0} u, v\right]_{p}\right|+\left|\operatorname{Re}[S u, v]_{p}\right| \\
& \leq\left(\left\|L_{0} u\right\|+\|S u\|\right)\|v\|^{p-1}, \\
\left|[A u, v]_{p}\right| & =\left|\operatorname{Re}[A u, v]_{p}+i \operatorname{Re}[A u, i v]_{p}\right| \\
& \leq\left|\operatorname{Re}[A u, v]_{p}\right|+\left|\operatorname{Re}[A u, i v]_{p}\right| \\
& \leq 2\left(\left\|L_{0} u\right\|+\|S u\|\right)\|v\|^{p-1} .
\end{aligned}
$$

Then $v \in D\left(A^{*}\right)$. As $S: H_{1} \rightarrow H$ is a bounded operator and $L_{1}=A^{*}-S$, we have $D\left(L_{1}\right) \supset D(L)$. As the operator $S$ is inversive, for arbitrary $v \in D\left(L_{0}\right)$ and $u \in D(L)$, then we have

$$
\begin{aligned}
\left|\left[v, L u-L_{1} u\right]_{p}\right| & =\left|\left[S^{1 / 2} v, S^{-1 / 2}\left(L u-L_{1} u\right)\right]_{p}\right| \\
& \leq\left\|S^{1 / 2} v\right\|\left(\left\|S^{1 / 2} L u\right\|_{2}+\left\|S^{1 / 2} L_{1} u\right\|_{2}\right)^{p-1}
\end{aligned}
$$

From the Riesz Representation Theorem of GSIP space, there exists $\check{f} \in \check{H}$, so that for any $v \in D\left(L_{0}\right)$, we have

$$
\left[v, L u-L_{1} u\right]_{p}=\left[S^{1 / 2} v, \check{f}\right]_{p}=\left[v, S^{1 / 2} \check{f}\right]_{p} .
$$

Hence $L u-L_{1} u=S^{1 / 2} \check{f}$, or $L u=L_{1} u+S^{1 / 2} \check{f}$. As $H_{0}=\overline{D\left(L_{0}\right)} \supset H_{1}$, then $L u=L_{1} u+S^{1 / 2} \check{f}, u \in H_{1}$.

Let $\widehat{u}, \widehat{v} \in \widehat{H}, \widehat{u}, \widehat{v}$ are the cosets of $\bar{u}, \bar{v}$ and $\bar{u}=\bar{u}_{+}+\bar{u}_{-}, \bar{v}=\bar{v}_{+}+\bar{v}_{-}, \bar{u}_{+}, \bar{v}_{+} \in \bar{H}_{+}$, $\bar{u}_{-}, \bar{v}_{-} \in \bar{H}_{-}$. From Theorem 4.2 we obtain

$$
\widehat{Q}(\widehat{u}, \widehat{v})=\overline{\bar{Q}}(\bar{u}, \bar{v})=\bar{Q}_{+}\left(\bar{u}_{+}, \bar{v}_{+}\right)+\bar{Q}_{-}\left(\bar{u}_{-}, \bar{v}_{-}\right)
$$

The following inequality holds:

$$
\begin{equation*}
\widehat{Q}(\widehat{u}, \widehat{v})-m\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\|\check{f}\|^{p} \leq 0 \tag{5.3}
\end{equation*}
$$

where $m$ is an fixed constant, $\bar{u}=\left\{u, L_{1} u\right\} \in G\left(L_{1}\right), \widehat{u}$ is a coset of $\bar{u}$. First, we prove

$$
\begin{equation*}
Q(\bar{u}, \bar{u})-m\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\|\check{f}\|^{p} \leq 0, \bar{u} \in G\left(L_{1}\right) \tag{5.4}
\end{equation*}
$$

As $Q(\bar{u}, \bar{u})=\operatorname{Re}\left[L_{1} u, u\right]_{p}-\operatorname{Re}[S u, u]_{p}$, then $\operatorname{Re}\left[L_{1} u, u\right]_{p}=Q(\bar{u}, \bar{u})-\operatorname{Re}[S u, u]_{p}$. As $\operatorname{Re}[L u, u]_{p} \leq 0$, then $\operatorname{Re}\left[L_{1} u+S^{1 / 2} \check{f}, u\right]_{p} \leq 0$,

$$
Q(\bar{u}, \bar{u})-\operatorname{Re}[S u, u]_{p}+\operatorname{Re}\left[S^{1 / 2} \check{f}, u\right]_{p} \leq 0 .
$$

Hence, to prove (5.2), we only have to prove

$$
\begin{equation*}
-m\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\|\check{f}\|^{p} \leq-[S u, u]_{p}+\operatorname{Re}\left[S^{1 / 2} \check{f}, u\right]_{p} \tag{5.5}
\end{equation*}
$$

In fact

$$
\begin{aligned}
\operatorname{Re}\left[-S^{1 / 2} u+\check{f}, S^{1 / 2} u\right]_{p} & \leq\left|\left[-S^{1 / 2} u+\check{f}, S^{1 / 2} u\right]_{p}\right| \\
& \leq\left\|S^{1 / 2} u-\check{f}\right\|\left\|S^{1 / 2} u\right\|^{p-1} \\
& \leq \frac{1}{p}\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\frac{1}{q}\left\|S^{1 / 2} u\right\|^{q(p-1)}
\end{aligned}
$$

where $1 / p+1 / q=1, p>1$.

The left-hand side of the above inequality is equal to $-[S u, u]_{p}+\operatorname{Re}\left[\check{f}, S^{1 / 2 u}\right]_{p}$. So

$$
[S u, u]_{p}-\operatorname{Re}\left[\check{f}, S^{1 / 2} u\right]_{p} \geq-\frac{1}{p}\|S 1 / 2 u-\check{f}\|^{p}-\frac{1}{q}\left\|S^{1 / 2} u\right\|^{p}
$$

By using the above inequality in (5.5), we have to prove
or

$$
-m\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\|\check{f}\|^{p}-\frac{1}{p}\left\|S^{1 / 2} u-\check{f}\right\|^{p}-\frac{1}{q}\left\|S^{1 / 2} u\right\|^{p} \leq 0
$$

$$
\begin{equation*}
\left(-m-\frac{1}{p}\right)\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\|\check{f}\|^{p} \leq \frac{1}{q}\left\|S^{1 / 2} u\right\|^{p} \tag{5.6}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\operatorname{Re}\left[-S^{1 / 2} u+\check{f}, \check{f}\right]_{p} & \leq\left|\left[-S^{1 / 2} u+\check{f}, \check{f}\right]_{p}\right|^{p} \\
& \leq\left\|-S^{1 / 2} u+\check{f}\right\|\|\check{f}\|^{p-1} \\
& \leq \frac{r}{p}\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\frac{1}{q r}\|\check{f}\|^{p}, \quad 1 / p+1 / q=1, \quad p>1, r>0 .
\end{aligned}
$$

Then

$$
\begin{gathered}
q \operatorname{Re}\left[-S^{1 / 2} u+\check{f}, \check{f}\right]_{p} \leq \frac{q r}{p}\left\|-S^{1 / 2} u+\check{f}\right\|^{p}+\frac{1}{r}\|\check{f}\|^{p}, \\
-q \operatorname{Re}\left[S^{1 / 2} u, \check{f}\right]_{p}+(q-1 / r)\|\check{f}\|^{p} \leq \frac{q r}{p}\left\|S^{1 / 2} u-\check{f}\right\|^{p}, \\
-\frac{q}{q-1} \operatorname{Re}\left[S^{1 / 2} u, \check{f}\right]_{p}+\frac{r q-1}{r(q-1)}\|\check{f}\|^{p} \leq \frac{q r}{p(q-1)}\left\|S^{1 / 2} u-\check{f}\right\|^{p}, \\
-\frac{q}{q-1} \operatorname{Re}\left[S^{1 / 2} u, \check{f}\right]_{p} \leq \frac{q r}{p(q-1)}\left\|S^{1 / 2} u-\check{f}\right\|^{p}-\frac{r q-1}{r(q-1)}\|\check{f}\|^{p} .
\end{gathered}
$$

As $1 / p+1 / q=1, q=p(q-1), p=q /(q-1)$, then

$$
\begin{gathered}
\frac{1-r q}{r(q-1)}\|\check{f}\|^{p}-r\left\|S^{1 / 2} u-\check{f}\right\|^{p} \leq p \operatorname{Re}\left[-S^{1 / 2} u, \check{f}\right]_{p} \leq p\left\|S^{1 / 2} u\right\|\|\check{f}\|^{p-1} \\
\leq p\left(\frac{l}{p}\left\|S^{1 / 2} u\right\|^{p}+\frac{1}{q l}\|\check{f}\|^{(p-1) q}\right)=l\left\|S^{1 / 2} u\right\|^{p}+\frac{p}{q l}\|\check{f}\|^{p}
\end{gathered}
$$

where $l>0$. By simplifying, we get

$$
\left(\frac{1-r q}{r(q-1)}-\frac{p}{q l}\right)\|f\|^{p}-r\left\|S^{1 / 2} u-\check{f}\right\|^{p} \leq l\left\|S^{1 / 2} u\right\|^{p}
$$

Let $r$ be such that $0<r<\min \left\{\frac{1}{q}, \frac{1}{\sqrt{2 q(q-1)}+q}, \frac{1}{q l+1}\right\}$. Then we have

$$
\begin{gathered}
t=\frac{1-r q}{r(q-1)}-\frac{p}{q l}>0 \\
\|\check{f}\|^{p} \leq \frac{r}{t}\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\frac{l}{t}\left\|S^{1 / 2} u\right\|^{p} .
\end{gathered}
$$

Using Inequality (5.6) it follows that

$$
\begin{equation*}
\left(-m-\frac{1}{p}+\frac{r}{t}\right)\left\|S^{1 / 2} u-\check{f}\right\|^{p} \leq\left(\frac{1}{q}-\frac{l}{t}\right)\left\|S^{1 / 2} u\right\|^{p} \tag{5.7}
\end{equation*}
$$

Next we choose $l>0$ such that $\frac{1}{q}-\frac{l}{t}>0$. In fact

$$
\frac{1}{q}-\frac{l}{t}=\frac{r(q-1)\left[l^{2}+\frac{1-r q}{q r(q-1)} l+\frac{1}{q(q-1)}\right]}{(r q-1)\left[l+\frac{1}{r q-1}\right]}
$$

Because

$$
r q-1<0, \quad r<\frac{1}{\sqrt{2 q(q-1)}+q}
$$

then the equation

$$
l^{2}+\frac{1-q}{q r(q-1)} l+\frac{1}{q(q-1)}=0
$$

exists with two real solutions $a_{1}$ and $a_{2}$ such that

$$
a_{1}, a_{2}=-\frac{1-r q}{q r(q-1)} \pm \sqrt{\frac{1}{q(q-1)}\left[\frac{(1-r q)^{2}}{q(q-1) r^{2}}-4\right]}
$$

Let $a_{3}=\frac{1}{1-r q}$. Then we change (5.7) into

$$
\frac{1}{q}-\frac{l}{t}=\frac{r(q-1)\left(l-a_{1}\right)\left(l-a_{2}\right)}{(r q-1)\left(l-a_{3}\right)}
$$

Hence we can choose $l>0$ such that $\frac{1}{q}-\frac{l}{t}>0$.
We take $m=-\frac{1}{p}+\frac{r}{t}$. At this time, the left-hand of (5.7) is 0 and the coefficient of the right-hand $=\frac{1}{q}-\frac{l}{t}>0$. Therefore (5.7) holds naturally. Hence (5.4) holds. So

$$
\begin{equation*}
Q(\bar{u}, \bar{u})-m\left\|S^{1 / 2} u-\check{f}\right\|^{p}+\|\check{f}\|^{p} \leq 0, \quad \bar{u}=\left\{u, L_{1} u\right\} \in G\left(L_{1}\right) . \tag{5.8}
\end{equation*}
$$

It follows that $Q(\bar{u}, \bar{u}) \leq 0$ and $\bar{u} \in \bar{H}_{-}, \overline{\bar{Q}}(\bar{u}, \bar{u})=\bar{Q}_{-}(\bar{u}, \bar{u})$. Hence $\widehat{Q}(\widehat{u}, \widehat{u})=$ $\overline{\bar{Q}}(\bar{u}, \bar{u})=\bar{Q}_{-}(\bar{u}, \bar{u})$. Equation (5.3) holds.

Analogous to the discussion of the form (1.20) in [12], if $v$ lies in $D\left(L_{0}\right)$, then $L v=L_{1} v=L_{0} v$; it follows from this that $\check{f}$ in $L u=L_{1} u+S^{1 / 2} \check{f}$ depends only on the $\widehat{H}$ boundary coset to which a belongs; that is, $\check{f}=\varphi(\widehat{u})$. Since $D\left(L_{0}\right)$ is dense in $H_{1}$ and $S^{1 / 2}$ is bounded on $H_{1}$ to $H$, we see that $S^{1 / 2} D\left(L_{0}\right)$ will be dense in $\check{H}$ and so will the image of $S^{1 / 2}$ acting on the first components of any boundary coset. Consequently, (5.3) holds for all $u$ in a given boundary coset only if it holds with the middle term omitted. In other words,

$$
\widehat{Q}(\widehat{u}, \widehat{u})+\|\varphi(\widehat{u})\|^{p} \leq 0, \quad u \in D(L) .
$$

Therefore $\{\{\widehat{u}, \varphi(\widehat{u})\}, \widehat{u} \in D(L)\}$ forms a negative subspace corresponding to $\widetilde{Q}$ in $\widetilde{H}$.
On the other hand if

$$
L u=L_{1} u+S^{1 / 2} \varphi(\widehat{u}), D(L)=\left\{u \in D\left(L_{1}\right), \widehat{u} \in \widehat{N}\right\}
$$

and $L$ is the extension of $L_{0}$ and $\{\{\widehat{u}, \varphi(\widehat{u})\} \mid u \in D(L)\}$ is a maximal negative subspace of $\widetilde{H}$, then we can show that (5.3) holds. So $L$ is a $p$ dissipative operator.

Therefore there exists one to one correspondence between the maximal negative subspace of $\widetilde{H}$ and the maximal dissipative extension representation (5.1), (5.2).

This completes the proof.

## 6. Application and Remark

In this section we study the maximal dissipative extension of the Schrödinger operator by means of the above theory.

According to [19], the Schrödinger operator is $-h \Delta+V(x)$, defined in $C^{\infty}(M), M$ is $C^{\infty}$ compact Riemann manifold. The Schrödinger operator has a unique dissipative extension in Sobolev space $H^{2}(M)$. But if the domain isn't a Riemann manifold, the operator becomes complex and the study of the Schrödinger equation becomes difficult (see [8, 9]. For this reason, we study the operator for the domain in Banach space, and give the maximal dissipative extension representation of the operator. Suppose $X=$ $L^{p^{\prime}}[0,2 \pi], p^{\prime} \neq 2,1<p^{\prime}<\infty$, its GSIP,$[\cdot, \cdot]_{p}$ as follows:

$$
[f, g]_{p}=\int_{0}^{2 \pi} f \bar{g}|g|^{p-2} d x, 1<p<\infty
$$

where $p$ may be different from $p^{\prime}$. Obviously the norm in $X$ is $\|f\|=[f, f]^{1 / p}$,

$$
\begin{aligned}
& L_{0} f=i f^{\prime \prime}-f, \\
& D\left(L_{0}\right)=\left\{f: f, f^{\prime}, f^{\prime \prime} \in L^{p^{\prime}}[0,2 \pi], \quad f(0)=f(2 \pi),\right. \\
& f^{\prime}(0)\left.=f^{\prime}(2 \pi), 1<p^{\prime}<\infty\right\},
\end{aligned}
$$

$L_{0}$ is a certain type of Schrödinger operator which will be studied.
Suppose $A: D\left(L_{0}\right) \rightarrow X, A f=i f^{\prime \prime}$. Let $G\left(L_{0}\right)=\left\{\left\{f, L_{0} f\right\}: f \in D\left(L_{0}\right)\right\}$. In $X \times X$, construct $Q(\cdot, \cdot)$ such that

$$
\begin{aligned}
Q(f, g) & =(f g)^{\prime}(2 \pi)-(f g)^{\prime}(0) \\
& =f^{\prime}(2 \pi) g(2 \pi)-f^{\prime}(0) g(0)+f(2 \pi) g^{\prime}(2 \pi)-f(0) g^{\prime}(0) .
\end{aligned}
$$

Let $\overline{\bar{H}}=\bar{H}_{+} \oplus \bar{H}_{-}$, where

$$
\bar{H}_{+}=\overline{\operatorname{span}}\{f \in X, Q(f, f) \geq 0\}, \quad \bar{H}_{-}=\overline{\operatorname{span}}\{f \in X, Q(f, f) \leq 0\} .
$$

Denote $\widehat{H}=\overline{\bar{H}} / G\left(L_{0}\right)$. Construct $\widehat{Q}=\widehat{Q}_{+}+\widehat{Q}_{-}$in $\widehat{H}$, where

$$
\begin{aligned}
\widehat{Q}_{+}\left(\widehat{f}_{+}, \widehat{g}_{+}\right) & =Q\left(\widehat{f}_{+}, \widehat{g}_{+}\right) \operatorname{sgn} Q\left(\widehat{g}_{+}, \widehat{g}_{+}\right), \quad \widehat{f}_{+}, \widehat{g}_{+} \in \bar{H}_{+} / G\left(L_{0}\right), \\
\widehat{Q}_{-}\left(\widehat{f}_{-}, \widehat{g}_{-}\right) & =Q\left(\widehat{f}_{-}, \widehat{g}_{-}\right)\left(-\operatorname{sgn} Q\left(\left(\widehat{g}_{-}, \widehat{g}_{-}\right)\right), \quad \widehat{f}_{-}, \widehat{g}_{-} \in \bar{H}_{-} / G\left(L_{0}\right) .\right.
\end{aligned}
$$

Suppose $\check{H}=\left\{f: f, f^{\prime} \in X\right\}, \widetilde{H}=\widehat{H} \oplus \check{H}$. For any $\widetilde{f}=\{\widehat{f}, \check{f}\}$, define $\widetilde{Q}$,

$$
\widetilde{Q}(\widetilde{f}, \widetilde{g})=\widehat{Q}(\widehat{f}, \widehat{g})+[\check{f}, \check{g}]_{p}
$$

It is easy to see that

$$
\widehat{H}=\left\{\widehat{u}=\left\{u(0), u(2 \pi), u^{\prime}(0), u^{\prime}(2 \pi)\right\}, u \in X\right\}
$$

Let $\widehat{u} \in \widehat{N}$, where $\widehat{N}$ is a maximal negative subspace in $(\widehat{H}, \widehat{Q})$. Then $\widehat{Q}(\widehat{u}, \widehat{u}) \leq 0$ and

$$
\widehat{Q}(\widehat{u}, \widehat{u})=2\left(u^{\prime}(2 \pi) u(2 \pi)-u^{\prime}(0) u(0)\right) \leq 0
$$

or

$$
\alpha u^{\prime}(2 \pi) u(2 \pi)+\beta u^{\prime}(0) u(0)=0 \quad \text { and }|\beta| \leq|\alpha| .
$$

Then $\widehat{N}$ is a two dimensional subspace.
Let $\widetilde{N}$ be a maximal negative subspace in $\widetilde{H}$. From Theorem $4.4, \widehat{N}$ which is a projection of $\widetilde{N}$ in $\widehat{H}$, is a maximal negative subspace. Because

$$
\check{u}=\varphi(\widehat{u}), \check{u} \in \check{H}, \quad \widehat{u} \in \widehat{H}, \quad \widetilde{u}=\{\widehat{u}, \quad \check{u}\} \in \widetilde{N}
$$

and $\varphi$ is a linear mapping, then the $\widehat{H}$ - component of $\widetilde{N}$ is linearly dependent on the $\widehat{H}$ - component by using Theorem 4.5. Thus $\widetilde{Q}(\widetilde{u}, \bar{u}) \leq 0$, or

$$
\widetilde{Q}(\widetilde{u}, \widetilde{u})=\widehat{Q}(\widehat{u}, \widehat{u})+[\check{u}, \check{u}]_{p} \leq 0 .
$$

Then

$$
-2(\beta / \alpha+1) u^{\prime}(0) u(0)+\|\varphi(\widehat{u})\|_{p}^{p} \leq 0 .
$$

Hence $\varphi(u)=u^{\prime}(0) u(0) f$, where $f$ satisfies

$$
-2(\beta / \alpha+1) u^{\prime}(0) u(0)+\|f\|^{p}\left|u^{\prime}(0) u(0)\right|^{p} \leq 0 .
$$

Theorem 6.1. The operator $A$ is a p symmetric operator in Banach space $L^{p^{\prime}}[0,2 \pi]$.
Proof. Let $f, g \in D\left(L_{0}\right)$, then

$$
\begin{aligned}
{[A f, g]_{p} } & =\int_{0}^{2 \pi} i f^{\prime \prime} \bar{g}|\bar{g}|^{p-2} d x=\int_{0}^{2 \pi} i f^{\prime \prime} \bar{g}\left((p-2) \int_{0}^{|g|} \alpha^{p-3} d \alpha\right) d x \\
& =\int_{0}^{2 \pi} i f^{\prime \prime} \bar{g}(p-2)\left(\int_{0}^{\infty} \chi_{[0,|g|]}(x) \alpha^{p-3} d \alpha\right) d x \\
& =(p-2) \int_{0}^{\infty} \alpha^{p-3}\left(\int_{0}^{2 \pi} i \bar{g} \chi_{[|g|>\alpha]}(x) d f^{\prime}\right) d \alpha \\
& =(p-2) \int_{0}^{\infty} \alpha^{p-3}\left(-\int_{0}^{2 \pi} i \overline{g^{\prime}} \chi_{[|g|>\alpha]}(x) d f\right) d \alpha \\
& =(p-2) \int_{0}^{\infty} \alpha^{p-3} \int_{0}^{2 \pi} f\left(\overline{-i g^{\prime \prime}}\right) \chi_{[|g|>\alpha]}(x) d x d \alpha \\
& =-(p-2) \int_{0}^{\infty} \alpha^{p-3}\left(\int_{0}^{2 \pi} f\left(\overline{-i g^{\prime \prime}}\right) \chi_{[|g|>\alpha]}(x) d x\right) d \alpha \\
& =-\int_{0}^{2 \pi} f\left(\overline{i g^{\prime \prime}}\right) \int_{0}^{\left|g^{\prime \prime}\right|} d\left(\alpha^{p-2}\right) d x \\
& =-\int_{0}^{2 \pi} f\left(\overline{i g^{\prime \prime}}\right)\left|g^{\prime \prime}\right|^{p-2} d x d=-[f, A g]_{p}
\end{aligned}
$$

where $\chi_{E}(x)$ denotes the characteristic function in $E$. Hence the operator $A$ is a $p$ symmetric operator.

Corollary 6.1. $\operatorname{Re}[A u, u]_{p}=0$ and $L_{0}$ is a $p$ dissipative operator in $L^{p^{\prime}}[0,2 \pi]$.
Theorem 6.2. $\left(\bar{H}_{+} / G\left(L_{0}\right), \widehat{Q}_{+}\right),\left(\bar{H}_{-} / G\left(L_{0}\right),-\widehat{Q}_{-}\right)$are GSIP spaces.
Theorem 6.3. In $X=L^{p \prime}[0,2 \pi], 1<p^{\prime}<\infty$, suppose the Schrödinger operator

$$
L_{0}=i f^{\prime \prime}-f, D\left(L_{0}\right)=\left\{f: f, f^{\prime}, f^{\prime \prime} \in X, f(0)=f(2 \pi), f^{\prime}(0)=f^{\prime}(2 \pi)\right\}
$$

If the maximal dissipative extension of $L_{0}$ is $L$, then there is one to one correspondence between the operator $L$ and the maximal negative subspace $\widetilde{N}$ of $\widetilde{H}$ and

$$
\begin{gathered}
L u=i u^{\prime \prime}-u+u^{\prime}(0) u(0) f \\
D(L)=\left\{u\left|u, u^{\prime}, u^{\prime \prime} \in X, \alpha u^{\prime}(2 \pi) u(2 \pi)+\beta u^{\prime}(0) u(0)=0,|\beta| \leq|\alpha|\right\},\right.
\end{gathered}
$$

where $f \in X$ satisfies the following:

$$
-2(\beta / \alpha+1) u^{\prime}(0) u(0)+\left|u^{\prime}(0) u(0)\right|^{p}\|f\|^{p} \leq 0 .
$$

By using Theorem 4.1, 4.3, 5.1, we easily prove Theorem 6.2, 6.3. Therefore we omit the proofs.

Remark 6.1. In $X=L^{p}[0,2 \pi], 1<p<\infty, p \neq 2$, we may define the $\operatorname{SIP}($ see $[4,15])$ as follows:

$$
[f, g]=\|g\| \int_{0}^{2 \pi} f\left(\frac{|g|}{\|g\|}\right)^{p-2} \operatorname{sgn} g d x, \quad g \in X
$$

Now consider following the operator $L_{1}$ in $X$ :

$$
\begin{gathered}
L_{1} f=i f^{\prime \prime}-\alpha f, \quad \alpha>0 \\
f \in D\left(L_{1}\right)=\left\{f: f, f,^{\prime} f^{\prime \prime} \in X, f(0)=f(2 \pi), f^{\prime}(0)=f^{\prime}(2 \pi), 1<p<\infty\right\}
\end{gathered}
$$

Similar to Theorem 6.1 and Corollary 6.1 we have that $L_{1}$ is a dissipative operator in $X$. Thus similarly we obtain Theorems 6.2 and 6.3.

Remark 6.2. Dissipative operators play an increasingly important role as research on nonselfadjoint operators proceeds. Many interesting initial value problems in partial differential equations are defined in Banach space. In the case considered here, we study the maximal dissipative extension representation of the operator in Banach space by introducing the GIIP space and researching the GSIP space. Especially we apply the theory to the Schrödinger operator. The Schrödinger operator $-h \Delta+V$ (or $i \Delta-i V$ ) is considered, where $V(x)$ is the potential. If $V(x)$ doesn't satisfy the $L^{2}$ integrable, it is $L^{p}$ integrable or $C[0,2 \pi]$ (see $[20,21]$ ). Then the particles in the Schrödinger equation will cause collision and scattering. Especially, for $i \Delta-i V, V(x)$ is a complex function in $L^{p}[0,2 \pi]$, and the particles cause scattering. It is difficult to study scattering in quantum mechanics at present. In the paper we try to study one of the Schrödinger operators in $L^{p}[0,2 \pi]$, where $V(x)$ is an imaginary number. Perhaps it is a new method to study the scattering of the Schrödinger equation in Banach space. But we don't know how to connect the maximal dissipative extension representation of the Schrödinger operator with the scattering of the Schrödinger equation in Banach space yet. This work will be on-going. Moreover research on the operator theory in GIIP space will be very meaningful.

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