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# ON PROPERTIES OF HYPERCHAOS: CASE STUDY*

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**ABSTRACT**: Some properties of hyperchaos are exploited by studying both uncoupled and coupled CML. In addition to usual properties of chaotic strange attractors, there are other interesting properties, such as: the number of unstable periodic points embedded in the strange attractor increases dramatically increasing and a large number of low-dimensional chaotic invariant sets are contained in the strange attractor. These properties may be useful for regarding the edge of chaos as the origin of complexity of dynamical systems.

**KEY WORDS**: hyperchaos, strange attractor, unstable periodic point, pattern formation

## 1 INTRODUCTION

The dynamical system with many degrees of freedom often exhibit complicated behaviors. Recently, this subject has drawn researchers more attention. The dynamical behavior of so called "hyperchaos" with more than one positive Lyapunov exponents often appears in the numerical simulation of a multi-degree-of-freedom system. So does in the numerical investigation on coupled maps of lattice (CML)[1]. In the analysis of zigzag phenomenon in CML, we have derived a two-dimensional map in which the states with two positive Lyapunov exponents occur. We conjecture that the hyperchaotic states of a multi-degree-of-freedo system has exerted great effects on the origin of complicated behavior. Therefore, it is necessary to study hyperchaos in multi-degree-of-freedom systems.

In order to simplify the discussion and to avoid mathematical difficulties, we introduce a simple uncoupled two-dimensional logistic map at first

$$L: \begin{cases} x_{n+1} = 4x_n(1-x_n) \\ y_{n+1} = 4y_n(1-y_n) \end{cases} \tag{1}$$

where $L: [0,1]\times[0,1] \to [0,1]\times[0,1]$. Model (1) may contain certain primary information of hyperchaos. Thus some important properties of hyperchaos can be summarised by analysing

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Eq.(1). Then a coupled two-dimensional map in [2] is treated as an example to illustrate these properties of hyperchaos by both theoretic analysis and numerical simulations.

## 2 THE DYNAMICAL BEHAVIOR AND THE STRUCTURE OF STRANGE ATTRACTOR OF MAP $L$

The dynamical behaviors of map $L$ are given as follow.

**Property 1**: $L$ is chaotic in the sense of Marotto[3~5].

Proof: Logistic map $f: x_{n+1} = 4x_n(1-x_n)$ is chaotic in the sense of Marotto, namely in a small neighborhood of the fixed point $\tilde{x}_0$, there is a point $\tilde{x}$ and a positive $M > 0$, such that $f^M(\tilde{x}) = \tilde{x}_0$ and $|Df^M(\hat{x})| \neq 0$. Now, $(x_0, y_0)$ is a fixed point of map $L$, where $x_0 = y_0 = \tilde{x}_0$. It is obvious that a point $(x, y)$, where $x = y = \tilde{x}$, can be found in a neighborhood of the fixed point, such that $L^M(x, y) = (x_0, y_0)$ and $|DL^M(x, y)| \neq 0$. So $L$ is chaotic in the sense of Marotto.

For Eq.(1), $\forall x \in [0,1]$ and $\forall y \in [0,1]$, a corresponding symbol series in $\Sigma_2$ can be defined. Then the shift map $\sigma: \Sigma_2 \times \Sigma_2 \to \Sigma_2 \times \Sigma_2$ can be set, namely $\forall s = (.s_0 s_1 s_2 \cdots) \in \Sigma_2$ and, $\forall t = (.t_0 t_1 t_2 \cdots) \in \Sigma_2$

$$\sigma(s, t) = (\sigma s, \sigma t) = (s', t')$$

where $s' = (.s_1 s_2 s_3 \cdots)$ and $t' = (.t_1 t_2 t_3 \cdots)$.

**Property 2**: $L|_{[0,1]\times[0,1]} \sim \sigma|_{\Sigma_2 \times \Sigma_2}$.

Proof: Property 2 is easily derived from the fact that map $f: x_{n+1} = 4x_n(1-x_n)$ possesses property $f|_{[0,1]} \sim \sigma|_{\Sigma_2}$.

As a result of the dynamical behavior of $\sigma$ on $\Sigma_2 \times \Sigma_2$, $L$ is topologically transitive and sensitively dependent on initial conditions on $[0,1] \times [0,1]$.

**Property 3**: $L$ has two positive Lyapunov exponents.

Proof: The result is obvious.

According to properties 1-3, the dynamical behavior of $L$ is hyperchaotic. Therefore it has a strange attractor with hyperchaotic behavior on $[0,1] \times [0,1]$. Based on the characters of one-dimensional logistic map, this strange attractor must spread all over the region $[0,1] \times [0,1]$. This strange attractor has the following properties.

**Property 4**: $A = \overline{W^u(p)}$ , where $p$ is an expanding fixed point of $L$ on $[0,1] \times [0,1]$.

Proof: $A$ is an invariant set of $L, p \in A$, so $W^u_{\text{loc}}(p) \subset A$

$$L^n(W^u_{\text{loc}}(p)) \subset A$$

$$\bigcup_{n\geq 0} L^n(W^u_{\text{loc}}(p)) \subset A \qquad \overline{\bigcup_{n\geq 0} L^n(W^u_{\text{loc}}(p))} \subset A$$

Hence $\overline{W^u(p)} \subset A$. On the other hand, $L$ is topological transitive on $A$, and $W^u(p)$ is an open set, so $A \subset \overline{\bigcup_{n\geq 0} L^n(W^u_{\text{loc}}(p))}$, namely $A \subset \overline{W^u(p)}$. Thus is $A = \overline{W^u(p)}$ proved.

**Property 5**: The periodic points in $A$ is dense.

Proof: $\forall (x_0, y_0) \in [0,1] \times [0,1]$, $V$ is an $\varepsilon$-neighborhood of $(x_0, y_0)$. $U$ is also a neighborhood of $(x_0, y_0)$, of which the diameter is less than $\varepsilon/\sqrt{2}$. Based on the fact that the periodic points of logistic map with $\lambda = 4$ is dense in $[0,1]$, there exist periodic points $\tilde{x}$ and $\tilde{y}$ of logistic maps $x_{n+1} = 4x_n(1-x_n)$ and $y_{n+1} = 4y_n(1-y_n)$ in the $U \cap \{x = x_0\}$ and $U \cap \{y = y_0\}$. Then $(\tilde{x}, \tilde{y}) \in V$ and $(\tilde{x}, \tilde{y})$ is a periodic point of $L$. Since $\varepsilon$ is arbitrary small, the periodic points of $L$ is dense in $A$.

From the above discussion, the geometric structure of the hyperchaotic strange attractors is almost same as that of usual chaotic strange attractors with only one positive Lyapunov exponent[6]. The difference between them is shown in their phase portraits. The former spreads all over some region in phase space, and the latter only converges to a low-dimensional manifold in phase space.

## 3 NEW PROPERTIES OF $L$

Besides the properties 1-5, which are common for usual chaotic strange attractors, there are two new properties for hyperchaotic strange attractors.

(1) The number of periodic points is countable square.

The one-dimensional logistic map with $\lambda = 4$ has countable unstable periodic points. Let $x_0$ be a periodic point of $x_{n+1} = 4x_n(1 - x_n)$ with period $p, y_0$ is also a periodic point of $y_{n+1} = 4y_n(1 - y_n)$ with period $q$, then $(x_0, y_0)$ is an expanding periodic point of $L$ with period $pq/k$ where $k$ is the largest common divisor of $p$ and $q$. Because the numbers of $x_0$ and $y_0$ are countable respectively, the number of $(x_0, y_0)$ is countable square. The number of periodic points does not attain uncountable, but it increases dramatically. In particular, various arrangements of periodic points in $(x, y)$ plane can occur. This provides a necessary condition to produce complicated patterns in physical space. Here periodic points with period 6 are taken as an example to illustrate these arrangements.

(i) The arrangement in a straight line parallel to a coordinate axis

The periodic orbit $\{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_1, y_4), (x_1, y_5), (x_1, y_6)\}$ is this kind of arrangement, where $\{x_1\}$ and $\{y_1, y_2, y_3, y_4, y_5, y_6\}$ are respectively a fixed point and a periodic orbit with period 6 of logistic map. The periodic orbit $\{(x_1, y_1), (x_2, y_1), (x_3, y_1),$ $(x_4, y_1)$, $(x_5, y_1)$, $(x_6, y_1)\}$ belongs to the same kind.

(ii) The arrangement in two straight lines parallel to $y$ axis

The periodic orbit $\{(x_1, y_1), (x_2, y_2), (x_1, y_3), (x_2, y_1), (x_1, y_2), (x_2, y_3)\}$ is this kind of arrangement, where $\{x_1, x_2\}$ and $\{y_1, y_2, y_3\}$ are respectively periodic orbits with period 2 and 3 of the logistic map. $\{(x_1, y_1), (x_2, y_2), (x_3, y_1), (x_1, y_2), (x_2, y_1), (x_3, y_2)\}$ is the same type of periodic orbits, which is arranged in two straight lines parallel to $x$ axis.

(iii) The arrangement in a diagonal line

The periodic orbit $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}$ is this arrangement, where $\{y_1, y_2, y_3, y_4, y_5, y_6\}$ and $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ are the same periodic orbits with period 6 of logistic map.

(iv) The other arrangement that is different from arrangements (i)~(iii)

The periodic orbit $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)\}$ is this arrangement, where $\{x_1, x_2, x_3, x_4, x_5, x_6\}$ and $\{y_1, y_2, y_3, y_4, y_5, y_6\}$ are different periodic orbits with period 6 of logistic map.

(2) There exist low-dimensional chaotic invariant sets in $A$.

For map (1), there are a lot of low-dimensional chaotic invariant sets.

$$A^{(k)} = \{(x_i, y_i) | i = 1, 2, \cdots; \{x_i\} \text{ (or } \{y_i\}) \text{ is a periodic orbit with period } k \text{ of logistic map and } \{y_i\} \text{ (or } \{x_i\}) \text{ is a chaotic orbit of logistic map } \}$$

It is obvious that $A^{(k)} \subset A$, and $A^{(k)}$ is an invariant set of $L$ consisting of $k$ straight lines parallel to $x$ axis (or $y$ axis) in $(x, y)$ plane. These invariant sets do not exist in usual chaotic strange attractors with only one positive Lyapunov exponent.

## 4 AN EXAMPLE OF A COUPLED TWO-DIMENSIONAL MAP

The zigzag pattern in CLM can be described by the coupled two-dimensional map[2]

$$Z : \begin{cases} x_{n+1} = 1 - a(x_n^2 + y_n^2) \\ y_{n+1} = -2a(1 - 2\varepsilon)x_n y_n \end{cases} \tag{2}$$

Fix $a = 1.95$. For $\varepsilon = 0.01$ and $\varepsilon = 0.20$, the attractors of map (2) obtained by numerical simulation are spreading ones[2].

The dynamical properties of map (2) are discussed at first. $Z$ is chaotic in the sense of Marotto, because if $y_n = 0$ map (2) is reduced to the logistic map that is chaotic in the sense of Marotto. In addition, numerical experiments show that Lyapunov exponents are $LE_1 = 0.575020$, $LE_2 = 0.573368$ for $\varepsilon = 0.01$, and $LE_1 = 0.480865$, $LE_2 = 0.479264$ for $\varepsilon = 0.20$. Hence $Z$ is hyperchaotic. To investigate the structure of the strange attractor corresponding to hyperchaos in the phase space, numerical research is carried out as follow. Let $(\tilde{x}, 0)$ be an expanding fixed point of map (2). Some points on the boundary of a sufficiently small neighborhood of $(\tilde{x}, 0)$. These points are iterated forward until the expanding processes are over. The results can be considered as the closure of the unstable manifold of the expanding fixed point. The case that $\varepsilon = 0.01$ ,and $\varepsilon = 0.20$ are respectively depicted in Fig.1 and Fig.2. Comparing with the results in [2], we infer that the closure of the unstable manifold of the expanding fixed point is identified with the strange attractor.

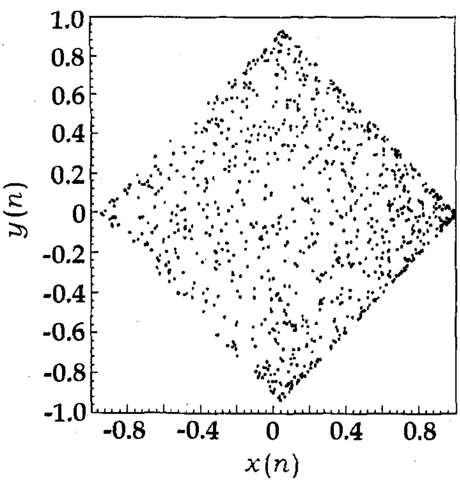


Fig.1 Closure of the unstable manifold of the expanding fixed point ($\varepsilon = 0.01$)

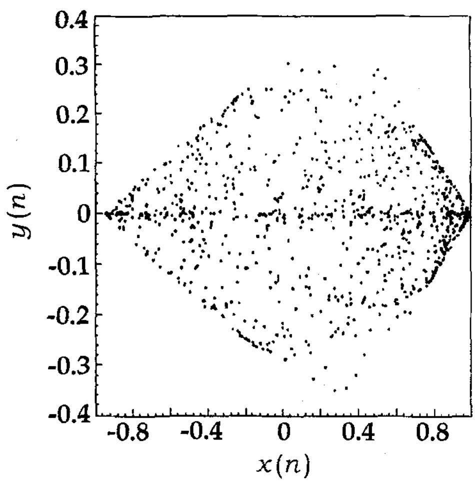


Fig.2 Closure of the unstable manifold of the expanding fixed point ($\varepsilon = 0.20$)

Map (2) is reduced to a chaotic logistic map if $y_n = 0$. Thus all periodic points of the logistic map are unstable ones embedded in the strange attractor of map (2). Besides these periodic points with $y_n = 0$, there are many other unstable periodic points embedded in the strange attractor. Period 1, 2, and 3 points are numerically determined, and the number of them are listed in Table 1. There are a lot of periodic points with $y_n \neq 0$. In addition, the strange attractor of map (2) contains a low-dimensional chaotic invariant set $A = \{(x_n, y_n), n \in N | y_n = 0, x_n \text{ such that } x_{n+1} = 1 - ax_n^2, a = 1.95\}$.

**Table 1 Number of unstable periodic points of map (2)**

| | $\varepsilon = 0.01$ | | $\varepsilon = 0.20$ | |
|---|---|---|---|---|
| | $y_n = 0$ | $y_n \neq 0$ | $y_n = 0$ | $y_n \neq 0$ |
| period 1 point | 2 | 2 | 2 | 2 |
| period 2 point | 2 | 10 | 2 | 1 |
| period 3 point | 7 | 52 | 6 | 6 |

By theoretic analysis and numerical simulation in the case study, we find that basically hyperchaos in coupled maps has the same properties as those in uncoupled maps, although we have not strictly proved it yet.

## 5 DISCUSSION

Based on the analysis above, some conclusions about hyperchaos can be drawn.

(1) For hyperchaotic states of a dynamical system, the number of positive Lyapunov exponent is considerably crucial to influencing its dynamical behaviors.

(2) The geometric structure of hyperchaotic strange attractor is basically similar to that of usual chaotic strange attractors. As the number of positive Lyapunov exponents increases, ordered spatiotemporal structures and the structures of order in space and chaos in time embedded in hyperchaotic strange attractors also increase greatly.

(3) From theoretic view point, the composition of the ordered spatiotemporal structures and the structures of order in space and chaos in time embedded in hyperchaotic strange is unstable and even expanding. For self-adjust systems under chaotic condition, this composition can become various patterns in physical space by various ways, such as OGY method[7], unstable controlling method[8], exact linearized method[9] and etc.[10]. Perhaps it is just what the term "edge of chaos"[11] implies.

(4) If the dimension of a multi-degrees-of-freedom system is very high, and thus the number of positive Lyapunov exponents can also be very large. In hyperchaotic strange attractors, there are a lot of unstable ordered structures (sometimes, these structures are chaotic in time), the number of these structures can come up to the power of countable. If the system can identify an order structure embedded in this strange attractor, the pattern in physical space can be formed. If the condition is slightly changed, this pattern will be out of control. As the system will realize other order structure, the new pattern is formed. The idea is coincided with the concept of complexity in [12]. Further work will be reported elsewhere.

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