Applied Mathematics and Mechanics
(English Edition, Vol 23, No 12, Dec 2002)

Published by Shanghai University,
Shanghai, China

Article ID: 0253-4827(2002)12-1407-06

ANCIENT CHINESE ALGORITHM: THE YING BUZU SHU (METHOD OF SURPLUS AND DEFICIENCY) VS NEWTON ITERATION METHOD *

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Abstract: *An exploratory discussion of an ancient Chinese algorithm, the Ying Buzu Shu, in about 2nd century BC, known as the rule of double false position in the West is given. In addition to pointing out that the rule of double false position is actually a translation version of the ancient Chinese algorithm, a comparison with well-known Newton iteration method is also made. If derivative is introduced, the ancient Chinese algorithm reduces to the Newton method. A modification of the ancient Chinese algorithm is also proposed, and some of applications to nonlinear oscillators are illustrated.*

Key words: ancient Chinese mathematics; Jiuzhang Suanshu (Nine Chapters); Newton iteration method; Duffing equation

CLC numbers: NO9; O112; O322　　**Document code**: A

1 An Introduction the Jiuzhang Suanshu (九章算术)

Essentially nothing of a primary nature has come down to West concerning ancient Chinese mathematics, little has been discussed on ancient Chinese mathematics in some of the most famous monographs on history of mathematics[1-3]. So the present author feels strongly necessary to give a basic introduction to the great classics of ancient Chinese mathematics for our Western colleges, who are unfamiliar with the Chinese language. This paper concerns briefly a famous ancient Chinese algorithm, *Ying Buzu Shu* (盈不足术, literally Method of Surplus and Deficiency) in *Jiuzhang Suanshu* (九章算术, Nine Chapters on the Art of Mathematics), which is comprised of nine chapters and hence its title. The *Nine Chapters* is the oldest and most influential work in the history of Chinese mathematics. It is a collection of 246 problems on agriculture, business procedure, engineering, surveying, solution of equations, and properties of right triangles. Rules of solution are given systematically, but there exists no proofs in Greek sense.

* **Received date**: 2001-07-19; **Revised date**: 2002-04-01
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As was pointed out by Dauben[4] that the *Nine Chapters* can be regarded as a Chinese counterpart to Euclid's *Elements*, which dominated Western mathematics in the same way, the *Nine Chapters* came to be regarded as the seminal work of ancient Chinese mathematics for nearly two millennia. Its author and time have been the issue of polemics for a long time, one of the tradition says it was finished during the period of Yellow Emperor, (黄帝 Huang Di), in the 27th century BC. But it is the recognized fact that it was published before 213 BC, when there is an infamous burning of all books ordered by the Emperor, Qin Shi Huang(秦始皇), 221 - 207 BC. The original version of the *Nine Chapters* no longer exists, and might have been burned at that time. The preserved version was reconstituted by unknown rescuers. Most texts in the *Nine Chapters* had been used in engineering before 213BC, for example, the *Gou-Gu* theory(勾股, known as Pythagorean theorem in the West) in Chapter 9 of the *Nine Chapters* had been widely applied in engineering before 1100 BC (see Zhou Bi Algorithm 周髀算经).

2 The Ying Buzu Shu, Method of Surplus and Deficiency (盈不足术)

The chapter 7 of the *Nine Chapters* is the Ying Buzu Shu (盈不足术, Method of Surplus and Deficiency), which is the oldest method for approximating real roots of an equation. To illustrate the basic idea of the method, we consider the first example in this chapter, which reads.

There is a thing of an unknown price, and people of an unknown number who buy together the thing, if each is assigned 8 *dollars, there is a surplus of* 3 *dollars; and if each is assigned* 7 *dollars, there is a deficiency of* 4 *dollars. What is the price and the number of people?*

The solution procedure is as follows (see Fig.1):

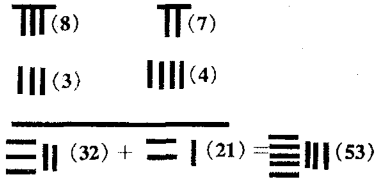

Fig.1 The solution procedure used by ancient Chinese mathematicians

1) Put the assigned dollars (8 and 7) in the first line;

2) Then put the surplus (3) and the deficiency (4) below;

3) The mixed products are 32 and 21, which leads to the sum of 53;

4) The difference of assignment is 4 - 3 = 1;

5) So we obtain the price 53/1 = 53.

In brief, and in modern form, the procedure is this:

Let x_1 and x_2 be the different assigned dollars (approximate prices), and R_1 and R_2 be the deviations to the exact price, then the price is

$$x = \frac{x_2 R_1 - x_1 R_2}{R_1 - R_2}, \tag{1}$$

and number of the people is

$$y = \frac{R_1 + R_2}{| x_1 - x_2 |}. \tag{2}$$

As pointed out by Bai[5], in *Sui* Dynasty (581 - 618 A D), the method was widely used by ancient mathematicians in Middle East, and al-Khowarizmi(in about 825AD, the word algorithm is originated from his name), an ancient famous Arabiann mathematician, who wrote the oldest algorithm book in Arabian, might well-known the *Nine Chapters* and other ancient Chinese mathematics monographs. The above method was called the *Khitai Method* (Khitai means China, and it was also written in the forms: Khatai, Chatayn, Chataain and that likes) in Middle East.

It is widely recognized that the *Khitai Method* was spread to the West by Leonardo Fibonacci (1170? ~ 1250?), who was the most talented Italian mathematician of Middle Ages. The record[2] says Fibonacci had subsequent extended trips with his father to Egypt, Sicily, Greece, and Syria, which brought him in contact with Eastern and Arabic mathematical practices of calculation. In 1202, shortly after his return home, Fibonacci published his famous work called the *Liber Abaci*. The above method was also introduced and called it *De Regulis el-Chatayn* (literally rule of China), the name must come from the Middle East, where the method was called *hisab el-Chataain*, Chataain refers to China. Some examples and the solution techniques in the *Liber Abaci* are exactly the same with those in ancient mathematical classics. For example, the problem discussed in section 23.8 (the Chinese version of this section is available in Ref. [5], p.202) is originated from an ancient book named Sun Zi Algorithms (孙子算经, in about 4th century A D):

There are things of an unknown number which when divided by 3 *leave* 2, *by* 5 *leave* 3, *and by* 7 *leave* 2. *What is the smallest number?*

The solution technique of this problem is the famous *Chinese Remainder Theory* of elementary number theory.

The Arabian *De Regulis el-Chatayn* or *De Regulis el-Chataieym* was translated into Latin as *Duarum Falsarum Posicionum Regula* in *Liber Abaci*. So in the West, the method was refereed to the method of double false position, which is exactly the same with that in the *Nine Chapters*. It certainly originates from China, as was already pointed out by Qian Bao-cong[6]. Unfortunately, most of our Western colleges guessed that the method might have originated in India and was used by the Arabians. The present author strongly suggests that the method be called Chinese Algorithm of Chinese Method.

3 Chinese Algorithm *vs* Newton Iteration Formulation

Consider an algebraic equation,

$$f(x) = 0. \tag{3}$$

Let x_1 and x_2 be the approximate solutions of the equation, which lead to the remainders $f(x_1)$ and $f(x_2)$ respectively, the improved approximate solution is

$$x_3 = \frac{x_2 f(x_1) - x_1 f(x_2)}{f(x_1) - f(x_2)}. \tag{4}$$

Some inequalities are included in the *Nine Chapters* in cases of 1) double surplus, i.e., $f(x_1) > 0$ and $f(x_2) > 0$; 2) double deficiency, i.e., $f(x_1) < 0$ and $f(x_2) < 0$; and 3) surplus and deficiency, i.e., $f(x_1) \cdot f(x_2) < 0$.

The process can be now applied with the appropriate pair (x_1, x_2) or (x_2, x_3) according to the established inequalities.

Now we re-write (4) in the form

$$x_3 = \frac{x_2 f(x_1) - x_1 (f_2)}{f(x_1) - f(x_2)} = x_1 - \frac{f(x_1)(x_1 - x_2)}{f(x_1) - f(x_2)}. \tag{5}$$

If we introduce the derivative $f'(x_1)$ defined as

$$f'(x_1) = \frac{f(x_1) - f(x_2)}{x_1 - x_2}, \tag{6}$$

then we have

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}, \tag{7}$$

which is the well-known Newton iteration formulation proposed by Newton (1642 – 1727). The Chinese algorithm seems to have some advantages over the well-known Newton iteration formula if the two points (x_1 and x_2) locate two sides of the exact root, i.e., $f(x_1)\cdot f(x_2) < 0$. In view of antique of Chinese algorithm, our Chinese mathematicians had made greatest contributions to the progress of science. The further development of Newton iteration formulation can be found in details in Refs. [7,8].

4 A Reliable Modification of the Chinese Algorithm

The ancient Chinese algorithm can be considered to linearly approximate solution by the two assumed solutions, see Fig. 2. To improve the accuracy of the Chinese Algorithm, we can use, in place of two points (x_1 and x_2), three points (x_1, x_2 and x_3). The present author first proposes the following new algorithm:

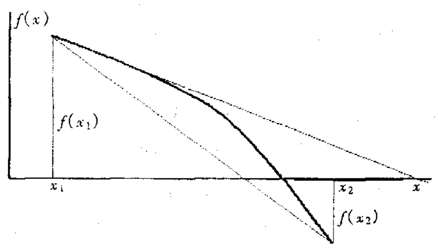

Fig.2 Chinese Algorithm *vs* Newton formula

$$x = \frac{x_1 f_2 f_3}{(f_1 - f_2)(f_1 - f_3)} + \frac{x_2 f_1 f_3}{(f_2 - f_1)(f_2 - f_3)} + \frac{x_3 f_1 f_2}{(f_3 - f_1)(f_3 - f_2)}, \tag{8}$$

where $f_i = f(x_i)$.

5 Application to Nonlinear Oscillators

The present author first finds that the ancient Chinese algorithm can be effectively applied to finding the angular frequency of nonlinear oscillators. Consider the Duffing equation:

$$R(u) = u'' + u + \varepsilon u = 0, \tag{9}$$

with initial conditions $u(0) = A$ and $u'(0) = 0$.

If we use the trial function $u(t) = A\cos\omega t$ to approximate the solution of (9), when $\omega_1^2 = 1$, we obtain the residual $R_1(t,\varepsilon) = \varepsilon A\cos t$; and when $\omega_2^2 = \omega^2$, the residual can be written in the form $R_2(t,\varepsilon) = (-\omega^2 + 1)A\cos\omega t + \varepsilon A\cos\omega t$. The present author proposes the following conjecture without proof.

The square of the angular frequency of a nonlinear oscillator can be approximated by

$$\omega^2 = \frac{\omega_1^2 R_2(t_0,\varepsilon) - \omega_2^2 R_1(t_0,\varepsilon)}{R_2(t_0,\varepsilon) - R_1(t_0,\varepsilon)}. \tag{10}$$

We always set $t_0 = 0$.

For Duffing equation, we have

$$\omega^2 = \frac{\omega_1^2 R_2(0,\varepsilon) - \omega_2^2 R_1(0,\varepsilon)}{R_2(0,\varepsilon) - R_1(0,\varepsilon)} = \frac{(1-\omega^2)A + \varepsilon A^3 - \varepsilon\omega^2 A^3}{(1-\omega^2)A} = 1 + \varepsilon A^2. \tag{11}$$

We, therefore, obtain the following approximate period:

$$T = \frac{2\pi}{\sqrt{1 + \varepsilon A^2}}. \tag{12}$$

The exact period is

$$T_{\text{ex}} = \frac{4}{\sqrt{1 + \varepsilon A^2}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k\sin^2 x}}, \quad k = \frac{\varepsilon A^2}{2(1 + \varepsilon A^2)}. \tag{13}$$

Table 1 Comparison of approximate period with exact one

εA^2	0	0.042	0.087	0.136	0.190	0.25
T_{ex} (exact)	6.283	6.187	6.088	5.986	5.879	5.767
T (Eq.10)	6.283	6.155	6.026 4	5.895	5.760	5.620

For small ε, the comparison of the approximate period with the exact one is illustrated in Table 1. It is obvious that our theory gives a remarkable accuracy in review of the crudeness of the trial function $u = A\cos\omega t$.

It is interesting to point out that no small assumption is made in this theory, so the obtained result might be valid regardless of the values of ε. In case $\varepsilon \to \infty$, we have

$$\lim_{\varepsilon \to \infty} \frac{T_{\text{ex}}}{T} = \frac{2}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = \frac{2}{\pi} \times 1.685\,75 = 1.073. \tag{14}$$

Therefore, our obtained result is valid for any values of ε. The 7.3% accuracy is remarkable good when $\varepsilon \to \infty$.

To further improve the accuracy of the approximate period, we write the trial function in the form

$$u = (A - B)\cos\omega t + B\cos 3\omega t. \tag{15}$$

When $\omega_1^2 = 1$, we have $R_1(0, \varepsilon) = \varepsilon A^3$; and when $\omega_2^2 = \omega^2$, we obtain

$$R_2(0, \varepsilon) = (-\omega^2 + 1)(A - B) + (-9\omega^2 + 1)B + \varepsilon A^3 = (-\omega^2 + 1)A - 8B\omega^2 + \varepsilon A^3.$$

By (10), we have

$$\omega^2 = \frac{\omega_1^2 R_2(0, \varepsilon) - \omega_2^2 R_1(0, \varepsilon)}{R_2(0, \varepsilon) - R_1(0, \varepsilon)} = \frac{(1 - \omega^2)A - 8B\omega^2 + \varepsilon A^3 - \varepsilon\omega^2 A^3}{(1 - \omega^2)A - 8B\omega^2} = 1 + \frac{\varepsilon A^3(1 - \omega^2)}{(1 - \omega^2)A - 8B\omega^2}, \tag{16}$$

which leads to the result

$$\omega = \sqrt{\frac{2A + 8B + \varepsilon A^3 - \sqrt{(2A + 8B + \varepsilon A^3)^2 - 4A(1 + \varepsilon A^2)}}{2(A + 8B)}}, \tag{17}$$

where B is a free parameter, which can be identified in view of the method of weighted residuals. The obtained result (17) is similar to those obtained by variational iteration method[9], homotopy perturbation method[10], and others[11].

6 Conclusion

China is one of the four countries with an ancient civilization. The ancient Chinese mathematicians had made greatest contributions to the development of human culture. The *Nine Chapters* is considered an oldest and greatest mathematical classic, it dates from far ancient time, and it is now regarded as a Chinese counterpart to Euclid's *Elements*. It is pointed out that the method of double false position known in the West is introduced from China by Arabian mathematician al-Khowarizmi (825 AD) and Italian mathematician Fibonacci (1201), and it should be corrected as Chinese Algorithm.

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