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General paper

Regularity of Strain Distribution in Short-Fiber/Whisker Reinforced Composites

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Abstract: Based on studies on the strain distribution in short-fiber/whisker reinforced metal matrix composites, a deformation characteristic parameter, λ is defined as a ratio of root-mean-square strain of the reinforcers identically oriented to the macro-linear strain along the same direction. Quantitative relation between λ and microstructure parameters of composites is obtained. By using λ, the stiffness moduli of composites with arbitrary reinforcer orientation density function and under arbitrary loading condition are derived.

The upper-bound and lower-bound of the present prediction are the same as those from the equal-strain theory and equal-stress theory, respectively. The present theory provides a physical explanation and theoretical base for the present commonly-used empirical formulae. Compared with the microscopic mechanical theories, the present theory is competent for stiffness modulus prediction of practical engineering composites in accuracy and simplicity.

Key words: *Short-fiber/Whisker reinforced composite, Strain distribution, Stiffness prediction, Anisotropy*

1. INTRODUCTION

As pointed by R.Hill [1], the stiffness tensor for heterogeneous materials can be formulated as long as the relation between local strains and overall strain is determined. Theoretically, it is very important to study the regularity of strain distribution in reinforcers in order to derive the stiffness tensor of composites.

In the present paper, such a relation will be numerically investigated first. Based on the strain distribution regularity of the reinforcers obtained, a new material model is proposed. Then an explicit stiffness tensor will be derived to predict stiffness moduli of composites with arbitrary whisker orientation density function and under arbitrary loading condition.

2. STRAIN DISTRIBUTION IN REINFORCERS

2.1. Singular Reinforcer in an Isotropic Medium

Let ε_x, ε_y, ε_z, γ_{yz}, γ_{zx}, and γ_{xy} denote the strain components in a rectangular coordinate. Without losing generality, z-axis of the local coordinate system is taken as the longitudinal direction of the reinforcer as shown in Fig. 1. The strain of the reinforcer can be written as

$$\varepsilon_f = f(\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy}) . \quad (1)$$

Transverse deformations of the reinforcer (ε_x, ε_y and γ_{xy}) are usually neglected if the aspect ratio is large enough. Moreover, shear strains γ_{yz} and γ_{zx} are anti-symmetric with respect to the longitudinal direction of the reinforcer, and have no contributions to ε_f. Therefore, ε_f depends only on the strain component ε_z, and Eq.(1) can be simplified as

$$\varepsilon_f = f(\varepsilon_z) = \lambda_f \varepsilon^{(1)} , \quad (2)$$

where $\varepsilon^{(1)}$ is the macro linear strain along the reinforcer, and λ_f, a strain heterogeneity factor. The subscript 'f' stands for the fiber-like reinforcers.

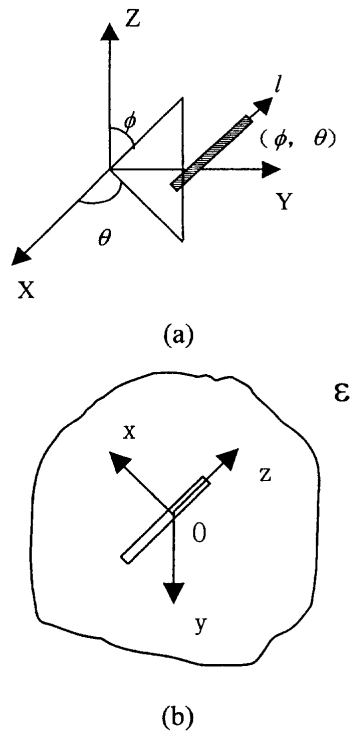

Fig. 1. The basic coordinate system (a) and the local coordinate system for the reinforcer (b).

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It has been proved with the shear lag theory [2] that λ_f is constant and is dependent on stiffness ratio E_m/E_f, aspect ratio L/d, and volume fraction V_f during elastic deformation, i. e.

$$\lambda_f = f(\frac{E_m}{E_f}, \frac{L}{d}, V_f). \tag{3}$$

2.2. Reinforcers Unidirectionally Oriented

Strain distribution of reinforcers unidirectionally oriented in the matrix is numerically investigated by the 2-D network model [3] shown in Fig. 2 where reinforcers are represented by black-short segments and have same length. ε_f is taken as the root-mean-square strain of the reinforcers. It is verified that though the strains in reinforcers are different, the ratio λ_f almost keeps unchanged during any elastic deformation.

Figure 3 shows a strain histogram of 2236 reinforcers in a direction. There are totally 288 combinations of microstructure parameters: 8 volume fractions, 6 stiffness ratios, and 6 aspect ratios listed in Table 1.

For various microstructure parameter combinations, Figure 4 shows the variation of λ_f with L/d and E_m/E_f when V_f=20%. And the curve of λ_f vs V_f with L/d=3 and E_m/E_f=1/5 is plotted in Fig. 5.

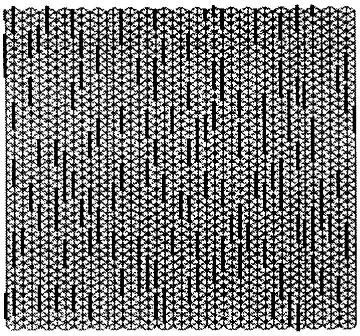


Fig. 2. Unidirectionally oriented reinforcers in the matrix.

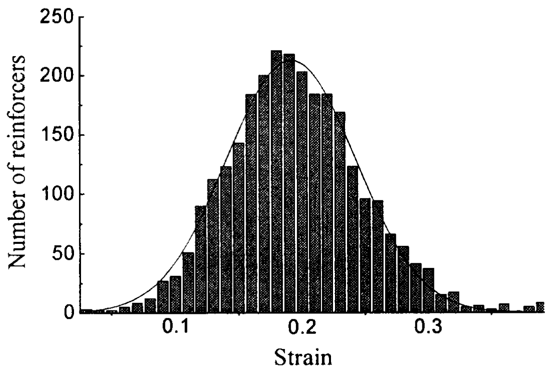


Fig. 3. Histogram of strain in reinforcers along a direction.

Table 1. Microstructure parameter combinations.

Parameters	Scales
Volume fraction V_f	5%, 10%, 15%, 20%, 25%, 30%, 35%, 40%
Moduli ratio E_m/E_f	1/3, 1/5, 1/8, 1/12, 1/18, 1/25
Aspect ratio L/d	3, 5, 8, 12, 18, 25

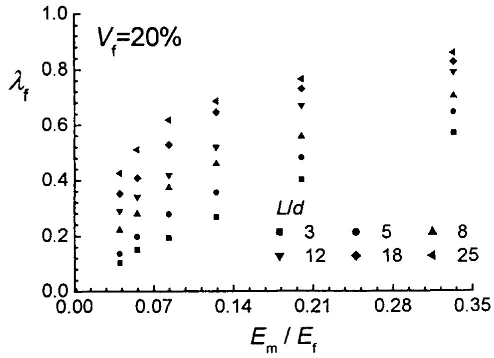


Fig. 4. λ_f versus E_m/E_f and L/d .

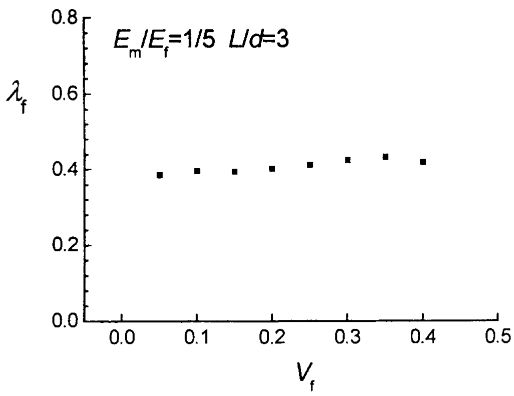


Fig. 5. λ_f versus V_f.

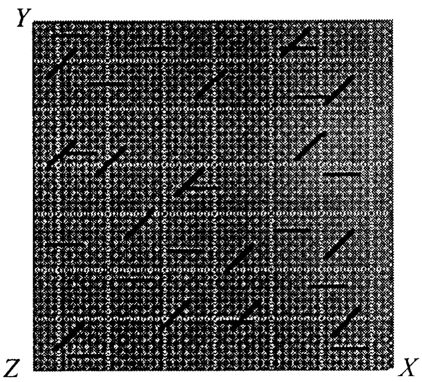


Fig. 6. Reinforcers oriented along 0° and 45°.

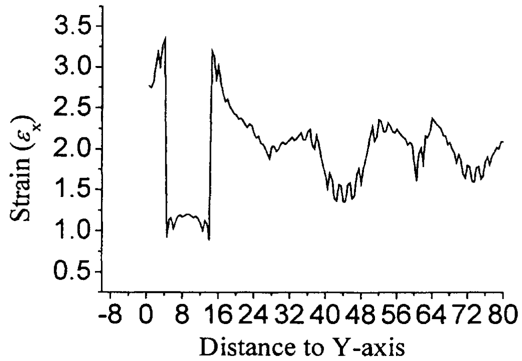


Fig. 7. ε_x obtained with FEM.

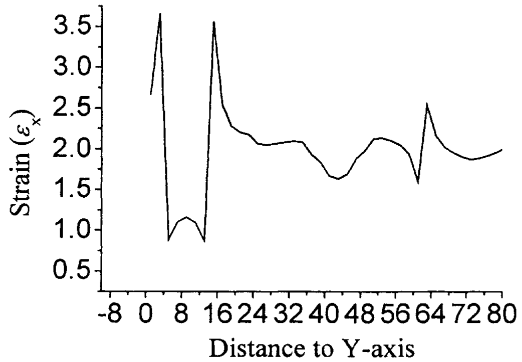
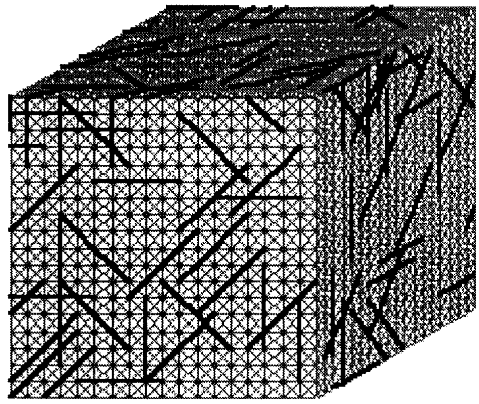


Fig. 8. ε_x obtained with network model.

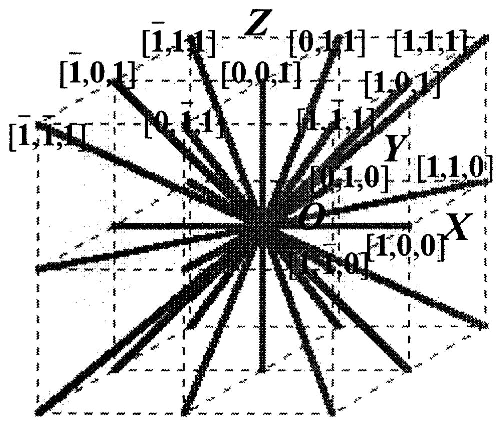
The strain distribution of reinforcers randomly oriented is numerically investigated with the network model and FEM respectively. As shown in Fig. 6, the volume fraction of short-fibers V_f is 5%, stiffness ratios E_m/E_f and aspect ratios L/d are 1/10 and 5 respectively. In Fig. 6, reinforcers, shown by black-short segments, locate in the matrix with orientation angles 0° and 45°. Just like in Fig. 2, reinforcers in Fig. 6 possess same length, too. The loading direction is along X-axis. Distribution of ε_x in a section in which Y is constant obtained by using finite element method (FEM) and network model are shown in Figs. 7 and 8 respectively. The two figures in which the abscissa represents the distance to Y-axis show that the result obtained by network model is proximately identical to the one by FEM. As a result, the two methods are able to predict the same macro-mechanical properties.

2.3. Reinforcers Oriented in Many Directions

The strain distribution of reinforcers oriented in many directions is numerically performed by a 3-D multi-scale network model [3] shown in Fig. 9 (a), in which 8462 short-fibers are randomly oriented in 13 directions. Figure 9 (b) gives the 13 directions with direction numbers of [1,0,0], [0,1,0], [0,0,1], [1,1,0], [1,0,1], [0,1,1], [1,-1,0], [-1,0,1], [0,-1,1], [1,1,1], [-1,1,1], [1,-1,1], [-1,-1,1]. In Fig. 9 (b), " $\bar{1}$ ", similar to others, means "-1". The loading direction is vertical. The strain distribution histogram of the reinforcers in each direction is similar to Fig. 3. The statistical analysis results of the root-mean-square strains $\bar{\varepsilon}_f^{(l)}$ and $\bar{\varepsilon}_f^{(l)}$-$\lambda\varepsilon^{(l)}$ in each of the 13 directions are given in Fig. 10.



(a)



(b)

Fig. 9. 3-D network model.

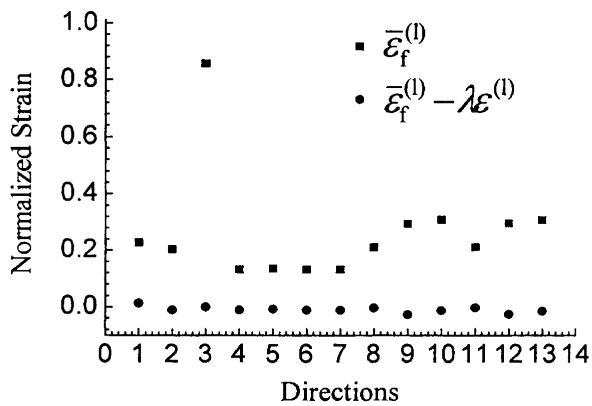


Fig. 10. Statistical regularity of strain distribution.

Figures 3 and 10 show that the root-mean-square strains of the reinforcers in the 13 orientations are different, but the ratio λ_f in different orientations is almost the same. For composites with reinforcer orientation density function ρ_f in 3-D space,

$$\lambda_{\mathrm{f}} = f(\frac{E_{\mathrm{f}}}{E_{\mathrm{m}}}, \frac{L}{d}, \rho_{\mathrm{f}}). \tag{4}$$

Therefore, λ_f is dependent on the orientation density function ρ_f. It is verified that when V_f<50% and the aspect ratio L/d<15, the interaction between the reinforcers in different directions may be neglected.

2.4. Expression of λ_f during Elastic Deformation

Based on the numerical results shown in Fig. 4, Fig. 5 and Fig. 10, λ_f can be expressed as

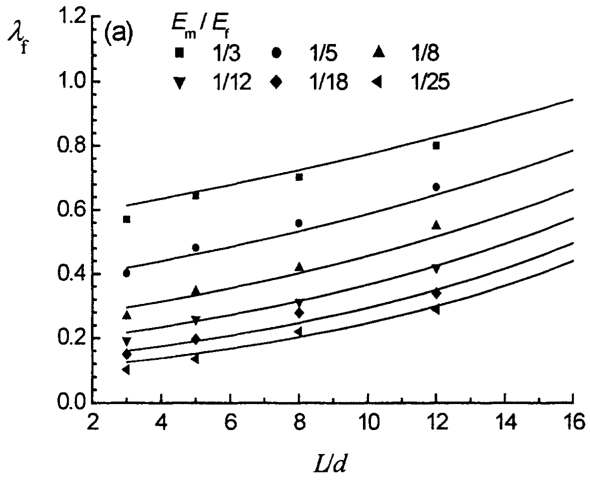
$$\lambda_{\mathrm{f}} = c(a + bV_{\mathrm{f}})(\frac{E_{\mathrm{m}}}{E_{\mathrm{f}}})^{d-e\frac{L}{d}}. \tag{5}$$

Let L/d=3, E_m/E_f=0.2, and V_f=20%. Parameters a=0.94, b=0.3, c=1.4, d=0.84 and e=0.03 are determined with the least square method. Now, Eq.(5) becomes

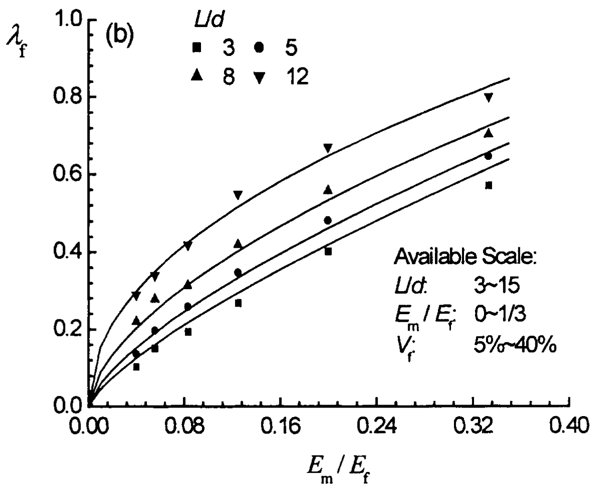
$$\lambda_{\mathrm{f}}^{(\mathrm{e})} = 1.4(0.94 + 0.3V_{\mathrm{f}})(\frac{E_{\mathrm{m}}}{E_{\mathrm{f}}})^{(0.84-0.03\frac{L}{d})}. \tag{6}$$

Equation (6) is suitable for $3 \le L/d \le 15$, $5\% \le V_f \le 40\%$ and $0 < E_m/E_f \le 1/3$. In Fig. 11, there is a comparison of predictions of Eq.(6) with the results shown in Figs. 4 and 5.

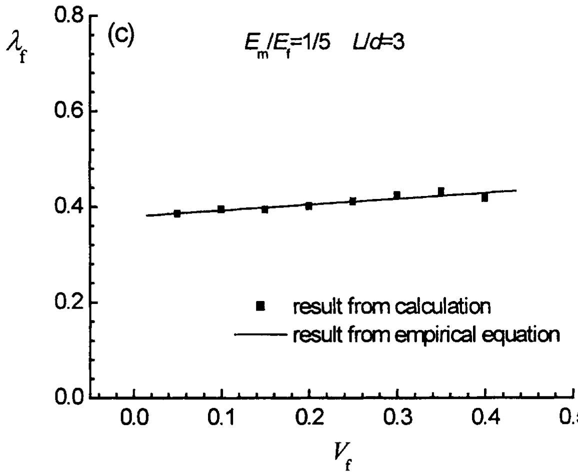
Liang, N.G. et al [4] investigated the precision of present theory, and pointed that the precision of network model is exactly the same as FEM's. The results obtained by present theory are also in good agreement with the ones by 2-D shear lag model.



(a)



(b)



(c)

Fig. 11. λ_f versus L/d (a), E_m/E_f (b) and V_f (c).

3. MATERIAL MODEL AND STIFFNESS TENSOR

3.1. Matrix and Fiber-Bundle

Consider the reinforcers in a direction l. The strain of the r^{th} reinforcer

$$\varepsilon_{\mathrm{f}}^{(r)} = \lambda_{\mathrm{f}}^{(r)} \varepsilon_{ij} l_i l_j, \tag{7}$$

where ε_{ij} is the macro strain tensor, and l_i, the cosine numbers of the fiber direction l.

The total strain energy of the reinforcers

$$\begin{aligned} W &= \frac{1}{2} E_{\mathrm{f}} \sum_r [\lambda_{\mathrm{f}}^{(r)}]^2 \varepsilon_{ij} \varepsilon_{st} l_i l_j l_s l_t V_{\mathrm{f}}^{(r)} \\ &= \frac{1}{2} \varepsilon_{ij} \varepsilon_{st} l_i l_j l_s l_t E_{\mathrm{f}} \sum_r [\lambda_{\mathrm{f}}^{(r)}]^2 V_{\mathrm{f}}^{(r)}. \end{aligned} \tag{8}$$

According to the above statistical analysis, fibers in direction l can be replaced by a fiber-bundle with an average stiffness modulus E_f, so

$$\sigma_f = E_f \varepsilon_f \ ,$$

where

$$\varepsilon_f = \lambda_f \varepsilon_{ij} l_i l_j \ , \tag{9}$$

where σ_f and ε_f denote the stress and strain of the fiber-bundle, ε_{ij}, the macro strain tensor. Provided that the matrix material can be regarded as an isotropic elastic medium, then

$$\sigma_{(m)rs} = K_{(m)ijrs} \varepsilon_{(m)ij} \ ,$$

where

$$\varepsilon_{(m)ij} = \lambda_m \varepsilon_{ij} \ , \tag{10}$$

where λ_m is the strain heterogeneity factor of the matrix and $K_{(m)ijrs}$, a stiffness tensor of the matrix material.

3.2. Stiffness Tensor of Composites

Since short-fiber/whisker reinforced composites are composed of the matrix material and the fiber-bundles, the total strain energy of a representative element volume of the composites

$$W_c = W_m + W_f \ . \tag{11}$$

In the coordinate system shown in Fig. 1 (a), the strain energy of the composites will be

$$W_c = \frac{1}{2} \lambda_m^2 V_m K_{(m)ijrs} \varepsilon_{ij} \varepsilon_{rs} + \frac{1}{2} V_f E_f \varepsilon_{ij} \varepsilon_{rs} \int_0^{\frac{\pi}{2}} \sin\phi \int_0^{2\pi} \lambda_f^2(\rho_f) \rho_f(\theta, \phi) l_i l_j l_r l_s d\theta \, d\phi. \tag{12}$$

Differentiating W_c with respect to ε_{ij} gives the stress tensor

$$\sigma_{(c)ij}^{(e)} = K_{(c)ijrs} \varepsilon_{rs} \ ,$$

where

$$K_{(c)ijrs} = \lambda_m^2 V_m K_{(m)ijrs} + V_f E_f \int_0^{\frac{\pi}{2}} \sin\phi \int_0^{2\pi} \lambda_f^2(\rho_f) \rho_f(\theta, \phi) l_i l_j l_r l_s d\theta \, d\phi, \tag{13}$$

where $K_{(c)ijrs}$ is the stiffness tensor of the composites.

When composites are isotropic, the stiffness tensor caused by the reinforcers in Eq.(13)

$$K_{(f)ijkl} = \begin{cases} \lambda_f^2 E_f / 5 & \text{when } i = j = k = l \\ \lambda_f^2 E_f / 15 & \text{when } i = j, k = l \text{ or } i = k, j = l \\ 0 & \text{in the other cases.} \end{cases} \tag{14}$$

When reinforcers are randomly oriented in a plane, such as a fiber-reinforced laminate, the stiffness matrix of composites will be

$$K_c = \frac{\lambda_m^2 V_m E_m}{(1 - 2\nu_m)(1 + \nu_m)} \begin{pmatrix} 1 - \nu_m & \nu_m & 0 \\ \nu_m & 1 - \nu_m & 0 \\ 0 & 0 & 1 - 2\nu_m \end{pmatrix} + \frac{\lambda_f^2 V_f E_f}{8} \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{15}$$

When reinforcers are unidirectionally oriented in x-direction, the Young's modulus of composites

$$K_{(c)1111}^{(0)} = \lambda_m^2 V_m K_{(m)1111} + \lambda_f^2 V_f E_f \ , \tag{16}$$

and the Young's modulus in y-axis is

$$K_{(c)2222}^{(90)} = \lambda_m^2 V_m K_{(m)2222} \ . \tag{17}$$

4. UPPER-BOND AND LOWER BOUND OF THE STIFFNESS MODULUS

For unidirectional fiber-reinforced composites, $\lambda_f = (E_m/(V_f E_m + V_m E_f))$ and $\lambda_m = (E_f/(V_f E_m + V_m E_f))$ make

$$\lambda_f E_f = \lambda_m E_m \ . \tag{18}$$

In this case, the stress in the reinforcers is the same as in the matrix. The stiffness tensor predicted by the present theory is the lower bound solution of Reuss (1929) [5].

It can also be verified that if $\lambda_f = \lambda_m = 1$ ($0 \le \lambda_f \le 1$), the Young's modulus given by (16) is

$$E_c = V_m E_m + V_f E_f \ . \tag{19}$$

Equation (18) means that the traditional mixed law holds. In this case,

$$\varepsilon_{(m)ij} = \varepsilon_{(f)ij} = \varepsilon_{ij} \ . \tag{20}$$

The stiffness tensor predicted by the present theory is just the upper bound solution of Voigt (1989) [6].

Figure 12 shows that the present theory can give stiffness moduli from the lower bound to the upper bound with proper parameters λ_f and λ_m.

5. COMPARISON WITH EXPERIMENTAL RESULTS

The elastic modulus of SiCw/2024Al unidirectional composite is calculated by the present theory. The volume fraction of SiC whisker is 10% and the aspect ratio is 4.5.

The elastic moduli of reinforcer and matrix materials are 480GPa and 80GPa, respectively.

Figure 13 shows the stiffness moduli of the composite calculated by the present theory and the experimental results in reference [7]. It can be seen that the results obtained by the present theory are excellently in good agreement with the experimental ones.

6. COMPARISON WITH EMPIRICAL FORMULAE

If the reinforcers are randomly oriented in a plane, comparing Eqs.(15-16) with Eq.(14) gives

$$K^{(d2)}_{(c)1111} = \frac{3}{8}K^{(0)}_{(c)1111} + \frac{5}{8}K^{(90)}_{(c)2222}, \qquad (21)$$

which is just the empirical formula given by Halpin & Pagano [8].

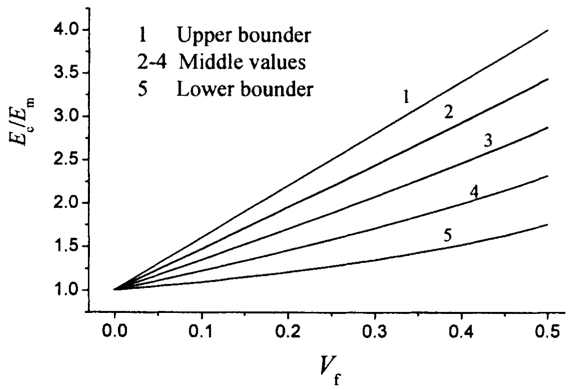


Fig. 12. The stiffness moduli predicted by present theory.

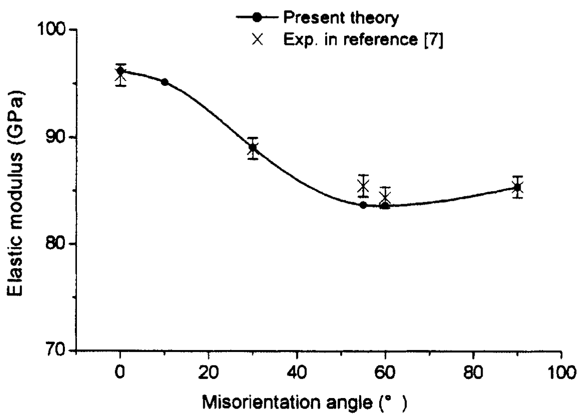


Fig. 13. Elastic moduli at different misorientation angles.

If the reinforcers are randomly oriented in 3-D space, comparing Eqs.(15-16) with Eqs.(13-13a) gives

$$K^{(d3)}_{(c)1111} = \frac{1}{5}K^{(0)}_{(c)1111} + \frac{4}{5}K^{(90)}_{(c)2222}, \qquad (22)$$

which is just the empirical formula by Nielsen & Landel [9].

7. CONCLUSIONS

(1) Statistical regularity of the strain in reinforcers of short-fiber/whisker reinforced composites is obtained by using the network model. Based on the strain distribution regularity, fiber-bundle reinforced composites can be regarded as the material model. The stiffness moduli of composites with arbitrary reinforcer orientation density function and under arbitrary loading condition can be predicted from the microstructure parameters of materials.
(2) The upper-bound and lower-bound of the present prediction are the same as those from the equal-strain theory and equal-stress theory, respectively. The present theory provides a physical explanation and theoretical base for the commonly used empirical formulae. Compared with the microscopic mechanical theories, the present theory is competent for modulus prediction of practical engineering composites in accuracy and simplicity.
(3) It is demonstrated that the network model [3] is a useful tool to simulate mechanical behavior of short-fiber/whisker reinforced composites.

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