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Preliminary report on the energy balance for nonlinear oscillations

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Abstract

In this paper, a reliable technique for calculating angular frequencies of nonlinear oscillators is developed. The new algorithm offers a promising approach by constructing a Hamiltonian for the nonlinear oscillator. Some illustrative examples are given.

Keywords: Nonlinear oscillator; Variational principle Hamilton; Asymptotic method; Energy balance

1. Introduction

In this paper, we will propose an heuristic approach called *the method of energy balance* to nonlinear oscillators. In this method, a variational principle for the nonlinear oscillation is established, then a Hamiltonian is constructed, from which the angular frequency can be readily obtained by collocation method. The results are valid not only for weakly nonlinear systems, but also for strongly nonlinear ones. Some examples reveal that even the lowest order approximations are of high accuracy.

Consider first the following generalized nonlinear oscillations without forced terms.

$$u'' + \omega_0^2 u + \varepsilon f(u) = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{1}$$

where f is a nonlinear function of u, u' and u''. In this preliminary report, we limit ourselves to the simplest case, i.e., f depends upon only the function of u.

If ε is a small parameter, then various perturbation techniques can be applied. In our study ε needs not to be small. There exist many new techniques to solve the above equation, for example, variational iteration method (He, 1999a), homotopy perturbation method (He, 1999b, 2000a), linearized perturbation method (He, 1999c), modified Lindstedt–Poincare methods (He, 2002a,b, 2001a), iteration perturbation method

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(He, 2001b), bookkeeping parameter perturbation method (He, 2001c), and other methods, a review on new techniques for strongly nonlinear oscillations can be found in details in He (2000b).

2. Basic idea

First we consider the Duffing equation

$$u'' + u + \varepsilon u^3 = 0, \quad u(0) = A, \quad u'(0) = 0. \tag{2}$$

Its variational principle can be easily obtained:

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 \right\} \mathrm{d}t. \tag{3}$$

Its Hamiltonian, therefore, can be written in the form

$$H = \frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 = \frac{1}{2}A^2 + \frac{1}{4}\varepsilon A^4, \tag{4a}$$

or

$$\frac{1}{2}u'^2 + \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4 - \frac{1}{2}A^2 - \frac{1}{4}\varepsilon A^4 = 0. \tag{4b}$$

In Eqs. (4a) and (4b) the kinetic energy (E) and potential energy (T) can be respectively expressed as $E = u'^2/2$, $T = u^2/2 + \varepsilon u^4/4$. Throughout the oscillation, it holds that $H = E + T = \text{constant}$.

We use the following trial function to determine the angular frequency ω:

$$u = A\cos\omega t. \tag{5}$$

Substituting (5) into (4b), we obtain the following residual equation:

$$R(t) = \omega^2\sin^2\omega t + \cos^2\omega t + \frac{1}{2}\varepsilon A^2\cos^4\omega t - 1 - \frac{1}{2}\varepsilon A^2. \tag{6}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (5) is only an approximation to the exact solution, R cannot be made zero everywhere. Collocation at $\omega t = \pi/4$ gives

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{7}$$

We can apply various other techniques, for examples, least square method, Galerkin method, to identify the constant ω.

Its period can be written in the form

$$T = 2\pi \Big/ \sqrt{1 + \frac{3}{4}\varepsilon A^2}. \tag{8}$$

The approximate period obtained by the traditional perturbation method reads (Nayfeh, 1985)

$$T_{\text{pert}} = 2\pi\left(1 - \frac{3}{8}\varepsilon A^2\right). \tag{9}$$

So our theory, in case $\varepsilon \ll 1$, gives exactly the same result with those obtained by perturbation method.

What is rather surprising about the remarkable range of validity of (8) is that the actual asymptotic period as $\varepsilon \to \infty$ is also of high accuracy.

$$\lim_{\varepsilon\to\infty} \frac{T_{ex}}{T_1} = \frac{2\sqrt{3/4}}{\pi} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - 0.5\sin^2 x}} = 0.9294.$$

The lowest order approximation given by (8) is actually within 7.6% of the exact frequency regardless of the magnitude of εA^2.

3. Some examples

3.1. Example 1

We consider the simple mathematical pendulum which can be written in the form (Hagedorn (1981))

$$u'' + \omega^2 \sin u = 0, \quad u(0) = A, \quad u'(0) = 0, \tag{10}$$

where u designates the deviation angle from the vertical equilibrium position, $\omega^2 = g/l$, where g is the gravitational acceleration, l the length of the pendulum.

Many of the mathematical methods employed in nonlinear problems, as mentioned by Hagedorn (1981), may be successfully tested on the simplest mathematical system. The variational principle for Eq. (10) can be written as

$$J(u) = \int_0^t \left\{ -\frac{1}{2}u'^2 - \omega^2 \cos u \right\} dt. \tag{11}$$

Its Hamiltonian can be expressed in the form

$$H = \frac{1}{2}u'^2 - \omega^2 \cos u = -\omega^2 \cos A. \tag{12}$$

By the trial functional $u = A\cot \Omega t$, where Ω is the frequency to be determined. We, therefore, have

$$\frac{1}{2}A^2\Omega^2 \sin^2 \Omega t - \omega^2 \cos(A\cos \Omega t) = -\omega^2 \cos A. \tag{13}$$

If we collocate at $\Omega t = \pi/2$, we obtain

$$\frac{\Omega}{\omega} = \sqrt{\frac{2(1-\cos A)}{A^2}} = \sqrt{\frac{2(A^2/2! - A^4/4! + \cdots)}{A^2}} = \sqrt{1 - \frac{1}{12}A^2 + \frac{1}{360}A^4 - \cdots}. \tag{14}$$

In case $A = \pi/2$, the value obtained from (14) is $T = T_0 A/\sqrt{2} = 1.11T_0$, the approximate period obtained by Hagedorn (1981) is $T_{ex} = 1.16T_0$, where $T_0 = 2\pi/\omega$. The 4.24% accuracy is also remarkable good in view of the lowest order approximation.

3.2. Example 2

Consider the motion of a ball-bearing oscillating in a glass tube that is bent into a curve such that the restoring force depends upon the cube of the displacement u. The governing equation, ignoring frictional losses, is (Acton and Squir, 1985)

$$\frac{d^2u}{dt^2} + \varepsilon u^3 = 0, \tag{15}$$

and the auxiliary conditions are that the ball-bearing is released from rest at a displacement u_0 when $t = 0$. Expressed mathematically, this is

$$u(0) = A, \quad u'(0) = 0. \tag{16}$$

In our study, the parameter ε needs not to be small, i.e. it follows $0 < \varepsilon < +\infty$. For this special example, the traditional perturbation methods cannot be applied even in case $0 < \varepsilon \ll 1$, for the unperturbed equation $u'' = 0$ cannot lead to a period solution. Similarly we obtain the following Hamiltonian

$$H = \frac{1}{2}u'^2 + \frac{1}{4}\varepsilon u^4 = \frac{1}{4}\varepsilon A^4. \tag{17}$$

Choosing the trial function $u = A\cos\omega t$, we obtain the following residual equation:

$$A^2\omega^2 \sin^2 \omega t + \frac{1}{2}\varepsilon A^4 \cos^4 \omega t = \frac{1}{2}\varepsilon A^4. \tag{18}$$

Collocate at $\omega t = \pi/4$ to find the constant ω:

$$\omega = \sqrt{\frac{3}{4}\varepsilon A^2}. \tag{19}$$

Its period, therefore, can be written as

$$T = \frac{4\pi}{\sqrt{3}}\varepsilon^{-1/2}A^{-1} = 7.25\varepsilon^{-1/2}A^{-1}. \tag{20}$$

Its approximate period obtained by Acton and Squir (1985) reads

$$T_{ex} = 7.4164\varepsilon^{-1/2}A^{-1}. \tag{21}$$

The maximal relative error is <2.2% for all $\varepsilon > 0$!

4. Conclusion

In this paper, we first propose a new technique called *energy balance* for nonlinear oscillation. The examples show that even the lowest order approximations obtained by the present theory are actually of high accuracy regardless of the magnitude of ε. This paper only gives the preliminary report on this theory, and the numerical results are not explicitly justified, we feel that there still leave much space to be further improved. Serious theory proof and further applications will be considered in other papers.

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