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Parkes' problems revisited— Application of response number

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Abstract

Response number $R_n(n)$, proposed in [3, 4], is an important independent dimensionless number for the dynamic response of structures [2]. In this paper, the response number is applied to the dynamic plastic response of the well-known Parkes' problem, i.e., beams struck by concentrated mass.

Keywords: Dynamic plastic response; Parkes' problems; Beams; Zhao's response number; Impact

List of symbols

D_n	Johnson's damage number
G	mass of projectile
H	thickness
I_0	impulse per unit area
L	characteristic dimension of a structure
m	mass per unit length of beam
M	mass of projectile
M_0	$\sigma_0 H/4$, fully plastic bending moment
n	positive real number
R_n	response number for $n=2$
$R_n(n)$	response number for a given n
t	time
V_0	initial impact velocity
w_f	final permanent transverse displacement
w_m	final permanent transverse displacement for mid-span of beam
w_t, w_{tf}	final permanent transverse displacement for beam tip
x	position in span for beam
α	mL/M, mass ratio
β	$\alpha\varphi$
φ	x/L
η	L/H
ρ	density
σ_0	yield stress

1. Introduction

Many engineering structures are composed of the basic structural elements, such as beams, plates, and shells. Until now, there have been a large number of theoretical and experimental studies on the dynamic plastic behaviour of these basic elements. In order to compare results presented in the studies on deformed structures with similar geometries, boundary conditions, and loading, it seems necessary to normalize all variables into dimensionless forms. A general dimensional analysis for structural mechanics has been discussed by Jones in [1], where important

physical quantities in the dynamic inelastic response are considered in developing a complete set of dimensionless numbers using Buckingham Π theorem. The dimensionless numbers obtained from dimensional analysis are useful for scaling purpose and for organizing experimental model tests and numerical calculations to avoid any unnecessary repetition of the results in dimensionless space [2].

Recently, a new dimensionless number, response number,

$$R_n(n) = \frac{I_0^2}{\rho\sigma_0 H^2}\left(\frac{L}{H}\right)^n, \qquad (1)$$

has been suggested by Zhao [3,4] for dynamic plastic response of structures made of rigid, perfectly plastic material. Here, I_0 is the impulse per unit area of the impact loading, ρ is the material density, V_0 is the impact velocity, σ_0 is the yield stress of the material, L is the half length of beams or plates and H is the thickness of beams or plates. When $n = 2$, $R_n(2)$ is abbreviated as R_n. For impulsive loading, the response number can be expressed as

$$R_n(n) = \frac{\rho V_0^2}{\sigma_0}\left(\frac{L}{H}\right)^n = D_n\left(\frac{L}{H}\right)^n, \qquad (2)$$

where D_n is Johnson's damage number [5]

$$D_n = \frac{\rho V_0^2}{\sigma_0}, \qquad (3)$$

for assessing the behaviour of various metal structures subjected to dynamic loading. Johnson's damage number is a basic dimensionless similarity parameter in material dynamics.

The response number $R_n(n)$ is an important independent dimensionless number [2] and might be used extensively for the dynamic plastic response of structure. Now $R_n(n)$ has been used for the dynamic plastic response of structural members in [2], structural bifurcation buckling in [4], plates in [6], and shells in [7] under uniformly distributed loading.

Concentrated impact is one of the important loading types in structural impact dynamics. Can $R_n(n)$ also be used for the dynamic plastic response of beams subjected to concentrated impact loading? In the presented paper, application of response number will be made for Parkes' problems [8,9], i.e., rigid, perfectly plastic beams subjected to mass impact.

2. Parkes' beams of mass impact

2.1 Impact of a mass on a fully clamped beam

In Ref. [8], Parkes has studied the dynamic plastic response of a built-in beam, which has length $2L$, thickness H and unit breadth, with a transversely moving mass striking at any position in the span.

When struck at the mid-span by a mass M traveling with an initial velocity V_0 as illustrated in Figure 1 (a), the mid-span of the beam travels with a velocity V_0 at the instant of impact, and the remainder of the beam is stationary. Therefore, to maintain dynamic equilibrium, a disturbance propagates away from the mid-span, while the

striker is assumed to remain in contact with the beam. In fact two distinct phases of motion occur.

A plastic hinge develops under the impact point at $t = 0$ and two plastic hinges propagate the disturbance away from the mid-span towards the supports and into the undeformed portions of motion, as indicated in Figure 1 (b). The plastic hinges remain stationary at the supports and the mid-span during the final phase of motion, as indicated in Figure 1 (c), until the beam and striker come to rest, when all the initial kinetic energy of the striking mass $MV_0^2/2$ is dissipated plastically.

The final permanent transverse displacement profile for the problem in Figure 1 (a) is obtained as follow [1]

$$w_f = \frac{M^2 V_r^2}{24 m M_0}\left\{\frac{\alpha-\beta}{(1+\alpha)(1+\beta)} + 2\ln\left(\frac{1+\alpha}{1+\beta}\right)\right\}, \quad (4)$$

where m is mass per unit length of beam, M is the projectile mass, $M_0 = \sigma_0 H^2/4$ is the full plastic bending moment, $\alpha = mL/M$, $\beta = mx/M$ and $0 \le \beta \le \alpha$.

When $M/mL >> 1$, the final permanent transverse displacement is

$$w_f = MV_0^2 L(1 - x/L)/8M_0. \quad (5)$$

Therefore, for the mid-span, we have

$$w_m = w_f\big|_{x=0} = \frac{MV_0^2 L}{8M_0}. \quad (6)$$

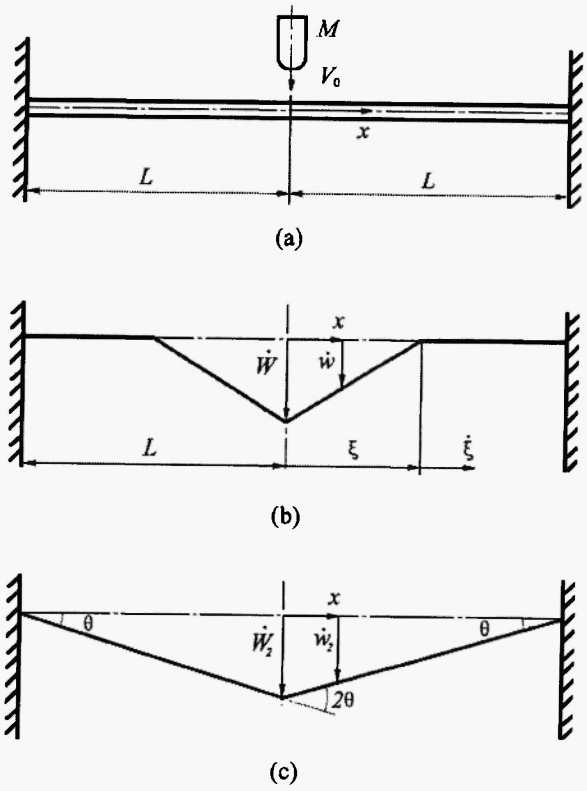

Figure 1 Impact of a mass on a fully clamped beam

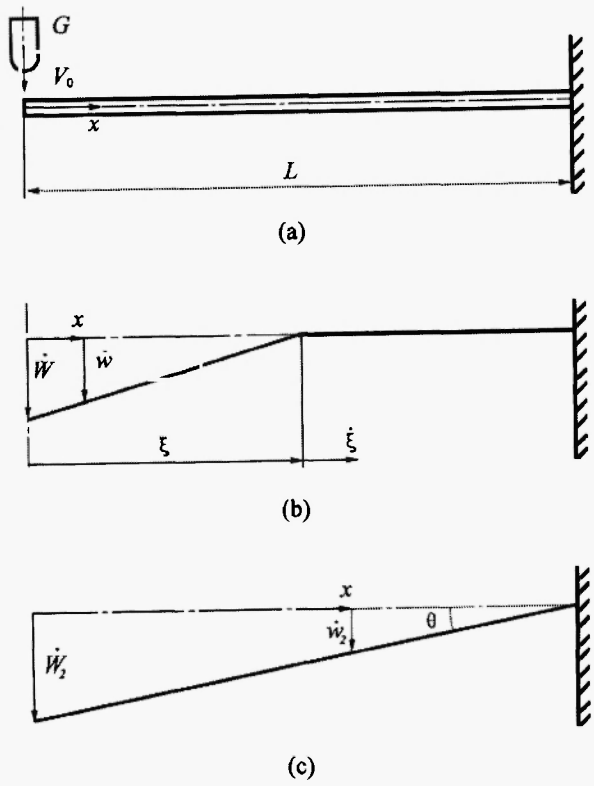

Figure 2 Impact of a mass on a cantilever beam

When $M/mL \ll 1$, the final permanent transverse displacement is

$$w_f = \frac{M^2 V_0^2 L}{12 m M_0} \ln\left(\frac{mL/M}{1+mx/M}\right), \tag{7}$$

and for the mid-span,

$$w_m = w_f\Big|_{x=0} = \frac{M^2 V_0^2 L}{12 m M_0} \ln(mL/M). \tag{8}$$

2.2 Impact of a mass on a cantilever beam

The dynamic response of a cantilever beam subjected to large dynamic loads, which produce an inelastic material behaviour, has been examined theoretically by many authors using rigid plastic methods of analysis. In particular, Parkes [9] studied the behaviour of a cantilever beam with length L which was struck at the tip by a mass G traveling with a velocity V_0, as shown in Figure 2 (a).

When a mass G strikes the tip of the cantilever beam, a disturbance develops immediately underneath the mass which propagates by means of a plastic hinge into the undeformed region of the beam, as shown in Figure 2 (b). The traveling plastic hinge eventually reaches the base of the cantilever at the end of the first phase of motion. The residual kinetic energy in the beam and mass is then dissipated at the plastic hinge which remains stationary at the support throughout a second phase of motion, as illustrated in Figure 2 (c).

The final permanent transverse displacement profile for the problem illustrated in Figure 2 is expressed as [1]

$$w_f = \frac{G V_0^2 L}{12 M_0}\left[\frac{1-x/L}{(1+\alpha)(1+\beta)} + \frac{2}{\alpha}\ln\left(\frac{1+\alpha}{1+\beta}\right)\right] \tag{9}$$

where $\alpha = mL/(2G)$, $\beta = mx/(2G)$, $G = M/2$.

At the beam tip, we have the expression as

$$w_t = w_f\Big|_{x=0} = \frac{G V_0^2 L}{12 M_0}\left[\frac{1}{1+\alpha} + \frac{2}{\alpha}\ln(1+\alpha)\right], \tag{10}$$

For one special case of heavy strikers (that is $G/mL \gg 1$, or $\alpha \to 0$), the final permanent transverse displacement profile can be expressed as

$$w_f = \frac{G V_0^2 L}{4 M_0}(1 - x/L), \tag{11}$$

and

$$w_t = w_f\Big|_{x=0} = \frac{G V_0^2 L}{4 M_0}. \tag{12}$$

For another special case of light strikers ($G/mL \ll 1$, or $\alpha \gg 1$), the final permanent transverse displacement profile is

$$w_f = \frac{G^2 V_0^2}{3 m M_0} \ln\left(\frac{mL/2G}{1+mx/2G}\right). \tag{13}$$

At the tip of beam, there is

$$w_t = w_f\Big|_{x=0} = \frac{G^2 V_0^2}{3 m M_0} \ln \alpha. \tag{14}$$

2.3 The recasted results on Parkes' problems

With the response number $R_n(n)$, we can make the following new forms for equations in the above section 2.1 and 2.2.

Noticing $m = \rho H$ and $M_0 = \sigma_0 H^2/4$, equation (4) can be rewritten as

$$w_f = \frac{R_n H}{6\alpha^2}\left\{\frac{\alpha-\beta}{(1+\alpha)(1+\beta)} + 2\ln\left(\frac{1+\alpha}{1+\beta}\right)\right\}, \tag{15}$$

then we have the final dimensionless permanent transverse deformation profile

$$\frac{w_f}{H} = \frac{R_n}{6\alpha^2}\left\{\frac{\alpha-\beta}{(1+\alpha)(1+\beta)} + 2\ln\left(\frac{1+\alpha}{1+\beta}\right)\right\}. \tag{16}$$

Equation (5) can be expressed as

$$w_f = \frac{R_n H}{2\alpha}(1 - x/L), \tag{17}$$

and then in dimensionless form

$$\frac{w_f}{H} = \frac{R_n}{2\alpha}(1 - x/L). \tag{18}$$

Equation (6) can be rewritten as

$$w_m = w_f\big|_{x=0} = \frac{R_n H}{2\alpha}, \tag{19}$$

and then in dimensionless form

$$\frac{w_m}{H} = \frac{R_n}{2\alpha}. \tag{20}$$

For equation (7), we have

$$w_f = \frac{R_n H}{3\alpha^2}\ln\left(\frac{\alpha}{1+\alpha\varphi}\right), \tag{21}$$

then in dimensionless form

$$\frac{w_f}{H} = \frac{R_n}{3\alpha^2}\ln\left(\frac{\alpha}{1+\alpha\varphi}\right), \tag{22}$$

where $\varphi = x/L$.

For equation (8), it can be reformulated as

$$w_m = \frac{R_n H}{3\alpha^2}\ln\alpha, \tag{23}$$

and then in dimensionless form

$$\frac{w_m}{H} = \frac{R_n}{3\alpha^2}\ln\alpha. \tag{24}$$

It is demonstrated by equations (20) to (24) that the dimensionless mid-span final deflection is determined by Zhao's response number and mass ratio.

In the same manner, equations (9) to (14) also can be reformulated as following new forms.

For equation (9), it can be expressed as

$$w_f = \frac{R_n H}{6\alpha}\left[\frac{1-\varphi}{(1+\alpha)(1+\alpha\varphi)} + \frac{2}{\alpha}\ln\left(\frac{1+\alpha}{1+\alpha\varphi}\right)\right], \tag{25}$$

hence,

$$\frac{w_f}{H} = \frac{R_n}{6\alpha}\left[\frac{1-\varphi}{(1+\alpha)(1+\alpha\varphi)} + \frac{2}{\alpha}\ln\left(\frac{1+\alpha}{1+\alpha\varphi}\right)\right]. \tag{26}$$

The equation (10) can be reformulated as

$$w_t = w_f\big|_{x=0} = \frac{R_n H}{6\alpha}\left[\frac{1}{1+\alpha} + \frac{2}{\alpha}\ln(1+\alpha)\right], \tag{27}$$

and in dimensionless form

$$\frac{w_t}{H} = \frac{R_n}{6\alpha}\left[\frac{1}{(1+\alpha)} + \frac{2}{\alpha}\ln(1+\alpha)\right]. \tag{28}$$

For equation (11), it can be expressed as

$$w_f = \frac{R_n H}{2\alpha}(1-\varphi), \tag{29}$$

then equation (12) can be recasted into

$$w_t = \frac{R_n H}{2\alpha}, \tag{30}$$

and in dimensionless form as follows

$$\frac{w_t}{H} = \frac{R_n}{2\alpha}. \tag{31}$$

For equation (13), we can recasted it into

$$w_f = \frac{R_n H}{3\alpha^2}\ln\left(\frac{\alpha}{1+\alpha\varphi}\right), \tag{32}$$

and then in dimensionless form

$$\frac{w_f}{H} = \frac{R_n}{3\alpha^2}\ln\left(\frac{\alpha}{1+\alpha\varphi}\right), \tag{33}$$

Equation (14) now can be expressed as

$$w_i = \frac{R_n H}{3\alpha^2}\ln\alpha, \tag{34}$$

then we have

$$\frac{w_t}{H} = \frac{R_n}{3\alpha^2}\ln\alpha. \tag{35}$$

It is also demonstrated that the dimensionless tip final deflection is determined by the Zhao's response number and mass ratio.

Compared with the expressions in section 2.1 and 2.2, the expressions in section 2.3 are more concise and it can be found that in the aforementioned two cases, with the response number $R_n^{\cdot}$, they have the same forms to describe the dynamic plastic response of beam under mass impact.

The mode solution of the final permanent transverse displacement of the tip of cantilever beam is given as

$$w_{tf} = \frac{V_0^2 L(G+mL/3)}{2M_0}, \tag{36}$$

where $M_0 = \sigma_0 H^2/4$ and $m = \rho H$.

Consequently, with the response number R_n and the mass ratio α, we can rewrite the equation (36) as

$$w_{tf} = \frac{R_n H}{3\alpha}(3+2\alpha), \tag{37}$$

and then in dimensionless form

$$\frac{w_{tf}}{H} = \frac{1}{3\alpha}R_n(3+2\alpha). \tag{38}$$

3. Impact of a mass on a long beam

For a long beam, when considering transverse shear effects, the following expression has been presented in Ref. [2]

$$\frac{w_f}{H} = \frac{G}{mL}\frac{\rho V_0^2}{\sigma_0}\frac{L}{H}\left(1+\frac{4}{3}\frac{G}{mL}\frac{L}{H}\right)\left(1+\frac{2}{3}\frac{G}{mL}\frac{L}{H}\right)^{-2} + \frac{2\rho V_0^2 L^2}{3\sigma_0 H^2}\left(\frac{G}{mL}\right)^2\left[1-\left(1+\frac{3}{2}\left(\frac{LG}{HmL}\right)^{-1}\right)^{-2}\right], \tag{39}$$

where G is the striker mass. With the response number $R_n(n)$, slenderness ration $\eta = L/H$, and mass ration $\alpha = mL/G$, the above equation can be easily recast as

$$\frac{w_f}{H} = \frac{3(3\alpha + 4\eta)}{(3\alpha + 2\eta)^2} R_n(1) + \frac{2\eta(3\alpha + 4\eta)}{\alpha(3\alpha + 2\eta)^2} R_n, \tag{40}$$

where

$$R_n(1) = \frac{\rho V_0^2}{\sigma_0}\left(\frac{L}{H}\right) \tag{41}$$

Equation (40) is composed of two parts. The first part involves the $R_n(1)$, which is related to the transverse shear effect, while the second part involves the R_n, which is related to the bending effect. It must be pointed out that the expression (40) can be reformulated more concisely as

$$\frac{w_f}{H} = \frac{R_n(1)}{\alpha}\left(\frac{3\alpha + 4\eta}{3\alpha + 2\eta}\right), \tag{42}$$

4. Discussion and concluding remarks

It has been shown that, with the Zhao's response number $R_n(n)$ or R_n and some other dimensionless numbers, such as α, η or φ, the aforementioned results on dynamic plastic response of beam subjected to a mass impact can be reformulated into new and more concise expressions, which are more physically meaningful and independent of dimensional units. Zhao's response number takes account of the geometrical influence of the structures on the dynamic response in addition to the inertia of the applied dynamic loading and the resistance ability of the material to the deformation due to the loading.

Including those expressions presented in Ref. [2, 3, 4, 6, 7], it has been demonstrated that the response number is an important dimensionless number extensively utilized for dynamic plastic response of structures made of rigid-perfectly plastic materials. Actually, it should be pointed out that Zhao's response number can be used to study the elastic, plastic and dynamic plastic problems, and this dimensionless number would have a more extensive utilization for structural dynamics.

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