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MICROMECHANICS ANALYSIS ON EVOLUTION OF CRACK IN VISCOELASTIC MATERIALS *

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Abstract: *A preliminary analysis on crack evolution in viscoelastic materials was presented. Based on the equivalent inclusion concept of micro-mechanics theory, the explicit expressions of crack opening displacement δ and energy release rate G were derived, indicating that both δ and G are increasing with time. The equivalent modulus of the viscoelastic solid comprising cracks was evaluated. It is proved that the decrease of the modulus comes from two mechanisms: one is the viscoelasticity of the material; the other is the crack opening which is getting larger with time.*

Key words: viscoelasticity; crack; micromechanics

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Introduction

As shown by ZHANG and XIONG (1997)[1], under constant loading the crack in viscoelastic materials has an incubation time, within it the crack opening displacement (COD) is changing larger with time, while the crack length keeps constant. Investigation on the gradual opening of the crack is meaningful to understand fracture behavior of viscoelastic materials. A preliminary research on crack opening displacement δ and energy release rate G of an embedded crack in viscoelastic material is presented.

In addition, although many studies on the effective modulus of cracked elastic solid have been carried out (Zhao, Tandon and Weng (1989)), rate investigation has been found for viscoelastic bodies comprising cracks. In the present article, an attempt is made to evaluate the changing tendency of the effective modulus of viscoelastic materials having embedded cracks.

1 Viscoelastic Constitutive Relationship

For three-dimensional problems, the viscoelastic constitutive relationships are usually decomposed into two equations for the hydrostatic and deviatoric parts respectively (Li and Weng (1994))[2]:

$$R(\mathrm{D})\sigma_{kk}(t) = S(\mathrm{D})\varepsilon_{kk}(t), \tag{1}$$

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Biography: ZHANG Shuang-yin (1938 –), Professor

$$P(\mathrm{D})\sigma_{ij}(t) = Q(\mathrm{D})\varepsilon_{ij}(t), \tag{2}$$

where $P(\mathrm{D})$, $Q(\mathrm{D})$, $R(\mathrm{D})$, $S(\mathrm{D})$ are operators, D stands for $\mathrm{d}/\mathrm{d}t$.
By introducing Laplace transformation

$$\hat{f}(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t.$$

Equations (1) and (2) change to those in Transformation Domain (*TD*)

$$R(s)\hat{\sigma}_{kk}(s) = S(s)\hat{\varepsilon}_{kk}(s), \tag{3}$$

$$P(s)\hat{\sigma}_{ij}(s) = Q(s)\hat{\varepsilon}_{ij}(s). \tag{4}$$

The bulk and shear moduli are

$$k^{TD}(s) = \frac{1}{3}\frac{S(s)}{R(s)}, \quad \mu^{TD}(s) = \frac{1}{2}\frac{Q(s)}{P(s)},$$

and the Young's modulus and Poisson's ratio are

$$E^{TD} = \frac{9k^{TD}\mu^{TD}}{3k^{TD} + \mu^{TD}}, \quad \nu^{TD} = \frac{3k^{TD} - 2\mu^{TD}}{2(3k^{TD} + \mu^{TD})}.$$

The Lame parameters are

$$\lambda^{TD} = 2\mu^{TD}\frac{\nu^{TD}}{1 - 2\nu^{TD}}.$$

The superscript TD expresses Transformed Domain. For one-dimensional case, the four-parameter model (Burgers model) gives the following Young's modulus:

$$E^{TD} = \frac{E_1\eta_1(E_2 + \eta_2 s)s}{E_1E_2 + [\eta_1E_2 + E_1(\eta_1 + \eta_2)]s + s^2\eta_2\eta_1},$$

where E_1, E_2, η_1, η_2 are two elastic moduli and two Newton's constants. For ED-6 resin, they are: $E_1 = 3.27\mathrm{GPa}$, $E_2 = 1.8\mathrm{GPa}$,, $\eta_1 = 8\ 000$ GPa.hr, $\eta_2 = 300$ GPa.hr. Through the inverse transformation $E(t)$ can be obtained (Li and Weng (1994))[2].

2 Equivalent Inclusion Simulation

In the following, the analysis is performed in the transformed domain. For the sake of concision, the superscript " *TD* " will be omitted. It is assumed that a penny shaped inclusion is embedded in an infinitely extended body. The space region of the inclusion, Ω can be expressed as (Fig.1):

$$\frac{x_1^2 + x_2^2}{r^2} + \frac{x_3^2}{c^2} \leqslant 1, \quad \frac{c}{r} \ll 1,$$

x_3 is perpendicular to the crack surface. The constants of matrix and inclusion are λ, μ and λ^*, μ^* respectively. A uniform traction, σ^∞ is applied in the remote boundary.

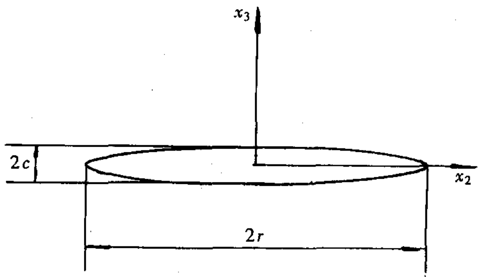

Fig.1 Schematic drawing of penny-shaped crack

Based upon Eshelby's equivalent inclusion theory, if the inclusion undergoes a plastic strain, $\varepsilon_{33}^p = \varepsilon_p$, the elastic stress field can be simulated by the following equations:

$$\sigma_{11} = \sigma_{22} = 2(\lambda^* + \mu^*)\varepsilon_{11} + \lambda^*(\varepsilon_{33} - \varepsilon_p) = 2(\lambda + \mu)(\varepsilon_{11} - \varepsilon_{11}^*) + \lambda(\varepsilon_{33} - \varepsilon_{33}^* - \varepsilon_p), \tag{5}$$

$$\sigma_{33} = 2\lambda^*\varepsilon_{11} + (\lambda^* + \mu^*)(\varepsilon_{33} - \varepsilon_p) = 2\lambda(\varepsilon_{11} - \varepsilon_{11}^*) + (\lambda + \mu)(\varepsilon_{33} - \varepsilon_{33}^* - \varepsilon_p), \tag{6}$$

where ε_{11}^*, $\varepsilon_{11}^* = \varepsilon_{22}^*$, ε_{33}^* are eigenstrain tensors. ε_{ij} is compatible strain tensor. For this case, we have

$$\varepsilon_{ij} = S_{ijkl}\varepsilon_{ij}^*, \tag{7}$$

where S_{ijkl} is Eshelby's tensor. For the thin penny shaped inclusion, the expressions of S_{ijkl} are given by Mura (1987)[3]:

$$S_{1133} = \frac{2\nu - 1}{8(1-\nu)}\frac{\pi c}{r}, \quad S_{3311} = \frac{\nu}{1-\nu}\left(1 - \frac{4\nu + 1}{8\nu}\frac{\pi c}{r}\right),$$

$$S_{1111} = -\frac{13 - 8\nu}{32(1-\nu)}\frac{\pi c}{r}, \quad S_{1122} = S_{2211} = \frac{8\nu - 1}{32(1-\nu)}\frac{\pi c}{r},$$

$$S_{3333} = 1 - \frac{(1 - 2\nu)}{4(1-\nu)}\frac{\pi c}{r}.$$

By putting (7) into (5) and (6) and omitting the terms involving $(c/r)^2$, we obtained

$$\varepsilon_{11}^* = \frac{\varepsilon_p}{D}\frac{\pi}{4(1-\nu)}\frac{c}{r}[\lambda^*\Delta\mu - 2\mu^*(2\Delta\lambda + \Delta\mu)](2\nu - 1), \tag{8}$$

$$\varepsilon_{33}^* = \frac{\varepsilon_p}{D}\frac{\pi}{4(1-\nu)}\frac{c}{r}[\lambda^*\Delta\mu(8\nu + 5) + 2\mu^*\{2\Delta\lambda^*(1 - 2\nu) + 3\Delta\mu\}], \tag{9}$$

where $\Delta\lambda = \lambda^* - \lambda$, $\Delta\mu = \mu^* - \mu$,

$$D = 2\lambda^*\left[\mu - 2\Delta\mu\frac{\nu}{1-\nu}\right] + 4\mu^*\left[\Delta\lambda\frac{\nu}{1-\nu} + \mu + \nu\right].$$

Substituting (8) and (9) into (5) and (6) yields

$$\sigma_{11} = \frac{1 + 4\nu}{4(1-\nu)}\frac{\pi c}{r}\varepsilon_p - 2\mu\frac{1+\nu}{1-\nu}\frac{\varepsilon_p}{D}\frac{\pi}{4(1-\nu)}\frac{c}{r} \times \{\Delta\mu\lambda^* - 2\mu^*(2\Delta\lambda + \Delta\mu)\}(2\nu - 1), \tag{10}$$

$$\sigma_{33} = -\frac{\mu}{2(1-\nu)}\frac{\pi c}{r}\varepsilon_p. \tag{11}$$

It can be seen that σ_{33} is independent on the elastic modulus of the inclusion. The similar conclusion can be found in the paper of Mori and Mura (1994)[4]. It implies that (11) is valid for any kind of inclusions or for crack.

The strain energy produced by ε_p is given by

$$E = \frac{\pi^2 r c^2}{3}\frac{\mu}{1-\nu}\varepsilon_p^2. \tag{12}$$

The potential of the applied load is

$$V = -\frac{4\pi r^2}{3}\sigma^\infty\varepsilon_p. \tag{13}$$

The total potential energy is the sum of E and V:

$$F = E + V. \tag{14}$$

Letting $\frac{\partial F}{\partial \varepsilon_p} = 0$, we obtain $\varepsilon_p = \sigma^\infty/[(\pi c/2r)(\mu/(1-\nu))]$. Comparing it with equation (11), it is obvious that $\sigma^\infty = -\sigma_{33}$. This indicates that the boundary condition of stress free on the crack surface is satisfied

$$(\sigma^{\infty} + \sigma_{33})\mid_{\Omega} = 0. \tag{15}$$

The strain energy release rate can be derived through the formula, $G = -\dfrac{\partial F}{\partial(\pi a^2)}$ and it is

$$G = \frac{2}{\pi}\frac{(1-\nu)}{\mu} r(\sigma^{\infty})^2. \tag{16}$$

Following the relation between stress intensity factor K and energy release rate G namely, $K^2 = 2\mu/(1-\nu)G$, the stress intensity factor was obtained

$$K = 2\sigma^{\infty}\sqrt{\frac{r}{\pi}}. \tag{17}$$

It is known that the crack opening displacement (COD) is calculated by equation $\delta = 2c\varepsilon_p$, thus,

$$\delta = \frac{4(1-\nu)}{\pi}\frac{r\sigma^{\infty}}{\mu}. \tag{18}$$

From the explicit expressions of (16) and (18), it is concluded that since μ is decreasing function of time (in the time domain), under constant applied load, both G and δ are gradually getting larger with time.

As is mentioned before, in equation (18), μ is the modulus in Laplace domain, by using the inverse transformation, $\delta(t)$ in time domain can be obtained. For (18) the inverse transformation can be conducted easily. For example, for the viscoelastic material of Maxwell type,

$$\mu(t) = \mu\exp\left(-\frac{t\mu}{\eta_0}\right),$$

where η_0, μ are the viscosity parameter and shear modulus of the Maxwell body. Thus

$$\delta(t) = \frac{4(1-\nu)}{\pi}\frac{r\sigma^{\infty}}{\mu\exp\left(-\dfrac{t\mu}{\eta_0}\right)}. \tag{18'}$$

It is obvious that $\delta(t)$ is increasing function of time.

3 Effective Modulus

According to Zhao, Tandon and Weng (1989)[5], for the elastic solids with unidirectionally aligned penny shaped cracks, the effective modulus in the perpendicular direction to the crack surface is given as

$$\frac{E_{33}}{E} = \frac{1}{1 + \dfrac{f_1}{f_0}\dfrac{4(1-\nu)r}{\pi c}}, \tag{19}$$

f_1, f_0 are the volume fractions of inclusion and matrix respectively. Introducing the symbol of crack density proposed by Budiansky and O'Connell (1976)[6], $\eta = Nr^3/\Omega$ (N is crack number, Ω is the total volume) and considering

$$f_1 = \frac{4}{3}\pi c r^2 N/\Omega = \frac{4c}{3r}\pi\eta,$$

equation (19) becomes

$$E_{33} = \frac{E}{1 + \dfrac{16}{3f_0}(1-\nu^2)\eta}. \tag{20}$$

Since $f_0 = 1 - f_1$ and $c = c_i + \delta(t)$, thus (20) is changed to

$$E_{33}(t) = \frac{3f_0 E(t)}{3f_0 + 16(1-\nu^2)\eta} = \frac{3\left(1 - \frac{4[c_i + \delta(t)]}{3r}\pi\eta\right)E(t)}{3\left(1 - \frac{4[c_i + \delta(t)]}{3r}\pi\eta\right) + 16(1-\nu^2)\eta}. \quad (21)$$

Where c_i is the initial COD of the crack. It is obvious that E_{33} is decreasing with time. Two mechanisms give the explanation: first, modulus of matrix E is decreasing function of time; secondly, COD is increased with time.

The brief analysis gives explicit expressions of G and δ. Since they are linear functions of μ, it is easy to obtain their inverse Laplace transformation. The equation (21) indicates that gradually opening of the cracks adds to the decreasing of effective modulus of viscoelastic materials comprising cracks.

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