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# ANALYSIS AND APPLICATION OF ELLIPTICITY OF STABILITY EQUATIONS ON FLUID MECHANICS *

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**Abstract**: *By using characteristic analysis of the linear and nonlinear parabolic stability equations (PSE), PSE of primitive disturbance variables are proved to be parabolic intotal. By using sub-characteristic analysis of PSE, the linear PSE are proved to be elliptical and hyperbolic-parabolic for velocity $U$, in subsonic and supersonic, respectively; the nonlinear PSE are proved to be elliptical and hyperbolic-parabolic for relocity $U + u$ in subsonic and supersonic, respectively. The methods are gained that the remained ellipticity is removed from the PSE by characteristic and sub-characteristic theories, the results for the linear PSE are consistent with the known results, and the influence of the Mach number is also given out. At the same time, the methods of removing the remained ellipticity are further obtained from the nonlinear PSE.*

**Key words**: compressible fluid; parabolic stability equation; characteristic; sub-characteristic

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## Introduction

In 1987, the parabolic stability equations (PSE) of fluid mechanics are presented by Herbert and co-workers[1], and they have become main methods to investigate the computation and theory of fluid flow. The analysis of computation of fluid flow shows: The diffusion parabolic stability equations have not been completely parabolized, there exists remained ellipticity[2~5]. Thanks to the remained ellipticity of the PSE, it not only has the high cost of computation to deal with the PSE directly, but also makes the streamwise marching of the solution ineffective. Haj-Hariri particularly investigated some mathematical details of the PSE[4], and analyzed the ellipticity of the PSE, and also gave some methods of removing the ellipticity of the PSE. Haj-Hariri and others suggest that each physical quantity $\Phi$ in the PSE is represented as a shape function $\phi$ and a wavelike component[2~5]. By using the unknown coefficient method, the sources of (unwanted) ellipticity in these equations are identified. Then, they removed the ellipticity from the PSE in acoustic and viscous. Therefore, PSE can be solved by using a marching procedure, which is

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economical and available. The goal of the present contribution is to analyze the remained ellipticity of the PSE by using characteristic and sub-characteristic which are used to study the grade structure equations of fluid mechanics in Ref.[2]. Mathematical characteristic and sub-characteristic imply that the PSE are not really parabolic, and only the name of the diffuse parabolic stability equations (DPSE) can reflect their mathematical and physical qualities. The advantages of the DPSE are that the description of the problem of stability of fluid mechanics is reasonable, and CPU hours and EMS memories are greatly economized for streamwise marching of the DPSE compared with the cost of the computations of the stability equations of fluid mechanics. The computations of the DPSE don't neen to regulate the outward flow boundary conditions of disturbance quantity, which is material simplification. Therefore, we further analyze mathematical characteristic of the PSE and remove the remained ellipticity of the PSE, our results for the linear PSE are consistent with that of Ref.[4], and we also give the influence of the Mach number out. At the same time, we further analyze the remained ellipticity from the nonlinear PSE, and also give out the methods of removing the remained ellipticity from the nonlinear PSE.

## 1 The Linear and Nonlinear Parabolic Stability Equations (PSE)

Let the fluid flow satisfy the following two-dimensional compressible basic equations of fluid mechanics (BEFM)[2]

$$S_t\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0,\tag{1a}$$

$$S_t\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{\rho Re}\left\{\frac{\partial}{\partial x}\left[\mu\left(\frac{4}{3}\frac{\partial u}{\partial x}-\frac{2}{3}\frac{\partial v}{\partial y}\right)\right]+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right\},\tag{1b}$$

$$S_t\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{\rho Re}\left\{\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{4}{3}\frac{\partial v}{\partial y}-\frac{2}{3}\frac{\partial u}{\partial x}\right)\right]+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial y}\right)\right\},\tag{1c}$$

$$\begin{aligned}&S_t\left(\rho C_p\frac{\partial T}{\partial t}-\frac{\partial p}{\partial t}\right)+\rho C_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)-\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)=\\&\quad\frac{C_p}{Re}\left[\frac{\partial}{\partial x}\left(\frac{\mu}{p_r}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{p_r}\frac{\partial T}{\partial y}\right)\right]+\frac{\mu}{Re}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\right.\\&\quad\left.2\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right],\end{aligned}\tag{1d}$$

$$p=\rho T.\tag{1e}$$

For the sake of argument, a parallel boundary layer over a flat plate is studied. Let the basic state be

$$(U(x,y,t),0,T(x,y,t),\bar\rho(x,y,t)),$$

which also satisfy the basic equations of fluid mechanics (1a) ~ (1e). The idea of small disturbance is that a little disturbance satisfying the basic equations of fluid mechanics is added into the fluid flow. Therefore, the parameter of fluid flow can be expressed by summation of the parameters of the basic flow and the disturbance flow. Assuming the linear compressible disturbances are $(u,v,\theta,\rho)$, and the velocity, temperature and density of the shear flow respectively are

$$\tilde{U} = U + u, \quad \tilde{V} = v, \quad \tilde{T} = T + \theta, \quad \tilde{\rho} = \bar{\rho} + \rho, \tag{2}$$

Let the formula (2) be replaced into Eqs. (1a) ~ (1d). The pressure $p$ of Eqs. (1a) ~ (1d) is eliminated by using the equation of state (1e). Then, we obtain the following two-dimensional compressible parabolic stability equations:

$$(U + u)\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + (\bar{\rho} + \rho)\frac{\partial u}{\partial x} + (\bar{\rho} + \rho)\frac{\partial v}{\partial y} = F_1, \tag{3a}$$

$$\frac{T + \theta}{\bar{\rho} + \rho}\frac{\partial \rho}{\partial x} + (U + u)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial x} - \frac{4\mu}{3(\bar{\rho} + \rho)Re}\frac{\partial^2 u}{\partial x^2} + \frac{2\mu}{3(\bar{\rho} + \rho)Re}\frac{\partial^2 v}{\partial x \partial y} - \frac{\mu}{3(\bar{\rho} + \rho)Re}\frac{\partial^2 v}{\partial y \partial x} - \frac{\mu}{(\bar{\rho} + \rho)Re}\frac{\partial^2 u}{\partial y^2} = F_2, \tag{3b}$$

$$\frac{T + \theta}{\bar{\rho} + \rho}\frac{\partial \rho}{\partial y} + (U + u)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial \theta}{\partial y} - \frac{\mu}{(\bar{\rho} + \rho)Re}\frac{\partial^2 v}{\partial x^2} + \frac{2\mu}{3(\bar{\rho} + \rho)Re}\frac{\partial^2 u}{\partial y \partial x} - \frac{\mu}{(\bar{\rho} + \rho)Re}\frac{\partial^2 u}{\partial x \partial y} - \frac{4\mu}{3(\rho + \bar{\rho})Re}\frac{\partial^2 v}{\partial y^2} = F_3, \tag{3c}$$

$$(\rho + \bar{\rho})(u + U)(C_p - 1)\frac{\partial \theta}{\partial x} + (\bar{\rho} + \rho)v(C_p - 1)\frac{\partial \theta}{\partial y} - (T + \theta)(u + U)\frac{\partial \rho}{\partial x} - (T + \theta)v\frac{\partial \rho}{\partial y} - \frac{C_p\mu}{PrRe}\frac{\partial^2 \theta}{\partial x^2} - \frac{C_p\mu}{PrRe}\frac{\partial^2 \theta}{\partial y^2} = F_4, \tag{3d}$$

where, $U, u, v$ are united by $U_e$; $x, y$ are united by $L$; $T, \theta$ and $C_p$ are united by gas constant $R$; $\rho$ and $\mu$ are united by $\rho_e$ and $\mu_e$, respectively. $Re = \rho_e U_e L/\mu_e$ is the Reynolds number. $F_1$, $F_2$, $F_3$ and $F_4$ denote all of the other terms except for all of the terms of $\partial/\partial x$ and $\partial/\partial y$ in the equations. In the remained parts of our paper, $F_1, F_2, F_3$ and $F_4$ always denote the same masning terms, and will not be explained any more (see Refs. [6 ~ 9])

By neglecting viscous partial derivative terms with respect to $x$ in Eqs. (3a) ~ (3d)), one obtains the following nonlinear diffusion parabolic stability equations:

$$(U + u)\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + (\bar{\rho} + \rho)\frac{\partial u}{\partial x} + (\bar{\rho} + \rho)\frac{\partial v}{\partial y} = F_1, \tag{4a}$$

$$\frac{T + \theta}{\rho + \bar{\rho}}\frac{\partial \rho}{\partial x} + (U + u)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial x} - \frac{\mu}{(\rho + \bar{\rho})Re}\frac{\partial^2 u}{\partial y^2} = F_2, \tag{4b}$$

$$\frac{T + \theta}{\rho + \bar{\rho}}\frac{\partial \rho}{\partial y} + (U + u)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{\partial \theta}{\partial y} - \frac{4\mu}{3(\rho + \bar{\rho})Re}\frac{\partial^2 v}{\partial y^2} = F_3, \tag{4c}$$

$$(\rho + \bar{\rho})(u + U)(C_p - 1)\frac{\partial \theta}{\partial x} + (\rho + \bar{\rho})v(C_p - 1)\frac{\partial \theta}{\partial y} - (T + \theta)(u + U)\frac{\partial \rho}{\partial x} - (T + \theta)v\frac{\partial \rho}{\partial y} - \frac{C_p\mu}{PrRe}\frac{\partial^2 \theta}{\partial y^2} = F_4. \tag{4d}$$

Furthermore, let the disturbances be very small, that is to say, $u \ll U, v \ll V, p \ll P$ and $\theta \ll T$, then one obtains the following linear diffusion parabolic stability equations:

$$U\frac{\partial \rho}{\partial x} + \bar{\rho}\frac{\partial u}{\partial x} + \bar{\rho}\frac{\partial v}{\partial y} = F_1, \tag{5a}$$

$$\frac{T}{(\rho + \bar{\rho})}\frac{\partial \rho}{\partial x} + U\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial x} - \frac{\mu}{(\rho + \bar{\rho})Re}\frac{\partial^2 u}{\partial y^2} = F_2, \tag{5b}$$

$$U\frac{\partial v}{\partial x} + \frac{T}{(\rho + \bar{\rho})}\frac{\partial \rho}{\partial y} - \frac{4\mu}{3(\rho + \bar{\rho})Re}\frac{\partial^2 v}{\partial y^2} + \frac{\partial \theta}{\partial y} = F_3, . \tag{5c}$$

$$- \theta U \frac{\partial \rho}{\partial x} + (C_p - 1)\bar{\rho} U \frac{\partial \theta}{\partial x} - \frac{C_p \mu}{PrRe} \frac{\partial^2 \theta}{\partial y^2} = F_4 . \tag{5d}$$

By the following analysis of characteristic and sub-characteristic, we know that Eqs. (4a) ~ (4d) and (5a) ~ (5d) are parabolic, and there exists remained ellipticity in the subsonic. Therefore, we always call Eqs. (4a) ~ (4d) and (5a) ~ (5d) the diffusion parabolic stability equations (DPSE).

## 2 Characteristic and Sub-Characteristic of the Linear PSE

By application of theories analysis of Ref. [1], there exist two sorts of the transmission fashion of information, one is the convection-diffusion transmission of information, the other is the convection-disturbances transmission of information. The first one is determined by the characteristic of the basic equations of fluid mechanics (BEFM), and the second one is determined by the characteristic of the sub-characteristic equations which are obtained by neglecting all viscous partial derivative terms of BEFM. In the following, in terms of characteristic and sub-characteristic theory for grade structure equations of fluid mechanics in Ref. [2], we will investigate characteristic and sub-characteristic of the PSE and study how to remove the remained ellipticity from the PSE. In order to obtain the characteristic of the linear PSE, let

$$\frac{\partial u}{\partial y} = U^{(y)}, \quad \frac{\partial v}{\partial y} = V^{(y)}, \quad \frac{\partial \theta}{\partial y} = \theta^{(y)} . \tag{6}$$

Then, the linear diffusion parabolic stability equations (5a) ~ (5d) can be denoted by the following first-order simulative linear partial differential equation on $Z = (\rho, u, U^{(y)}, v, V^{(y)}, \theta, \theta^{(y)})$

$$A \frac{\partial Z}{\partial x} + B \frac{\partial Z}{\partial y} = F, \tag{7}$$

where $Z$ and $F$ are seventh-order vectors, $A$ and $B$ are $7 \times 7$ matrixes. The determinant equation is

$$\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = 0, \tag{8}$$

where

$$\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = \begin{vmatrix} U\sigma_1 & \bar{\rho}\sigma_1 & 0 & \bar{\rho}\sigma_2 & 0 & 0 & 0 \\ \frac{T}{\rho + \bar{\rho}}\sigma_1 & U\sigma_1 & \frac{-\mu}{(\rho + \bar{\rho})Re}\sigma_2 & 0 & 0 & \sigma_1 & 0 \\ \frac{T}{\rho + \bar{\rho}}\sigma_2 & 0 & 0 & U\sigma_1 & \frac{-4\mu}{3(\rho + \bar{\rho})Re}\sigma_2 & \sigma_2 & 0 \\ -TU\sigma_1 & 0 & 0 & 0 & 0 & \bar{\rho}U\sigma_1(C_p - 1) & \frac{-C_p\mu}{PrRe}\sigma_2 \\ 0 & \sigma_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_1 & 0 \end{vmatrix} =$$

$$- \frac{4 C_p \mu^3}{3 Pr(\rho + \bar{\rho})^2 Re^3} \sigma_1^4 \sigma_2^3 .$$

Then the characteristic values are

$$\sigma_1^4 = 0, \quad \sigma_2^3 = 0. \tag{9}$$

All the characteristic values are zero. Therefore, the diffusion parabolic stability equations (5a) ~ (5d) are parabolic. In a similar way, one can gain all characteristic values of two-dimensional compressible stability equations (3a) ~ (3d) which are

$$\sigma_1^3 = 0, \quad \lambda_4 = \frac{v}{U + u}, \quad \lambda_{5,6} = \pm i, \quad \lambda_{7,8} = \pm i, \quad \lambda_{9,10} = \pm i, \tag{10}$$

where $\lambda = -\sigma_1/\sigma_2$. Third-order 0-characteristic roots are related to the three equations with the main portions $\partial u/\partial x$, $\partial v/\partial x$ and $\partial T/\partial x$, respectively. They are independent of the other seven equations of Eq. (7), the other seven characteristic roots are complex except for one real root. Therefore, two-dimensional compressible stability equations (3a) ~ (3d) are elliptical.

Let us now turn to consider the relation of sub-characteristic and the Mach number. In fact, parts of ellipticity for the linear PSEs (5a) ~ (5d) are remained in subsonic regions. Removing all of the viscous terms of the linear parabolic stability equations (5a) ~ (5d), then the governing equations become the following sub-characteristic stability equations:

$$U\frac{\partial \rho}{\partial x} + \bar{\rho}\frac{\partial u}{\partial x} + \bar{\rho}\frac{\partial v}{\partial y} = F_1, \tag{11a}$$

$$\frac{T}{(\rho + \bar{\rho})}\frac{\partial \rho}{\partial x} + U\frac{\partial u}{\partial x} + \frac{\partial \theta}{\partial x} = F_2, \tag{11b}$$

$$U\frac{\partial v}{\partial x} + \frac{.T}{\rho + \bar{\rho}}\frac{\partial \rho}{\partial y} + \frac{\partial \theta}{\partial y} = F_3, \tag{11c}$$

$$-\theta U\frac{\partial \rho}{\partial x} + (C_p - 1)\bar{\rho}U\frac{\partial \theta}{\partial x} = F_4. \tag{11d}$$

Ley $\boldsymbol{Z} = (\rho, u, v, V^{(y)}, \theta)$. Then, sub-characteristic stability equations (11a) ~ (11d) can be denoted by the following first-order simulative linear partial differential equation:

$$\boldsymbol{A}\frac{\partial \boldsymbol{Z}}{\partial x} + \boldsymbol{B}\frac{\partial \boldsymbol{Z}}{\partial y} = \boldsymbol{F}. \tag{12}$$

Then the sub-characteristic equation is

$$\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = \bar{\rho}^2 U^2 (C_p - 1)\sigma_1^2[(U^2 - a^2)\sigma_1^2 - a^2\sigma_2^2] = 0, \tag{13}$$

where $TC_p/(C_p - 1) = a^2$, $a$ is the velocity of sound. From the sub-characteristic equation (13), we have

$$\sigma_1^2 = 0, \quad \lambda_{3,4} = \pm\frac{1}{\sqrt{M_U^2 - 1}}, \tag{14}$$

where $\lambda = -\sigma_1/\sigma_2$, $M_U = U/a$ is the Mach number of the non-disturbed flow. It is evident that the sub-characteristic of linear PSE is related to $M_U$. If $M_U > 1$, the sub-characteristic values are real which implies that linear PSEs (5a) ~ (5d) are hyperbolic-parabolic. If $M_U < 1$, the sub-characteristic values are complex which implies that linear PSEs (5a) ~ (5d) exist remained ellipticity. It is obvious that the remained ellipticity is not caused by the viscous diffusion terms. We will talk about how to remove the ellipticity from the linear PSE in Section 4.

## 3 Characteristic and Sub-Characteristic of the Nonlinear PSE

Next we consider the characteristic and sub-characteristic of nonlinear PSE, then consider the difference of characteristic and sub-characteristic of linear and nonlinear PSE, and the relation of the sub-characteristic of nonlinear PSE and the Mach number. Let

$$Z = (\rho, u, U^{(y)}, v, V^{(y)}, \theta, \theta^{(y)}). \tag{15}$$

Then, nonlinear PSEs (4a) ~ (4d) can be denoted by the following first-order simulative linear partial differential equation on $Z = (\rho, u, U^{(y)}, v, U^{(y)}, \theta, \theta^{(y)})$:

$$A\frac{\partial Z}{\partial x} + B\frac{\partial Z}{\partial y} = F. \tag{16}$$

The determinant equation is

$$\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = -\frac{4C_p\mu^3}{3Pr(\rho+\bar{\rho})^2 Re^3}[(U+u)\sigma_1 + v\sigma_2]\sigma_1^3\sigma_2^3 = 0. \tag{17}$$

Then the characteristic values are

$$\lambda_1 = \frac{V}{U+u},\ \sigma_1^3 = 0,\ \sigma_2^3 = 0. \tag{18}$$

All the characteristic values are real, therefore nonlinear PSEs (4a) ~ (4d) are hyperbolic-parabolic. Let us now turn to consider the relation of sub-characteristic and the disturbance Mach number $M_{u+U}$, a part of ellipticity of the nonlinear PSEs (4a) ~ (4d) virtually remained in subsonic regions.

Removing all of the viscous terms of nonlinear PSEs (4a) ~ (4d), one obtains the sub-characteristic stability equations

$$(u+U)\frac{\partial \rho}{\partial x} + (\bar{\rho}+\rho)\frac{\partial u}{\partial x} + (\bar{\rho}+\rho)\frac{\partial v}{\partial y} + v\frac{\partial \rho}{\partial y} = F_1, \tag{19a}$$

$$\frac{T+\theta}{\bar{\rho}+\rho}\frac{\partial \rho}{\partial x} + (u+U)\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial \theta}{\partial x} = F_2, \tag{19b}$$

$$(u+U)\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{T+\theta}{\rho+\bar{\rho}}\frac{\partial \rho}{\partial y} + \frac{\theta}{\partial y} = F_3, \tag{19c}$$

$$-(T+\theta)(u+U)\frac{\partial \rho}{\partial x} - (T+\theta)v\frac{\partial \rho}{\partial y} + (C_p-1)(\rho+\bar{\rho})(u+U)\frac{\partial \theta}{\partial x} + (C_p-1)(\rho+\bar{\rho})v\frac{\partial \theta}{\partial y} = F_4. \tag{19d}$$

The nonlinear equations (19a) ~ (19d) can be translated into the following united first-order simulative linear partial differential equations on $Z = (\rho, u, v, \theta)$:

$$A\frac{\partial Z}{\partial x} + B\frac{\partial Z}{\partial y} = F. \tag{20}$$

And the sub-characteristic values are solved out to be

$$\det(\sigma_1 a_{ij} + \sigma_2 b_{ij}) = (\rho+\bar{\rho})[-(u+U)\sigma_1 + v\sigma_2]^2(C_p-1)\times \{[(u+U)^2 - a^2]\lambda^2 + 2v(u+U)\lambda - a^2\} = 0, \tag{21}$$

where $\lambda = -\sigma_1/\sigma_2$, $a^2 \cong a'^2 = (T+\theta)C_p/(C_p-1)$, $a'$ is sound velocity. By using sub-characteristic equation (21), four sub-characteristic values are solved to be

$$\lambda_{1,2} = \frac{v}{u + U}, \quad \lambda_{3,4} = \frac{M_v M_{u+U} \pm \sqrt{M_v^2 M_{u+U}^2 + M_{u+U}^2 - 1}}{\sqrt{M_{u+U}^2 - 1}}, \tag{22}$$

where $M_v = v/a$ is the Mach number which is related to the normal direction, $M_{u+U} = (u + U)/a$ the Mach number which is related to the flow direction. By using characteristic values $\lambda_{3,4}$, the sub-characteristic of nonlinear PSE is related to the Mach numer $M_{u+U}$. If $M_{u+U} > 1$, the sub-characteristic values are real which implies that nonlinear PSEs (4a) ~ (4d) are hyperbolic-parabolic. If $M_{u+U} < 1$, the sub-characteristic values are complex which implies that nonlinear PSEs (4a) ~ (4d) exist remained ellipticity. It is obvious that the remained ellipticity is not caused by the viscous diffusion terms. In the following, we will talk about how to remove the remained ellipticity from the nonlinear PSE.

## 4 Application of Characteristic and Sub-Characteristic Theories in Removing the Ellipticity from the PSE

From the analysis of Section 2 and Section 3, we gain the methods of removing the remained ellipticity from the PSE. From the linear PSE, the method of removing its ellipticity is: The characteristic values are real of the linear PSE. Therefore, the linear PSE are completely hyperbolic-parabolic. Then, sub-characteristic of the linear PSE is related to the non-disturbance flow Mach number $M_U$. If $M_U > 1$, sub-characteristic values are real, and the sub-characteristic equations are hyperbolic-parabolic. If $M_U < 1$, sub-characteristic values are complex, and there exists remained ellipticity for the linear PSE. It is obvious that the remained ellipticity is not caused by the viscous diffusion terms. By using the characteristic and sub-characteristic theories, if the partial term $\bar{\rho}\partial u/\partial x$ is neglected on disturbance velocity $\boldsymbol{u}$ of base flow for Eq. (5a) (or the term $(T/(\rho + \bar{\rho}))(\partial \rho/\partial x)$, for Eq. (5b), or the term $(T/(\rho + \bar{\rho}))(\partial \rho/\partial y)$, for Eq. (5c)), we can remove the remained ellipticity for the linear PSE. By using the above "neglected" operation, the characteristic values of the linear PSE are

$$\sigma_1^2 = 0, \ \sigma_2^2 = 0, \tag{23}$$

That the characteristic values are real implies that the linear PSE are parabolic. Therefore, the linear PSE have heen completely parabolized, and our results are consistent with that of Ref. [4].

By using characteristic and sub-characteristic theories, we can further talk about how to remove the remained ellipticity for the nonlinear PSE. Firstly, sub-characteristic of the nonlinear PSE is related with the disturbance flow Mach number $M_{u+U}$. If $M_{u+U} > 1$, sub-characteristic values are real, and the sub-characteristic equations are hyperbolic-parabolic. If $M_{u+U} < 1$, sub-characteristic values are complex, and there exists remained ellipticity for the nonlinear PSE. If the partial term $(\bar{\rho} + \rho)\partial u/\partial x$ is neglected on disturbance velocity $u$ of base flow for equation (4a), we can remove the remained ellipticity for the nonlinear PSE. By using the above "neglected" operation, the characteristic values of the nonlinear PSE are

$$\begin{cases} \lambda_{1,2} = \dfrac{v}{u + U}, \\ \lambda_3 = \dfrac{(C_p - 1)^{1/2} v - (T + \theta)^{1/2} C_p^{1/2}}{(C_p - 1)^{1/2} v}, \\ \lambda_4 = \dfrac{(C_p - 1)^{1/2} v + (T + \theta)^{1/2} C_p^{1/2}}{(C_p - 1)^{1/2} v}, \end{cases} \tag{24}$$

That the characteristic values are real implies that the sub-characteristic equations are hyperbolic-parabolic, and they are not related to the disturbance Mach number $M_{u+U}$. Therefore, the nonlinear PSE have been completely parabolized.

## 5 Conclusion

In this paper, characteristic and sub-characteristic theories[2] are used to analyze the mathematical characteristic and remove the remained ellipticity of the parabolized stability equations (PSE)[3,4]. By using our "neglected" operation, the linear or nonlinear PSE have been completely parabolized. By using characteristic and sub-characteristic theories, we can remove the remained ellipticity from the PSE, and our results for the linear PSE are consistent with that of Ref. [4], and we also give the influence of the Mach number out.

**References:**


[1] Herbert Th, Bertolotti F P. Stability analysis of non-parallel boundary layers[J]. *Bull American Phys Soc*, 1987, **32**(8):2097.

[2] GAO Zhi. Grade structure theory for the basic equations of fluid mechanics (BEFM) and the simplied Navier-Stokes equations (SNSE)[J]. *Acta Mechanica Sinica*, 1988, **20**(2), 107 - 116. (in Chinese)

[3] Herbert Th. Nonlinear stability of parallel flows by high-order amplitude expansions[J]. *AIAA J*, 1980, **18**(3):243 - 248.

[4] Haj-Hariri H. Characteristics analysis of the parabolic stability equations[J]. *Stud Appl Math*, 1994, **92**(1): 41 - 53.

[5] Chang C L, Malik M R, Erleracher G, *et al*. Compressible stability of growing boundary layers using parabolic stability equations[Z]. *AAIA*91 - 1636, New York: AAIA, 1991.

[6] GAO Zhi. Grade structure of simplified Navier-Stokes equations and its mechanics meaning and application[J]. *Science in China, Ser A*, 1987, **17**(10):1058 - 1070. (in Chinese)

[7] GAO Zhi, ZHOU Guang-jiong. Some advances in high Reynolds numbers flow theory, algorithm and appplication[J]. *Advances in Mechanics*, 2001, **31**(3):417 - 436.

[8] GAO Zhi, SHEN Yi-qing. Discrete fluid dynamics and flow numerical simulation[A]. In: F Dubois, WU Hua-mu Eds. *New Advances in Computational Fluid Dynamics*[C]. Beijing: Higher Education Press, 2001, 204 - 229.

[9] Schlichting H. *Boundary-Layer Theory*[M]. 7th ed. New York: McGraw-Hill, 1979.