Vol 13 No 10, October 2004
1009-1963/2004/13(10)/1631-08

Chinese Physics

Instability of the vertically forced surface wave in a circular cylindrical container*

Jian Yong-Jun(菅永军)[a)b)†] and E Xue-Quan(鄂学全)[b)]

[a)] *First Institute of Oceanography, State Oceanic Administration, Qingdao* 266061, *China*

[b)] *Institute of Mechanics, Chinese Academy of Sciences, Beijing* 100080, *China*

(Received 2 December 2003; revised manuscript received 25 May 2004)

The nonlinear amplitude equation, which was derived by Jian Yongjun employing expansion of two-time scales in inviscid fluids in a vertically oscillating circular cylindrical vessel, is modified by introducing a damping term due to the viscous dissipation of this system. Instability of the surface wave is analysed and properties of the solutions of the modified equation are determined together with phase-plane trajectories. A necessary condition of forming a stable surface wave is obtained and unstable regions are illustrated. Research results show that the stable pattern of surface wave will not lose its stability to an infinitesimal disturbance.

Keywords: instability, vertically forced oscillation, amplitude equation, phase-plane trajectories
PACC: 0340G, 4754, 5235M, 9530L

1. Introduction

Faraday[1] (1831) waves can be excited on the free surface of a fluid layer that is periodically vibrated in the direction normal to the surface at rest when the amplitude of the driving acceleration is large enough to overcome the dissipative effect of fluid viscosity (Miles and Henderson[2] (1990)). These surface waves have a frequency equal to half that of the excitation and belong to subharmonic resonance. With a sinusoidal driving force, different wave patterns can be excited depending on the fluid properties and the driving amplitude or frequency. Many spatially periodic patterns, such as parallel stripes (Edwards and Fauve[3] (1994), Daudet *et al*[4] (1995)), triangular pattern (Müller[5] (1993)), competing hexagons and equilateral triangles(Kumar and Bajaj[6] (1994)), square (Ciliberto *et al*[7] (1991)), and hexagonal, eightfold, and tenfold (Binks and Water[8] (1991)) patterns, have been observed in experiment.

E and Gao[9−11] (1996, 1996, 1998) carried out the flow visualization of surface wave patterns in a circular cylindrical vessel by vertical external vibrations. They obtained very beautiful photographs of free surface patterns in wider driven frequencies, and most of them have not been reported before.

Benjamin and Ursell[12] (1954) showed that the linear dynamics of the amplitudes of the surface modes is governed by Mathieu's equation. Miles[13−15] (1976, 1984, 1993) studied nonlinear effects of this problem adopting a variational approach in inviscid fluids. Viscous effects are usually included heuristically, proportional to the kinematic viscosity ν. This approximation ignores viscous boundary layers along the container walls and beneath the surface, where additional dissipation occurs.

Recently, an approximate theoretical treatment associated with the experiments of Refs.[9–11] was established by Jian and E[16,17] (2003), from which the second-order free surface displacements and their contours were obtained by two-time scale singular perturbation expansion in ideal fluids. Later on, the influence of the surface tension and weak viscosity was considered by Jian and E[18,19] (2004), and the theoretical result approaches experiment results much more than that in the case of no surface tension and viscosity. An approximate expression for the damping coefficient was determined analytically in Ref.[19] by solving outer potential flow and the flow in the inner boundary layer region with the perturbation technique.

*Project supported by the National Natural Science Foundation of China (Grant Nos 19772063, 19772068).

†Correspondence author: E-mail: jianyongjun@yahoo.com.cn

In this paper, the nonlinear evolution equation in Refs.[16] and [17] is modified by introducing the above linear damping, and the stability of the modified amplitude equation is studied together with phase-plane trajectories. Unstable regions are determined by stability analysis with respect to the infinitesimal disturbance to the equilibrium solution.

2. The modification of the amplitude equation

The nonlinear amplitude equation in Refs.[16] and [17] is modified by adding linear damping, which was obtained from Ref.[19], and we obtain the modified equation

$$\mathrm{i}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}+\beta_1\right)p(\tau)=M_1p^2(\tau)\bar{p}(\tau)+M_2\mathrm{e}^{2\mathrm{i}\sigma\tau}\bar{p}(\tau), \quad (1)$$

where i is the unit of imaginary number, $p(\tau)$ is called the slowly variable complex amplitude and $\bar{p}(\tau)$ denotes the complex conjugate of $p(\tau)$; τ is a slowly varying time scale, σ denotes the difference between the surface wave frequency and the forced frequency, β_1 is damping coefficient, and real coefficients M_1 and M_2 can be found in Ref.[16]. $\beta_1=\mathrm{Re}(\beta)>0$, its detailed expression is

$$\beta_1=\left[\frac{\lambda[\sinh(2\lambda h/R)+2\lambda h/R]}{8\Omega\cosh^2(\lambda h/R)}+\frac{\lambda^2}{4\Omega\cosh^2(\lambda h/R)}+\frac{\lambda^2\Omega}{2(\lambda^2-m^2)}\right]\sqrt{\frac{2\nu}{\Omega}}. \quad (2)$$

Note that, for the sake of clarity, all of the damping factors associated with the side-wall, bottom, and meniscus term have been lumped into a single coefficient equation (2).

For the convenience of solving the modified amplitude equation (1), we make a transformation for an unknown function $p(\tau)$. Let

$$q(\tau)=p(\tau)\mathrm{e}^{-\mathrm{i}\sigma\tau}, \quad (3)$$

then Eq.(1) becomes

$$\mathrm{i}\frac{\mathrm{d}q(\tau)}{\mathrm{d}\tau}=-\mathrm{i}\beta_1q(\tau)+\sigma q(\tau)+M_1q^2(\tau)\bar{q}(\tau)+M_2\bar{q}(\tau). \quad (4)$$

The physical meanings of all the terms in the right-hand side of Eq.(4) can be explained as follows. The first term denotes the damping of the surface wave, and can lead to energy dissipation in the externally excited system. The second term means the difference between half the forcing frequency and the surface wave frequency. It reflects the approximate degree of the surface wave to Faraday resonance. The third term describes the influence of nonlinearity, and this parameter determines the nonlinear intensity of the surface wave. The last term indicates the energy entering the system via external oscillation, and has an important effect on mode selection and instability of the surface wave.

The stable properties of the amplitudes $p(\tau)$ and $q(\tau)$ are equivalent, which can be proved from Eq.(3). That is to say, if Eq.(4) is stable, then Eq.(1) is still stable, and vice versa. Divide the unknown variable into real and imaginary parts, and the amplitude equation (4) yields the following simultaneous nonlinear ordinary differential equations:

$$\frac{\mathrm{d}q_1(\tau)}{\mathrm{d}\tau}=-\beta_1q_1(\tau)+(\sigma-M_2)q_2(\tau)+M_1q_2(\tau)[q_1^2(\tau)+q_2^2(\tau)], \quad (5)$$

$$\frac{\mathrm{d}q_2(\tau)}{\mathrm{d}\tau}=-\beta_1q_2(\tau)-(\sigma+M_2)q_1(\tau)-M_1q_1(\tau)[q_1^2(\tau)+q_2^2(\tau)], \quad (6)$$

where $q_1(\tau)$ and $q_2(\tau)$ are the real and imaginary parts of $q(\tau)$ respectively.

Simultaneous equations (5) and (6) are our original modelling equations. In the following we investigate the stability of their linearization.

3. The stability of the linearized equation

Neglect the nonlinear terms in the right-hand sides of Eqs.(5) and (6), and make $\hat{q}_1(\tau)$ and $\hat{q}_2(\tau)$ the infinitesimal disturbances associated with the zero solutions of $q_1(\tau)$ and $q_2(\tau)$. The disturbances $\hat{q}_1(\tau)$ and $\hat{q}_2(\tau)$ satisfy

$$\frac{\mathrm{d}\hat{q}_1(\tau)}{\mathrm{d}\tau}=-\beta_1\hat{q}_1(\tau)+(\sigma-M_2)\hat{q}_2(\tau), \quad (7)$$

$$\frac{\mathrm{d}\hat{q}_2(\tau)}{\mathrm{d}\tau}=-(\sigma+M_2)\hat{q}_1(\tau)-\beta_1\hat{q}_2(\tau). \quad (8)$$

The eigenfunction of Eqs.(7) and (8) can be given as

$$\begin{vmatrix}\delta+\beta_1 & -(\sigma-M_2)\\ \sigma+M_2 & \delta+\beta_1\end{vmatrix}=0,$$

namely

$$(\delta+\beta_1)^2=M_2^2-\sigma^2. \quad (9)$$

The eigenvalue δ can be obtained easily from Eq.(9): when

$$M_2^2\geq\sigma^2, \quad \text{then} \quad \delta=-\beta_1\pm(M_2^2-\sigma^2)^{1/2};$$

however, when

$$M_2^2 < \sigma^2, \text{ then } \delta = -\beta_1 \pm \mathrm{i}(\sigma^2 - M_2^2)^{1/2}.$$

Instability will happen if the real part of the eigenvalue δ is larger than zero: namely, when

$$M_2^2 > \sigma^2 + \beta_1^2, \tag{10}$$

the surface wave appears at the free surface. However, the free surface remains planar if the real part of the eigenvalue δ is smaller than zero. Under this condition, the eigenvalue yields

$$M_2^2 < \sigma^2, \text{ or } \sigma^2 < M_2^2 < \sigma^2 + \beta_1^2. \tag{11}$$

From the physical point of view, conditions (10) and (11) indicate that when the external forced energy is larger than that of the viscous dissipation, the surface wave appears due to the instability of the free surface. On the other hand, when the external forced energy is smaller than that of the viscous dissipation, the surface wave cannot be produced.

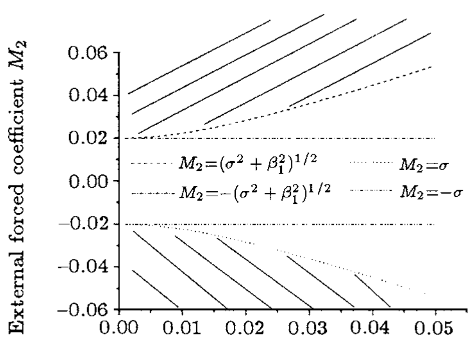


Fig.1. Unstable region determined by damping coefficient β_1 and excited coefficient M_2 (σ=0.02).

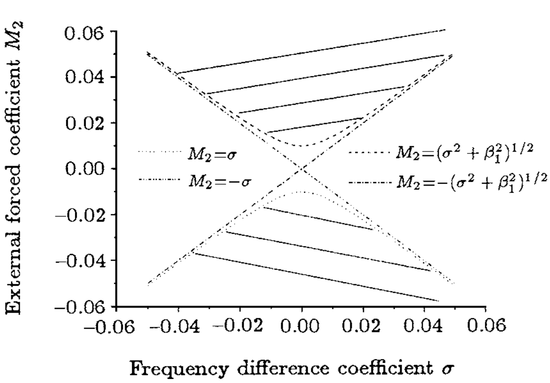


Fig.2. Unstable region determined by frequency difference coefficient σ and excited coefficient M_2 (β_1=0.01).

The results of instability are illustrated in Figs.1 and 2, and the unstable regions are determined by inequality (10). The shaded regions in Figs.1 and 2 are the unstable regions. When the parameters enter these regions, the surface wave can be excited due to first instability. In contrast, when the parameters locate outside of shaded regions, the surface waves cannot be excited.

In the next section, the instability of the nonlinear amplitude equations (5) and (6) are studied by linear stable theory.

4. Linear stability of the non-zero solution

4.1. Instability condition for appearance of stable surface wave

In order to discuss the stability of the nonlinear amplitude equations (5) and (6), we seek for the nonzero equilibrium solution of Eq.(4). The analysis of the instability associated with the finite-amplitude solution is so called 'secondary instability'. In Eq.(4), let the derivative with respect to time equal zero; the equilibrium solution yields

$$\mathrm{i}\beta_1 q(\tau) = \sigma q(\tau) + M_1 q^2(\tau)\bar{q}(\tau) + M_2\bar{q}(\tau). \tag{12}$$

Let $q_0 = a_0\mathrm{e}^{\mathrm{i}\vartheta} \neq 0$ (where a_0 is a real number) be an equilibrium solution of Eq.(12), and insert the expression q_0 into Eq.(3): we have

$$a_0 = \left[\frac{-\sigma \pm \sqrt{M_2^2 - \beta_1^2}}{M_1}\right]^{1/2}, \ \sin 2\vartheta = -\frac{\beta_1}{M_2}. \tag{13}$$

Assuming $q'(\tau)$ is an infinitesimal disturbance associated with the equilibrium solution q_0, substituting disturbed expression $q_1(\tau) = q'(\tau) + q_0$ into Eq.(4), and ignoring the nonlinear term of infinitesimal disturbance, the disturbance equation can be written as

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}q'(\tau)}{\mathrm{d}\tau} =& -\mathrm{i}\beta_1 q'(\tau) + \sigma q'(\tau) + M_2\bar{q}'(\tau)\\ &+ M_1(q_0^2\bar{q}'(\tau) + 2|q_0|^2 q'(\tau)).\end{aligned} \tag{14}$$

We separate Eq.(14) into real and imaginary parts, let $q'(\tau) = A_1\ (\tau) + \mathrm{i}A_2(\tau)$, and substituting it into Eq.(14), we have the following simultaneous ordinary differential equations:

$$\begin{aligned}\frac{\mathrm{d}A_1}{\mathrm{d}\tau} =& -\beta_1\left[1+\frac{-\sigma\pm\sqrt{M_2^2-\beta_1^2}}{M_2}\right]A_1\\ &+\left[(\sigma-M_2)+(-\sigma\pm\sqrt{M_2^2-\beta_1^2})\right.\\ &\left.\times\left(2\pm\frac{\sqrt{M_2^2-\beta_1^2}}{M_2}\right)\right]A_2,\end{aligned}\tag{15}$$

$$\begin{aligned}\frac{\mathrm{d}A_2}{\mathrm{d}\tau} =& \left[-(\sigma+M_2)-(-\sigma\pm\sqrt{M_2^2-\beta_1^2})\right.\\ &\left.\times\left(2\pm\frac{\sqrt{M_2^2-\beta_1^2}}{M_2}\right)\right]A_1\\ &-\beta_1\left[1-\frac{-\sigma\pm\sqrt{M_2^2-\beta_1^2}}{M_2}\right]A_2.\end{aligned}\tag{16}$$

The eigenfunction of Eqs.(15) and (16) can be given as

$$(\delta+\beta_1)^2=-4\left[M_2^2-\frac{5}{4}\beta_1^2\mp\sigma\sqrt{M_2^2-\beta_1^2}\right].\tag{17}$$

Since we consider the equilibrium solution of finite amplitude now, the condition (10) must be realized. Subsequently, we study the stability of the equilibrium solution q_0. Two kinds of circumstances ($M_1>0$ and $M_1<0$) are discussed below.

4.1.1. When $M_1>0$

(a) For the equilibrium solution

$$a_0=[(-\sigma+(M_2^2-\beta_1^2))/M_1]^{1/2},$$

when $\sigma>0$, since a_0 is a real number, the equilibrium solution q_0 exists and Eq.(17) yields

$$(\delta+\beta_1)^2=-4\left[M_2^2-\frac{5}{4}\beta_1^2-\sigma\sqrt{M_2^2-\beta_1^2}\right].\tag{18}$$

According to the condition (10), we can assume

$$M_2^2=\sigma^2(1+s)+\beta_1^2,\quad\text{where}\quad s>0,\tag{19}$$

and Eq.(18) can be expressed as

$$(\delta+\beta_1)^2=4\left\{-\sigma^2[1+s-(1+s)^{1/2}]+\frac{1}{4}\beta_1^2\right\}.\tag{20}$$

Since $1+s>(1+s)^{1/2}$, the eigenvalue satisfies $(\delta+\beta_1)^2\le\beta_1^2$. Hence, under this condition, the real part of the eigenvalue is smaller than zero, and the equilibrium solution is stable.

When $\sigma<0$, the equilibrium solution q_0 still exists. From inequality (10), we let

$$s_1=(M_2^2-\beta_1^2)/\sigma^2,\quad\text{where}\quad s_1>0,\tag{21}$$

and the eigenfunction of Eq.(18) is written as

$$(\delta+\beta_1)^2=4\left[-\sigma^2(s_1+s_1^{1/2})+\frac{1}{4}\beta_1^2\right].\tag{22}$$

Since $s_1+s_1^{1/2}\ge0$, we have $(\delta+\beta_1)^2\le\beta_1^2$. Hence, under this condition, the real part of the eigenvalue is smaller than zero, and the equilibrium solution is stable.

(b) For the equilibrium solution

$$a_0=[(-\sigma-(M_2^2-\beta_1^2))/M_1]^{1/2},$$

when $\sigma>0$, from inequality (10), we can assume

$$M_2^2=\sigma^2(1+s)+\beta_1^2,\quad\text{where}\quad s>0,\tag{23}$$

and we have $[-\sigma-(M_2^2-\beta_1^2)]<0$. Hence, we deduce that a_0 is not a real number and the equilibrium solution will not exist. Similarly, when $\sigma<0$, $[-\sigma-(M_2^2-\beta_1^2)]<0$, and the equilibrium solution will not exist.

4.1.2. When $M_1<0$

(a) For the equilibrium solution

$$a_0=[(-\sigma+(M_2^2-\beta_1^2))/M_1]^{1/2},$$

when $\sigma>0$, we can use the same approach to know

$$[(-\sigma+(M_2^2-\beta_1^2))/M_1]<0.$$

Thus a_0 is not a real number, and the equilibrium solution

$$a_0=[(-\sigma+(M_2^2-\beta_1^2))/M_1]^{1/2}$$

will not exist.

(b) For the equilibrium solution

$$a_0=[(-\sigma-(M_2^2-\beta_1^2))/M_1]^{1/2},$$

when $\sigma>0$, a_0 is a real number, and this equilibrium solution is stable, which can be confirmed by using the same method.

In summary, whenever $M_1>0$ or $M_1<0$, if the condition $M_2^2>\sigma^2+\beta_1^2$ is fulfilled, stable surface waves can be formed.

4.2. Stability of the surface wave

Will a stable surface wave mode lose its stability to an infinitesimal disturbance? In this case, the condition (10) must be satisfied. It can be shown that the equilibrium mode will not lose its stability to an infinitesimal disturbance using reductio ad absurdum as follows.

We assume that the equilibrium mode will lose its stability to an infinitesimal disturbance. That is to say, the eigenvalue of expression (17) has a positive real part, namely

$$\delta = -\beta_1 \pm \sqrt{-4\left[M_2^2 - \frac{5}{4}\beta_1^2 \mp \sigma\sqrt{M_2^2-\beta_1^2}\right]} > 0, \tag{24}$$

and the inequality

$$-4\left[M_2^2 - \frac{5}{4}\beta_1^2 \mp \sigma\sqrt{M_2^2-\beta_1^2}\right] \geq 0, \tag{25}$$

will be satisfied. If inequality (25) cannot be yielded, then the value on the left-hand side of inequality (25) is smaller than zero. Thus the real part of the eigenvalue (24) is $-\beta_1$, which is contrary to the assumption of the appearance of the instability. We can obtain the following expression from Eqs.(24) and (25):

$$0 < M_2^2 - \beta_1^2 < \sigma^2. \tag{26}$$

Since inequality (26) is contrary to the condition (10), the equilibrium mode will not lose its stability to an infinitesimal disturbance.

In the following section, the solution properties of the nonlinear amplitude equations (5) and (6) are studied by numerical computation. The validity of our theoretical analysis is proved.

5. The properties of modified amplitude equation

5.1. The validity of the linear stability

Some numerical computations for the simultaneous modified amplitude equations (5) and (6) are presented by the fourth order Runge–Kutta approach with equivalent time-step. The evolution of the amplitude with time is investigated in different parameter regions. The properties of the amplitude are illustrated by phase-plane trajectory.

The evolutions of the amplitude with time and the phase-plane trajectory are depicted in Figs.3(a) and 3(b) respectively. The choice of the parameters satisfies the first condition of inequality (11). It can be easily seen from Fig.3 that the amplitude decreases gradually and tends to zero eventually on the prescribed initial conditions.

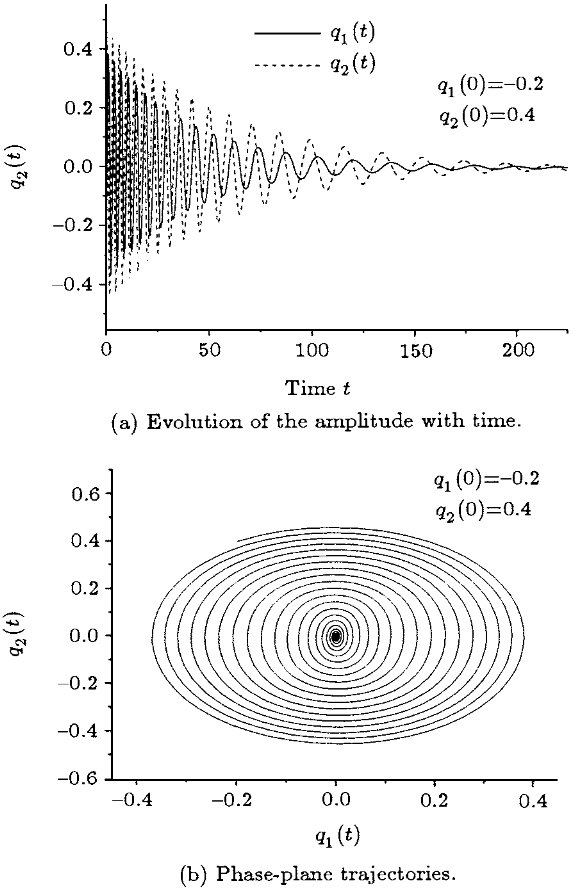
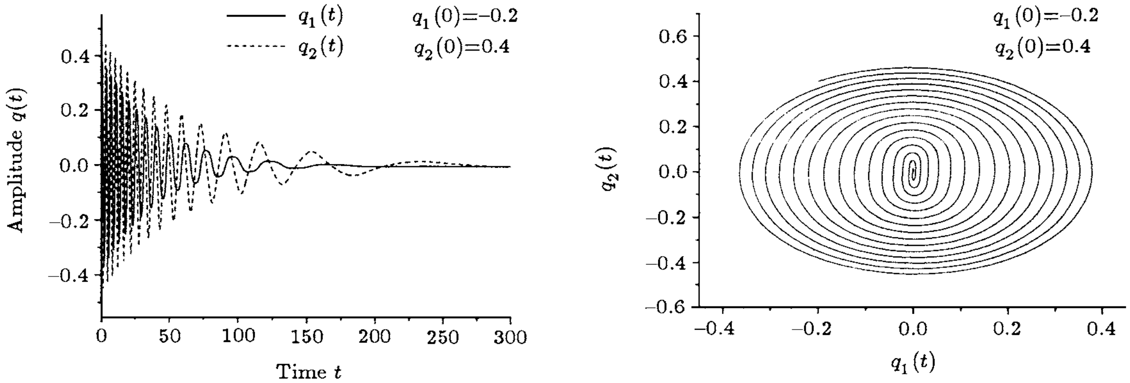


Fig.3. The evolution of the amplitude with time and phase- plane trajectories (σ=0.5, M_1=10, M_2=0.4, β_1=0.02).

This indicates that the externally driven energy can not overcome the viscous dissipation, and the stable surface wave cannot be formed.

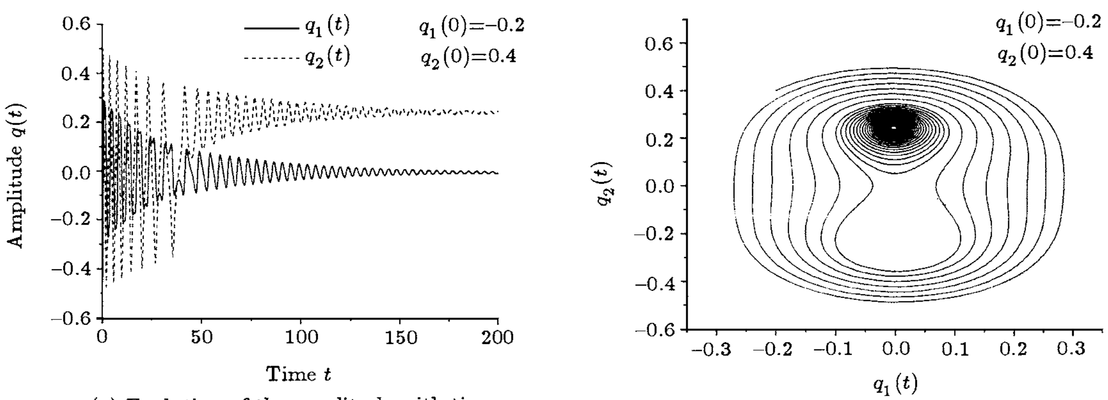
Similarly, when the parameters fulfill the second condition of inequality (11), the stable surface wave still cannot be formed. This situation is plotted in Figs.4(a) and 4(b).

However, when the parameters yield the condition (10), the instability will happen and the surface wave will appear.



(a) Evolution of the amplitude with time. (b) Phase-plane trajectories.

Fig.4. The evolution of the amplitude with time and phase-plane trajectories (σ=0.4, M_1=10, M_2=0.4001, β_1=0.02).



(a) Evolution of the amplitude with time. (b) Phase-plane trajectories.

Fig.5. Evolution of amplitude with time and phase- plane trajectories (σ=0.4, M_1=10, M_2=1, β_1=0.02).

Figure 5 illustrates the evolution of the amplitude with time and corresponding phase-plane trajectory.

It can be seen respectively from Figs.5(a) and 5(b) that the amplitude tends to a constant and a fixed point with the passage of time. Under this condition, the stable surface wave is formed.

5.2. The influence of the initial conditions on the amplitude

We find computationally that the shape of the phase-plane trajectory is sensitive to the initial conditions of the amplitude equations (5) and (6). Figures 6(a)–6(e) show different phase-plane trajectories when the initial conditions are changed.

There are two equilibrium solutions deduced from Eq.(13). We can see that if the initial conditions are located in the neighborhood of one equilibrium solution, then the solution of the amplitude equation will be attracted to the region of the equilibrium solution. In contrast, the solution of the amplitude equation will be attracted to the region of another equilibrium solution.

The evolution of the transformed amplitude $q(\tau)$ with time has been studied. In the next section, we investigate the evolution of the original amplitude $p(\tau)$ with time.

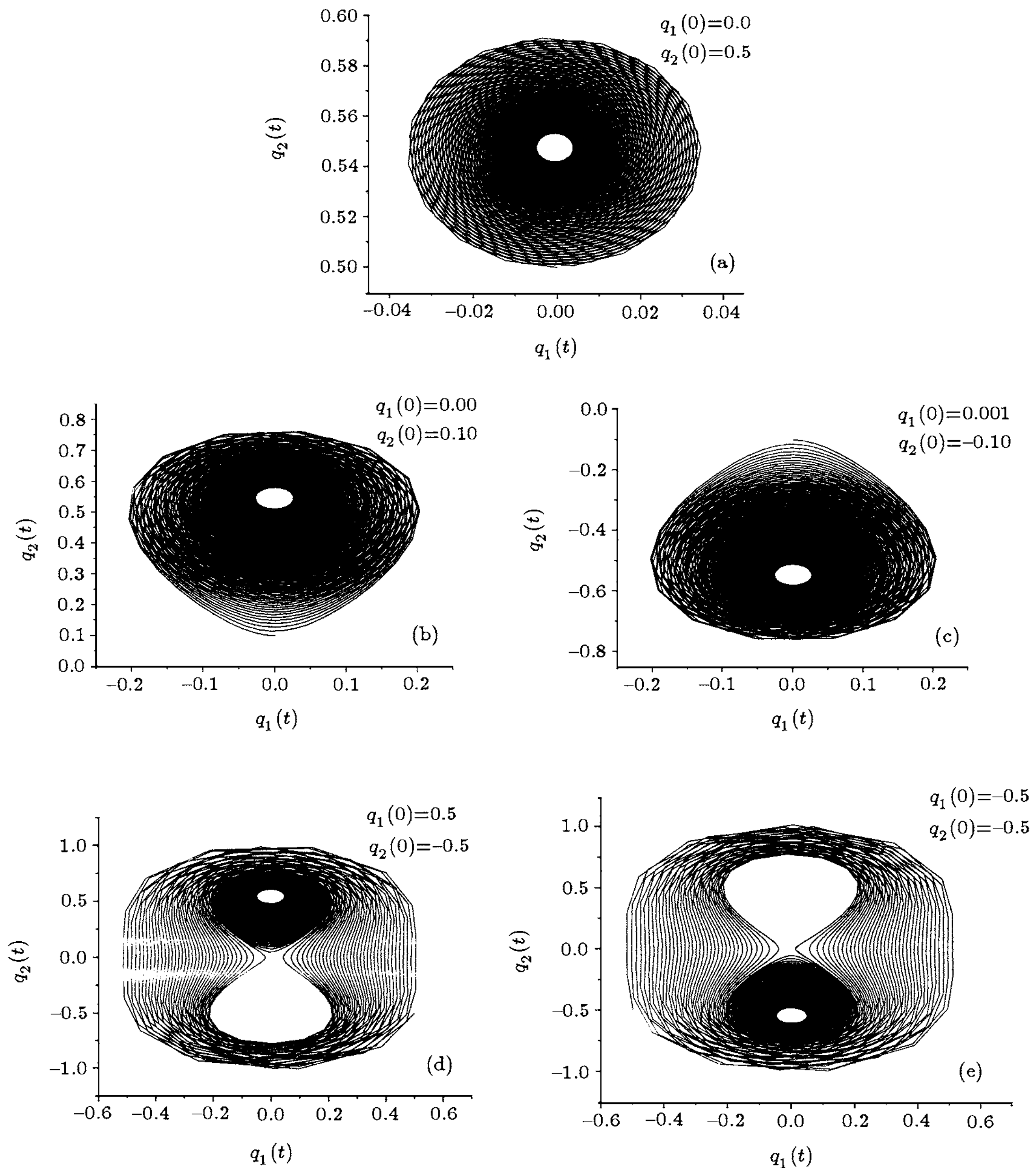


Fig.6. The effect of initial conditions on the phase-plane trajectories (σ=2, M_1=10, M_2=5, β_1=0.01).

5.3. The evolution of the original amplitude variable with time

Figures 7(a)–7(d) illustrate the evolution of the transformed and the original amplitudes with time. We can see from Figs.7(a), 7(b) that the transformed amplitude $q(\tau)$ tends to a fixed point with the evolution of time, and the surface wave is stable. Figures 7(a), 7(b) show that the surface wave is oscillatory at first and a stable periodical solution can be formed with the passage of time. Similarly, Figs.7(c), 7(d) indicate that the surface wave evolves into a stable limited cycle with the passage of time.

The parameters that can give rise to stable surface waves as mentioned above satisfy the unstable condition (10). However, not all of these parameters that satisfy the unstable condition (10)can insure the formation of stable surface wave. That is to say, the condition (10) is a necessary rather than a sufficient condition.

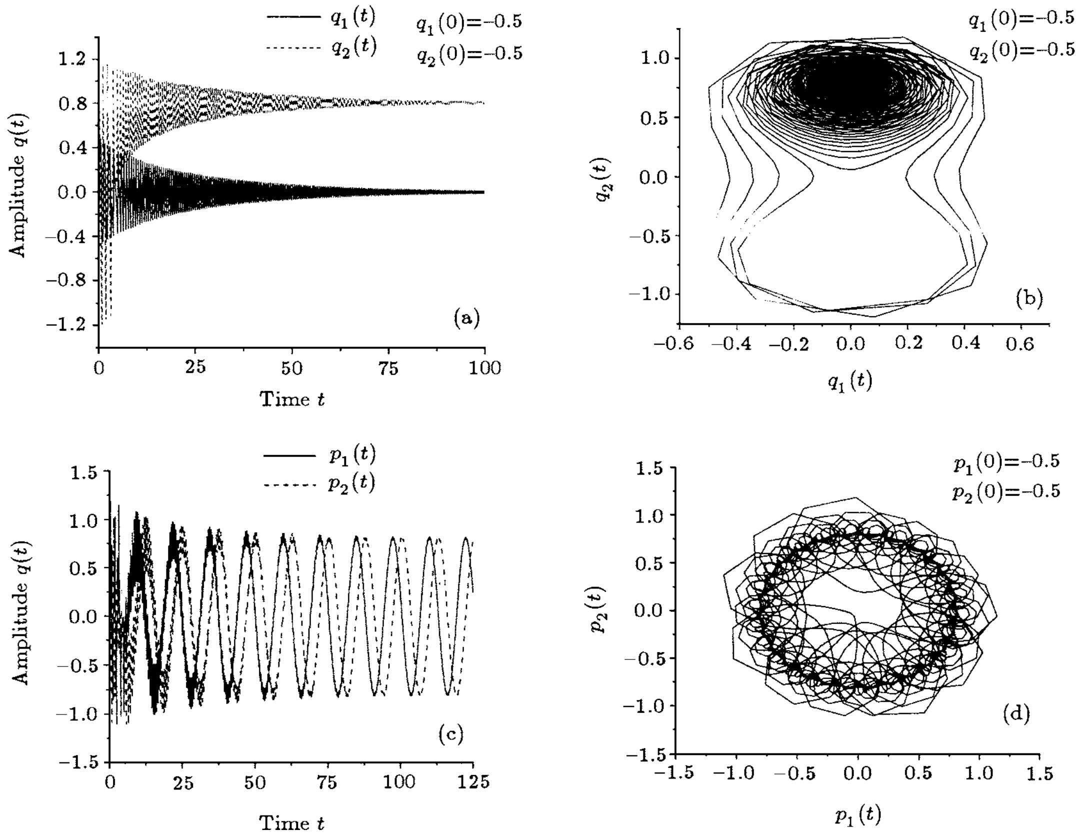


Fig.7. Evolution of amplitude with time and phase- plane trajectories (σ=0.5, M_1=10, M_2=7, β_1=0.02).

6. Conclusions

From the above analyses, the following results can be obtained:

1. The modified amplitude equation is more reasonable physically to describe the motion of vertically excited surface waves.

2. A necessary condition of producing stable surface waves is derived, and the unstable regions are determined by the instability analysis.

3. The theoretical results are proved by the numerical computation of the modified amplitude.

Acknowledgement

The authors are grateful to Professor Zhou Xian-Chu for discussion on the mathematical treatment, and Dr Bai Wei for assistance in the numerical work.

References

[1] Faraday M 1831 *Phil. Trans. R. Soc. Lond.* **121** 319
[2] Miles J W and Henderson D 1990 *Ann. Rev. Fluid Mech.* **22** 143
[3] Edwards W S and Fauve S 1994 *J. Fluid Mech.* **278** 123
[4] Daudet L *et al* 1995 *Europhys. Lett.* **32** 313
[5] Müller H W 1993 *Phys. Rev. Lett.* **71** 3287
[6] Kumar K and Bajaj K 1994 *Phys. Rev.* E **52** R4606
[7] Ciliberto S *et al* 1991 *Europhys. Lett.* **15** 23
[8] Binks D and van de Water W 1997 *Phys. Rev. Lett.* **78** 4043
[9] E X Q and Gao Y X 1996 *Commun. Nonlinear Sci. Numer. Simul.* **1** 1
[10] E X Q and Gao Y X 1996 *Proc. 4th Asian Symposium on Visualization Beijing* 653
[11] Gao Y X and E X Q 1998 *Exp. Mech.* **13** 326 (in Chinese)
[12] Benjamin T B and Ursell F 1954 *Proc. R. Soc. Lond.* **255** 505
[13] Miles J W 1976 *J. Fluid Mech.* **75** 419
[14] Miles J W 1984 *J. Fluid Mech.* **148** 451
[15] Miles J W 1993 *J. Fluid Mech* **248** 671
[16] Jian Y J *et al* 2003 *Appl. Math. Mech.* **24** 1194
[17] Jian Y J and E X Q 2003 *J. Hydrodynam.* **18** 135 (in Chinese)
[18] Jian Y J and E X Q 2004 *Adv. Mech.* **34** 61 (in Chinese)
[19] Jian Y J and E X Q 2004 *Chin. Phys.* **13** 1191