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STRESS INTENSITY FACTORS CALCULATION IN ANTI-PLANE FRACTURE PROBLEM BY ORTHOGONAL INTEGRAL EXTRACTION METHOD BASED ON FEMOL ⋆

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ABSTRACT For an anti-plane problem, the differential operator is self-adjoint and the corresponding eigenfunctions belong to the Hilbert space. The orthogonal property between eigenfunctions (or between the derivatives of eigenfunctions) of anti-plane problem is exploited. We developed for the first time two sets of radius-independent orthogonal integrals for extraction of stress intensity factors (SIFs), so any order SIF can be extracted based on a certain known solution of displacement (an analytic result or a numerical result). Many numerical examples based on the finite element method of lines (FEMOL) show that the present method is very powerful and efficient.

KEY WORDS anti-plane problem, Hilbert space, eigenvalue, eigenfunction, orthogonal relationship, stress intensity factor, finite element method of lines

I. INTRODUCTION

The stress analysis in practical engineering applications inevitably encounters stress singularities caused by sudden changes in geometry, e.g. around re-entrant corners (notches) or, more severely, around crack tips. Their presence causes great difficulty to the numerical solutions that have to be invoked when the analytical solutions are not available. From Williams[1−−3], who provided a general solution to the two-dimensional stress and displacement fields for the case of the planar crack problem, interest was mainly given to the eigenfunction expansion method. Hartranft and Sih[4] used eigenfunction expansions in the general solution to three-dimensional crack problems, and Liu[5] gave eigenfunction expansions of general displacements and general stresses in the Reissner plate with crack. Most interest was focused on SIF calculation with the eigenfunction expansion method. In Refs.[6–10], the boundary collocation technique is used to calculate the SIFs, while in Refs.[11,12], the advantage of boundary integral method is taken to solve the problem of a bending beam with a notch. Stern et al.[13,14] exploited the contour integral method based on Betti's reciprocal work. Long[15] presented the sub-region generalized variational principle, which was extended by Long et al.[16,17] to fracture problems. Most numerical methods make direct or indirect use of the eigensolutions available in calculating SIFs for cracks/notches

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so that singularity can be treated in a more efficient, accurate and reliable way. Recently, Xu and Yuan et al.[18−−23] have proposed some effective methods for accurate and reliable computation of completely real or complex eigensolutions in two-dimensional notch/crack singularities with multiple materials, arbitrary opening angles and various surface conditions. The resulting algorithm is robust and can be employed by any numerical method that makes use of singular solutions. A numerical recipe for accurate and efficient computation of stress singularity factors (SIFs) usually consists of two major ingredients, namely, a powerful numerical method for general stress analysis and a novel approach to obtaining the desired SIFs which may include a special treatment of various singularities. Instead of giving an extensive review of various existing numerical approaches, the discussion is confined to a brief introduction to the major ingredients adopted in the present papers.

From an anti-plane problem, the differential operator is self-adjoint and the corresponding eigenfunctions belong to the Hilbert space. The orthogonal property between eigenfunctions (or between the derivatives of eigenfunctions) of the anti-plane problem is exploited. According to the orthogonal relationship, we developed for the first time two sets of radius-independent orthogonal integrals for extraction of stress intensity factors (SIFs), so any order SIF can be extracted based on a certain known solution to displacement (an analytic result or a numerical result). The background numerical method employed in this paper is the finite element method of lines (FEMOL)[24−−27], which is a general-purpose semi-analytical method. With this method, a partial differential equation defined in an arbitrary domain is semi-discretized, by finite element techniques based on the energy theorems or variational principles, into a system of ordinary differential equations (ODEs) defined on straight or curved mesh lines. It can be meshed easily with a group of lines radiating from the vertex of cracks/notches for fracture problems, such as anti-plane cracks/notches of multi-materials, arbitrary opening angles and different surface conditions. At present, the resulting ODE system is solved directly and efficiently by a state-of-the-art ODE solver, e.g. COLSYS[28, 29] is exclusively adopted in the present paper. These solvers have built-in self-adaptability features so the accuracy of the ODE solutions satisfies the user pre-specified error tolerances, and have no need for re-meshing. Using these solvers, FEMOL has been proved to be a remarkable numerical method with efficient adaptability in the mesh line directions automatically built in, so its inherent semi-analytical characteristics are well preserved. Its power and versatility have been demonstrated by a series of theoretical analyses and computational applications to various linear and nonlinear problems. A general-purpose computer code FEMOL92[30] that is capable of static and vibration analysis of various linear elastic structures has been developed. A more detailed and systematic descriptions of FEMOL see Ref.[31].

The anti-plane problem is considered in this paper, and without loss of generality, body forces are not included and only homogeneous displacement boundary conditions are taken into account. A number of illustrative numerical examples, including bi-material notches/cracks problem, are given in the paper to show the generally excellent performance of the proposed SIFs computation method. Many numerical examples based on the finite element method of lines (FEMOL) show that the method presented in this paper is very powerful and efficient.

II. PRELIMINARY CONSIDERATION

Figure 1 shows an anti-plane notch with N different wedged materials around the notch tip. The body forces are assumed to be negligible. $G_n(n = 1,\ 2,\ \ldots,\ N)$ are the shear module of the materials, θ_1 and θ_{N+1} the two boundaries of the notch which can be either stress free (F) or displacement clamped (C). $\theta_n(n = 2,\ \ldots,\ N)$ are the interface of two materials. The differential equation can be expressed by the stresses in the n-th wedged material.

$$\frac{\partial\left(r\tau_{rz_n}\right)}{\partial r}+\frac{\partial\tau_{\theta z_n}}{\partial\theta}=0,\quad n=1,\ 2,\ ...,\ N \tag{1}$$

The stress-strain relations are

$$\tau_{\theta z_n}=\frac{1}{r}G_n\frac{w_n}{\partial\theta},\quad \tau_{rz_n}=G_n\frac{w_n}{\partial r},\qquad 1\le n\le N \tag{2}$$

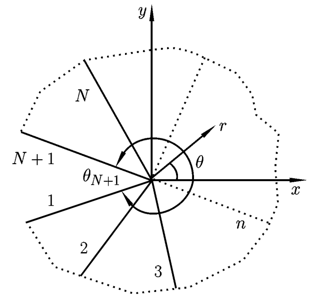


Fig. 1. N-material notch problem.

The equilibrium equation can be expressed by the displacement w_n in the n-th wedge

$$\frac{\partial^2 w_n}{\partial r^2}+\frac{1}{r}\frac{\partial w_n}{\partial r}+\frac{1}{r^2}\frac{\partial^2 w_n}{\partial \theta^2}=0, \qquad n=1,\,2,\,...,\,N \tag{3}$$

It is well known that the Williams expression of w_n can be written as

$$w_n=\alpha\, r^{\lambda} f_n\,(\theta)$$

Substituting w_n into Eq.(3) yields an eigenproblem in ordinary differential equations (ODEs) as

$$f_n''+\lambda^2 f_n=0, \qquad n=1,\ 2,\ ...,\ N \tag{4}$$

The boundary conditions (BCs) of notch/crack can be written as

$$\text{F: } f_1'\,(\theta_1)=0, \quad f_N'\,(\theta_{N+1})=0; \qquad \text{C: } f_1\,(\theta_1)=0, \quad f_N\,(\theta_{N+1})=0 \tag{5}$$

The displacement continuity and stress equilibrium conditions for the interface $\theta=\theta_n$can be given as

$$f_n\,(\theta_{n+1})=f_{n+1}\,(\theta_{n+1})\,, \quad G_n f_n'\,(\theta_{n+1})=G_{n+1} f_{n+1}'\,(\theta_{n+1})\,, \qquad n=1,\ 2,\ \ldots,\ N-1 \tag{6}$$

III. EIGENFUNCTION

To solve the eigenproblem defined by Eqs.(4)-(6), the well known explicit form of eigenfunction $f_n\,(\theta)$ can be employed, i.e.

$$f_n\,(\theta)=A_n\cos\,(\lambda\theta)+B_n\sin\,(\lambda\theta)\,, \qquad n=1,\,2,\,...,\,N \tag{7}$$

From Eq.(6) we have the relationship with respect to $\{A_n,\,B_n\}$ between the two adjoining materials as

$$\begin{Bmatrix} A_{n+1} \\ B_{n+1} \end{Bmatrix}=[\varDelta_n]\begin{Bmatrix} A_n \\ B_n \end{Bmatrix}, \qquad n=1,\,2,\,...,\,N-1 \tag{8}$$

where

$$[\varDelta_n]=\begin{bmatrix} \cos\,(\lambda\theta_{n+1}) & -\sin\,(\lambda\theta_{n+1}) \\ \sin\,(\lambda\theta_{n+1}) & \cos\,(\lambda\theta_{n+1}) \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \dfrac{G_n}{G_{n+1}} \end{bmatrix}\begin{bmatrix} \cos\,(\lambda\theta_{n+1}) & \sin\,(\lambda\theta_{n+1}) \\ -\sin\,(\lambda\theta_{n+1}) & \cos\,(\lambda\theta_{n+1}) \end{bmatrix} \tag{9}$$

The matrix $[\varDelta_n]$ is not singularity, i.e. $|\varDelta_n|=G_n/G_{n+1}\neq 0$, which implies that any set of $\{A_n,\,B_n\}$ can be represented by other sets, namely, if we know one set of $\{A_n,\,B_n\}$ in the n-th wedge, we can calculate the others one by one from Eq.(9).

For any order eigenvalue, solving the corresponding eigenfunction $f_n\,(\theta)$ $(n=1,2,3,\ldots,N-1)$ is equivalent to computing the coefficients $\{A_n,\,B_n\}$. For example, when the boundary θ_1 is stress free (F), A_1 and B_1 are not all zero. If $\cos\,(\lambda\theta_1)\neq 0$, then $A_1\neq 0$, A_1 is normalized and we have

$$A_1=1, \qquad B_1=\frac{\sin\,(\lambda\theta_1)}{\cos\,(\lambda\theta_1)} \tag{10}$$

If $\cos\,(\lambda\theta_1)=0$, then $A_1=0$, B_1 is normalized and we have

$$A_1=0, \qquad B_1=1 \tag{11}$$

Then we can uniquely calculate $\{A_n,\,B_n\}$ from Eq.(9).

IV. ORTHOGONAL PROPERTY OF THE EIGENFUNCTIONS

Denote

$$f(\theta) = \{ f_n(\theta) \mid \theta_n \le \theta \le \theta_{n+1}, \quad n = 1, 2, \ldots, N\} \tag{12}$$

and define the inner product of any functions $f(\theta)$ and $g(\theta)$ in $C[\theta_1, \theta_{N+1}]$ as

$$(f,g)^* = \sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} G_n f_n(\theta) g_n(\theta) \mathrm{d}\theta \tag{13}$$

and the induced norm as

$$\|f\|_2^* = \sqrt{(f,f)^*} \tag{14}$$

It is well known that $\left\{C[\theta_1, \theta_{N+1}], \|\cdot\|_2^*\right\}$ is bellowed to Hilbert space, denoted as $L_2^*[\theta_1, \theta_{N+1}]$.

Denoting the function set $\mathcal{D}(T)$ in which all functions satisfy $f(\theta) \in L_2^*[\theta_1, \theta_{N+1}]$, the corresponding boundary conditions (BCs) of notch/crack (5) and the interface conditions (6), and introducing the differential operator T which maps $\mathcal{D}(T)$ to $L_2^*[\theta_1, \theta_{N+1}]$. When $f \in \mathcal{D}(T)$

$$Tf = -\mathcal{D}^2 f(\theta) = \{-f_n(\theta) \mid \theta \in (\theta_n, \theta_{n+1}), \ n = 1, 2, ..., N\} \tag{15}$$

By using the part integral method, it is easy to prove that T is a self-adjoint and nonnegative operator.

According to Eq.(15), Eqs.(4)-(6) can be turned into solving the eigenvalue λ^2 and the eigenfunction f of the operator T

$$Tf - \lambda^2 f = 0 \tag{16}$$

which represents a standard eigenproblem in ODEs. Denoting that the eigenspace is $M_\lambda = \mathrm{Span}\left\{f \mid Tf = \lambda^2 f, \ f \ne 0\right\}$ and the eigenvaule set is $\sigma_p(T)$ (which is called spectrum of T).

According to the self-adjoining operator T in Hilbert space T , we have the following three lemmas

Lemma 1. All eigenvalues λ^2 of T are nonnegative real number.

Lemma 2. There is orthogonal property between eigenfunctions corresponding to different eigenvalues.

Lemma 3. There is orthogonal property between the derivatives of the eigenfunction corresponding to different eigenvalues.

The proof of the above lemmas can be found in many literatures, such as Ref.[32].

V. SOLUTIONS

According to the well known Williams expression, the displacement shown in Fig.1 can be written as

$$w = \left\{ w_n = \sum_{i=0}^{\infty} \alpha_i r^{\lambda_i} f_{ni} \,\middle|\, \theta \in (\theta_n, \ \theta_{n+1}), \ 1 \le n \le N \right\} \tag{17}$$

Assuming f_k be the eigenfunction of the corresponding eigenvalue λ_k,

$$\begin{aligned}(w, f_k)^* &= \sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} G_n w_n f_{nk} \mathrm{d}\theta = \sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} G_n \sum_{i=0}^{\infty} \alpha_i r^{\lambda_i} f_{ni} f_{nk} \mathrm{d}\theta \\ &= \sum_{i=0}^{\infty} \alpha_i r^{\lambda_i} \sum_{n=1}^{N} \int_{\theta_n}^{\theta_{n+1}} G_n f_{ni} f_{nk} \mathrm{d}\theta = \sum_{i=0}^{\infty} \alpha_i r^{\lambda_i} (f_i, f_k)^* \\ &= \alpha_k r^{\lambda_k} (f_k, f_k)^* \end{aligned} \tag{18}$$

The coefficient α_k can be extracted from Eq.(18) as the following orthogonal integral equation

$$\alpha_k = r^{-\lambda_k} \frac{(w, \ f_k)^*}{(f_k, f_k)^*} = r^{-\lambda_k} \frac{\sum\limits_{n=1}^{N} \int\limits_{\theta_n}^{\theta_{n+1}} G_n w_n f_{nk}}{(f_k, f_k)^*}, \qquad k = 0, 1, 2, 3, ... \tag{19}$$

α_i can also be thought as the expanded coefficients of a general Fourier series.

Moreover, we can get another set of α_k based on the expansion from the derivatives of the eigenfunction as

$$\alpha_k = r^{-\lambda_k} \frac{(w_{,\theta},\ f_{k,\theta})^*}{(f_{k,\theta}\,, f_{k,\theta})^*} = r^{-\lambda_k} \frac{\sum\limits_{n=1}^{N} \int\limits_{\theta_n}^{\theta_{n+1}} G_n w_{n,\theta} f_{nk,\theta}}{(f_{k,\theta}\,, f_{k,\theta})^*}, \qquad k = 1,\ 2,\ 3,\ ... \tag{20}$$

Difference between the above two orthogonal integral extraction equations (19) and (20) is that the constant term (namely, the zero order coefficient) has been excluded at the latter case. For any order eigenvalue, based on the displacement field calculated from a certain numerical method as the finite element method of lines (FEMOL) used in this paper, the coefficient α_k can be gotten from Eqs.(19) and (20). Furthermore, the stress intensity factors can be obtained.

VI. NUMERICAL EXAMPLES

To assess the performance of the proposed method, several numerical examples based on the FEMOL are given in this section. The following notations are used: r_0—radius of the circle contour, p— polynomial degree used for element displacements in FEMOL, Tol—tolerance specified for ODE solutions, E, G, v— Young's modulus, shear modulus, Poisson's ratio.

In the subsequent examples, the singular mapping technique of FEMOL is available. Only one of the two sets of orthogonal integral extraction equations, (19) and (20), is list, which means that the numerical results are identical within the given number of digits. All of the examples are reckoned on Pentium 586-100 computer.

Example 1. Single edge crack problem

In this example, a single edge crack anti-plane problem is studied. The boundary conditions and the FEMOL meshes are shown in Fig.2. We take $p = \tilde{p} = 3$ and Tol$= 0.1\%$. The computed results of coefficients α_1 and α_3 along different radii r_0 are tabulated in Table 1.

Table 1. **Computed results of α_1 and α_3**

r_0	α_1 ($\lambda_1 = 0.5$)	α_3 ($\lambda_3 = 1.5$)
0.001	1.08087	0.33437
0.1	1.08105	0.33420
0.25	1.08108	0.33416
0.5	1.08111	0.33413
Ref.[33]	1.081	

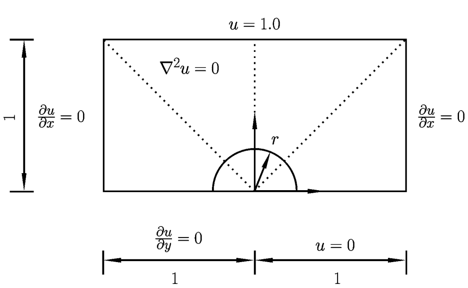


Fig. 2. Single edge crack anti-plane problem.

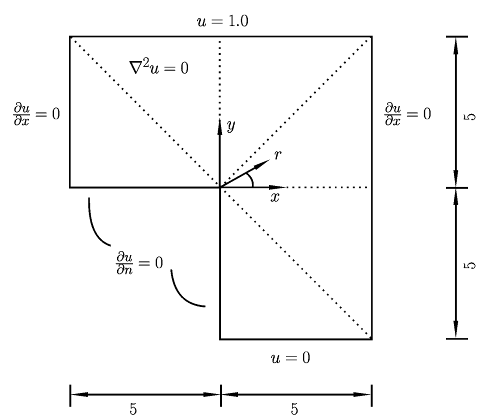


Fig. 3. L-shaped anti-plane problem.

Example 2. L-shaped anti-plane problem

In this example, a L-shaped anti-plane problem with a notch of 90° open angle is studied. The boundary conditions and the FEMOL meshes are shown in Fig.3. We take $p = \tilde{p} = 3$ and Tol= 0.1%. The well known Williams expression of displacement u in this example can be written as

$$u = \sum_{i=1,3,5,\cdots} \alpha_i r^{\lambda_i} \sin(\lambda_i \theta) + \sum_{j=0,2,4,\cdots} \alpha_j r^{\lambda_j} \cos(\lambda_j \theta)$$

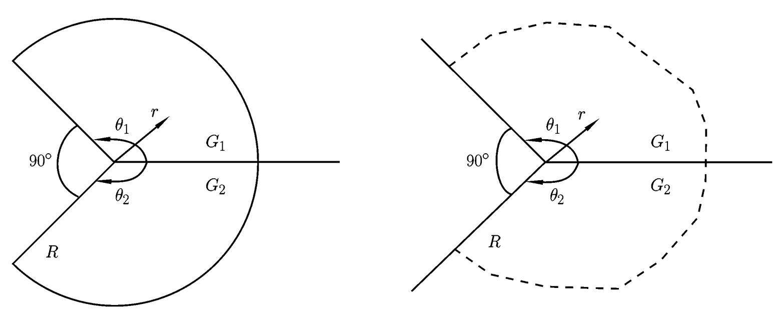
The first three lower order coefficients α_0, α_1 and α_2 along different radii r_0 are computed and tabulated in Table 2.

Table 2. **Computed result of L-shape anti-plane problem**

r_0	$\alpha_0(\lambda_0 = 0)$	$\alpha_1(\lambda_1 = 2/3)$	$\alpha_2(\lambda_2 = 4/3)$
0.001	0.6666666667	0.15060	0.025665
0.01	0.6666666667	0.15143	0.025400
0.1	0.6666666669	0.15460	0.025142
1.0	0.6666666675	0.15459	0.025137
2.0	0.6666666672	0.15459	0.025135
Best known [31]	2/3	0.1546	0.02513

Example 3. Bi-material anti-plane disk with a notch

In this paper, a bi-material anti-plane disk with a right angle notch as shown in Fig.4(a) is considered. The ratio of the shear moduli of materials is taken as $G_1 : G_2 = 10$, the open angle between two different materials is $\theta_1 = \theta_2 = 135°$, and the radius of the disk is taken as $R = 10$.



(a) bi-material anti-plane disk with a notch (b) bi-material in finite body with a notch

Fig. 4. Bi-material disk with a notch.

We firstly consider an infinite body with a notch shown in Fig.4(b), and the displacement is assumed to be

$$w = \sum_{i=1}^{\infty} \alpha_i r^{\lambda_i} f_i^n(\theta), \quad f_i^n(\theta) = \left[a_i^n \cos(\lambda_i \theta) + b_i^n \sin(\lambda_i \theta)\right], \quad 1 \le n \le 2$$

$$\text{If } |\cos(\lambda_i \theta_1)| \le |\sin(\lambda_i \theta_1)|, \quad \text{then } b_1^i = 1, \quad a_1^i = \cos(\lambda_i \theta_1)/\sin(\lambda_i \theta_1)$$

$$\text{If } |\sin(\lambda_i \theta_1)| \le |\cos(\lambda_i \theta_1)|, \quad \text{then } a_1^i = 1, \quad b_1^i = \sin(\lambda_i \theta_1)/\cos(\lambda_i \theta_1)$$

The other coefficients $(a_2^i,\ b_2^i)$ can be calculated from the interface conditions. We take the first three terms and let $\alpha_1 = 0.01$, $\alpha_2 = 0.1$, $\alpha_3 = 1.0$, then the displacement on the circle of $r = 10$ can be determined and used as the boundary condition for the disk as shown in Fig.4(a). Both of the anti-plane disk with a notch and the infinite body with a notch have the same coefficients α_1, α_2, α_3.

We take four FEMOL elements and $p = \tilde{p} = 4$ and Tol= 0.01%. The first three lower order coefficients α_1, α_2 and α_3 along different radii r_0 are computed and tabulated in Table 3.

Table 3. **Computed results of α_1, α_2 and α_3**

r_0	α_1	α_2	α_3
0.001	0.00999996	0.100003	1.012677
0.01	0.00999996	0.100002	1.000452
1.0	0.00999996	0.100003	1.000290
4.0	0.00999996	0.100003	1.000331
8.0	0.00999996	0.100003	1.000352
exact results	0.01	0.1	1.0

VII. CONCLUSIONS

The following conclusions can be drawn:

(1) i-th order SIFs directly extraction: The orthogonal relationships between eigenfunctions are exploited and two sets of orthogonal integral extraction algorithms for SIF calculations are developed. It is crucial that any order characteristic coefficient or SIF can be extracted directly.

(2) Contour integral radius independence: The two sets of orthogonal integral extraction algorithms show that the present methods are independent of the contour integral radius. A good proposal is that the radius for contour integral can not be too small or too large, because most of the methods based on stress analytical solution could not give a satisfied field in too small or too large radius domains.

(3) Self verification: The computed results can be verified by itself by the two orthogonal integral extraction algorithms and the characteristic of the independent contour integral radius.

(4) Generality: The present algorithms are general SIFs computing method and are applicable to anti-plane problem of the crack/notches with arbitrary opening angles, multiple materials and notch boundary surface conditions.

(5) Accuracy: Since the present algorithms have not led to any error, so the accuracy of the final results only relies on the based numerical methods. The FEMOL used in this paper is a semi-analytic method and is based on the ODE solver COLSYS, in which the self-adaptability is automatically built, and the accuracy is fully controlled by the user with a desired error tolerance specified to the solver.

(6) Reliability: Because of the two sets of orthogonal integral extraction relationship, the SIFs results are almost independent of the radius.

(7) Efficiency: The present algorithms only need to be integrated along an arc, so it is better to avoid too small radius or too large radius.

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