

大挠度微曲矩形薄板的动力性质¹⁾

董智法

(中国科学院力学研究所, 北京 100080)

摘要 本文导出了由位移表示的带初始挠度的粘弹性大挠度矩形薄板的运动方程. 在简支边界条件下, 应用分歧理论和 Melnikov 方法, 研究在周期外力作用下系统的动力行为. 给出了出现次谐分枝和马蹄态的条件.

关键词 粘弹性, 大挠度, 分歧集, 次谐分枝, Melnikov 方法

一、前 言

在工程结构中, 由于结构的临界承载能力敏感地依赖于初始缺陷^[1], 带缺陷结构的稳定性研究一直是一个非常活跃的领域. 关于有初始挠度曲板的非线性理论, 最初由 Marguerre^[2] 在 1938 年用能量法导出的, 这一理论仍然在浅壳非线性分析中广泛地应用着. 有初始挠度曲板的理论要比平板理论复杂得多. 带初始变形板的大挠度问题的精确解也仅仅有一些特殊问题的解. 如 Timoshenko^[3] 关于无限长初挠度板的大变形精确解, 以及 Coan^[4] 关于小初始变形板的后屈曲的傅利叶分析解. 其他近似结果很多, 文 [5] 给出了一些有关的文献目录. 其中主要的有动力松弛法和有限元法. 在文 [6] 中用动力松弛法给出了各种边界条件下的大挠度数值解, 以及板的后屈曲解. 在文 [7] 中用有限元法给出了一般板壳结构的动态响应的一些有意义的结果.

本文试图针对粘弹性带初始挠度板的大挠度动力振动问题, 运用非线性动力学理论, 给出一些分析及计算结果, 并且考虑了初始挠度对板的动力特性的影响.

二、基 本 方 程

设未变形板的中面形状为 $\xi(x, y)$, 从中面测量的任意点的垂直距离为 $z(x, y)$, 取微曲板中的一小变元 $h dx dy$. 这一变元的位置矢量为

$$\mathbf{r}_0 = x\mathbf{i} + y\mathbf{j} + (\xi + z)\mathbf{k}$$

其中, $\mathbf{i}, \mathbf{j}, \mathbf{k}$ 分别是坐标 x, y, z 方向的单位矢量. z 为垂直于板的中面的法向座标, x, y 为板的平面的二个坐标, h 为板的厚度. 采用 Kirchhoff 假设, 横向正应力与其他应力分量相比可以忽略, 即 $\sigma_z = 0$; 在弯曲时, 初时垂直于中面的直线纤维仍保持

¹⁾ 中国自然科学基金资助项目.

本文于 1992 年 11 月 6 日收到第一稿, 1993 年 4 月 1 日收到修改稿.

直线和垂直. 则板中任一点的位移 u, v, w , 相对于中面的位移 (u^0, v^0, w^0) 为 (以下标的逗号表示偏微分)

$$u = u^0 - zw_{,x}, \quad v = v^0 - zw_{,y}, \quad \omega = w^0$$

则小变元的位置矢量为

$$\mathbf{r} = (x + u^0 - zw_{,x})\mathbf{i} + (y + v^0 - zw_{,y})\mathbf{j} + (\xi + z + w)\mathbf{k}$$

考虑变元的微线元 ds_0 , 未变形时

$$\begin{aligned} ds_0^2 &= d\mathbf{r}_0 \cdot d\mathbf{r}_0 = \{1 + [(\xi + z)_{,x}]^2\} dx^2 \\ &\quad + 2(\xi + z)_{,x}(\xi + z)_{,y} dx dy + \{1 + [(\xi + z)_{,y}]^2\} dy^2 \end{aligned}$$

变形后

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r} = \frac{\partial x_i^*}{\partial x_j} \cdot \frac{\partial x_i^*}{\partial x_k} dx_j dx_k$$

其中, $x_1^* = x + u^0 - zw_{,x}$, $x_2^* = y + v^0 - zw_{,y}$, $x_3^* = \xi + z + w$.

由应变的定义

$$ds^2 - ds_0^2 = 2(\varepsilon_x dx^2 + \varepsilon_{xy} dx dy + \varepsilon_y dy^2)$$

其中, 忽略高阶小量

$$\left. \begin{aligned} \varepsilon_x &= u_{,x}^0 + \frac{1}{2}[(u_{,x}^0)^2 + (v_{,x}^0)^2 + w_{,x}^2] + \xi_{,x} w_{,x} - zw_{,xx} \\ \varepsilon_{xy} &= u_{,y}^0 + v_{,x}^0 + u_{,x}^0 u_{,y}^0 + v_{,x}^0 v_{,y}^0 + w_{,x} w_{,y} \\ &\quad + \xi_{,y} w_{,x} + \xi_{,x} w_{,y} - 2zw_{,xy} \\ \varepsilon_y &= v_{,y}^0 + \frac{1}{2}[(v_{,y}^0)^2 + (w_{,y}^0)^2 + w_{,y}^2] + \xi_{,y} w_{,y} - zw_{,yy} \end{aligned} \right\} \quad (1)$$

可以看到, 上式中 u^0 和 v^0 的二次项比其他项都要小得多, 可以忽略. 记

$$\left. \begin{aligned} \varepsilon_x^0 &= u_{,x}^0 + \frac{1}{2}w_{,x}^2 + \xi_{,x} w_{,x} \\ \varepsilon_{xy}^0 &= v_{,x}^0 + u_{,y}^0 + w_{,x} w_{,y} + \xi_{,y} w_{,x} + \xi_{,x} w_{,y} \\ \varepsilon_y^0 &= v_{,y}^0 + \frac{1}{2}w_{,y}^2 + \xi_{,y} w_{,y} \end{aligned} \right\} \quad (2)$$

得到带初挠度的应变公式

$$\left. \begin{aligned} \varepsilon_x &= u_{,x}^0 + \frac{1}{2}w_{,x}^2 + \xi_{,x} w_{,x} - zw_{,xx} = \varepsilon_x^0 - zw_{,xx} \\ \varepsilon_y &= v_{,y}^0 + \frac{1}{2}w_{,y}^2 + \xi_{,y} w_{,y} - zw_{,yy} = \varepsilon_y^0 - zw_{,yy} \\ \varepsilon_{xy} &= u_{,y}^0 + v_{,x}^0 + w_{,x} w_{,y} + \xi_{,y} w_{,x} + \xi_{,x} w_{,y} - 2zw_{,xy} \\ &= \varepsilon_{xy}^0 - 2zw_{,xy} \end{aligned} \right\} \quad (3)$$

再考虑应变对时间的微商，记为 $\dot{\epsilon}_{ij}$ ，则

$$\left. \begin{aligned} \dot{\epsilon}_x &= u_{,xt}^0 + w_{,x}w_{,xt} + \xi_{,x}w_{,xt} - zw_{,xxt} = \dot{\epsilon}_x^0 - zw_{,xxt} \\ \dot{\epsilon}_y &= v_{,yt}^0 + w_{,y}w_{,yt} + \xi_{,y}w_{,yt} - zw_{,yyt} = \dot{\epsilon}_y^0 - zw_{,yyt} \\ \dot{\epsilon}_{xy} &= u_{,yt}^0 + v_{,xt}^0 + w_{,xt}w_{,y} + w_{,x}w_{,yt} + \xi_{,y}w_{,xt} \\ &\quad + \xi_{,x}w_{,yt} - 2zw_{,xyt} = \dot{\epsilon}_{xy}^0 - 2zw_{,xyt} \end{aligned} \right\} \quad (4)$$

忽略板的体力，转动和平面惯性作用，则板的力平衡方程为^[5]

$$\left. \begin{aligned} \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} &= 0 \\ \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} &= 0 \\ \frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2(w + \xi)}{\partial x^2} \\ &\quad + 2N_{xy} \frac{\partial^2(w + \xi)}{\partial x\partial y} + N_y \frac{\partial^2(w + \xi)}{\partial y^2} + q = \rho w_{,tt} \end{aligned} \right\} \quad (5)$$

其中 N_x, N_{xy}, N_y 是薄膜力， M_x, M_{xy}, M_y 是弯矩和扭矩，如图 1. q 是横向力，是 (x, y, t) 的函数， ρ 为板的密度.

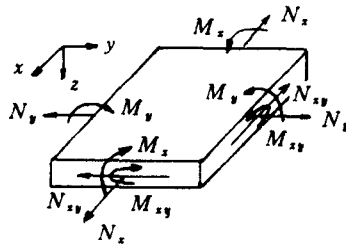


图 1 板单元内的薄膜力，弯矩和扭矩

Fig.1 Plate element with membrane forces, bending and twisting moments

假设所考虑材料是各向同性粘弹性材料，其本构方程为

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [C] \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_{xy} \end{Bmatrix} + x[C] \begin{Bmatrix} \dot{\epsilon}_x \\ \dot{\epsilon}_y \\ \dot{\epsilon}_{xy} \end{Bmatrix} \quad (6)$$

这里采用最简非线性 Kelvin-Voigt 模型. x 为粘弹性系数. 矩阵 $[C]$ 为

$$[C] = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

ν 为泊松比, E 为弹性模量.

薄膜力的计算公式为

$$[N_x, N_y, N_{xy}] = \left\{ \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \sigma_{xy}] dz \right\} \quad (7)$$

弯矩和扭矩的计算公式为

$$[M_x, M_y, M_{xy}] = \int_{-h/2}^{h/2} [\sigma_x, \sigma_y, \sigma_{xy}] z dz \quad (8)$$

令 $k_x = -w_{,xx}$, $k_y = -w_{,yy}$, $k_{xy} = -2w_{,xy}$ 和 $(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C_{ij}(1, z, z^2) dz$ 则

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \varepsilon^0 \\ k \end{Bmatrix} + x \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \dot{\varepsilon}^0 \\ \dot{k} \end{Bmatrix} \quad (9)$$

把 (3), (4) 式代入 (9) 中, 再把 (9) 式中各分量代入到 (5) 中去, 得到由位移 u^0, v^0, w 表示的运动方程:

$$\begin{aligned} & x u_{,xxt}^0 + u_{,xx}^0 + d_1(x u_{,yyt}^0 + u_{,yy}^0) + d_2(x v_{,xyt}^0 + v_{,xy}^0) \\ &= -d_2 w_{,xy}(x w_{,yt} + w_{,y}) - (w_{,xx} + d_1 w_{,yy})(x w_{,xt} + w_{,x}) \\ & \quad - x d_2 w_{,y} w_{,xyt} - d_2 \xi_{,xy}(x w_{,yt} + w_{,y}) \\ & \quad - d_2 \xi_{,y}(x w_{,xyt} + w_{,xy}) - (\xi_{,xx} + d_1 \xi_{,yy})(x w_{,xt} + w_{,x}) \\ & \quad - \xi_{,x}(x w_{,xxt} + w_{,xx} + d_1(x w_{,yyt} + w_{,yy})) \\ & \quad - x w_{,x}(w_{,xxt} + d_1 w_{,yyt}) \end{aligned} \quad (10)$$

$$\begin{aligned} & d_1(x v_{,xxt}^0 + v_{,xx}^0) + x v_{,yyt}^0 + v_{,yy}^0 + d_2(x u_{,xyt}^0 + u_{,xy}^0) \\ &= -d_2 w_{,xy}(x w_{,xt} + w_{,x}) - (w_{,yy} + d_1 w_{,xx})(x w_{,yt} + w_{,y}) \\ & \quad - x d_2 w_{,x} w_{,xyt} - d_2 \xi_{,xy}(x w_{,xt} + w_{,x}) \\ & \quad - d_2 \xi_{,x}(x w_{,xyt} + w_{,xy}) - (\xi_{,yy} + d_1 \xi_{,xx})(x w_{,yt} + w_{,y}) \\ & \quad - \xi_{,y}(x w_{,yyt} + w_{,yy} + d_1(x w_{,xxt} + w_{,xx})) \\ & \quad - x w_{,y}(w_{,yyt} + d_1 w_{,xxt}) \end{aligned} \quad (11)$$

$$\begin{aligned}
& D\nabla^4 w + xD\nabla^4 w_{,t} + \rho w_{,tt} \\
= & q(x, y, t) + \frac{Eh}{1-\nu^2} [(u_{,x}^0 + \frac{1}{2}w_{,x}^2 + \xi_{,x}w_{,x})(w_{,xx} + \xi_{,xx} + \nu(w_{,yy} + \xi_{,yy})) \\
& + (v_{,y}^0 + \frac{1}{2}w_{,y}^2 + \xi_{,y}w_{,y})(w_{,yy} + \xi_{,yy} + \nu(w_{,xx} + \xi_{,xx})) \\
& + x(u_{,xt}^0 + w_{,x}w_{,xt} + \xi_{,x}w_{,xt})(w_{,xx} + \xi_{,xx} + \nu(w_{,yy} + \xi_{,yy})) \\
& + x(v_{,yt}^0 + w_{,y}w_{,yt} + \xi_{,y}w_{,yt})(w_{,yy} + \xi_{,yy} + \nu(w_{,xx} + \xi_{,xx})) \\
& + (1-\nu)((w_{,xy} + \xi_{,xy})(u_{,y}^0 + v_{,x}^0 + w_{,x}w_{,y} + \xi_{,y}w_{,x} + \xi_{,x}w_{,y}) \\
& + x(w_{,xy} + \xi_{,xy})(u_{,yt}^0 + v_{,xt}^0 + w_{,xt}w_{,y} + w_{,x}w_{,yt} + \xi_{,y}w_{,xt} + \xi_{,x}w_{,yt}))] \quad (12)
\end{aligned}$$

其中 $d_1 = \frac{1-\nu}{2}$, $d_2 = \frac{1+\nu}{2}$, $D = \frac{Eh^3}{12(1-\nu^2)}$, $\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}$.

本文着重以四边简支, 边界不能移动的边界条件为例, 求解基本方程 (10)-(12). 其他边界条件下也可相应地求解, 见文 [5].

边界条件: x, y, z 坐标取在板的中心, 板长为 a , 宽为 b , 则

$$\left. \begin{aligned}
u^0 = v^0 = w = w_{,xx} = 0, \quad \xi = \xi_{,xx} = 0, \quad x = \pm \frac{a}{2} \\
u^0 = v^0 = w = w_{,yy} = 0, \quad \xi = \xi_{,yy} = 0, \quad y = \pm \frac{b}{2}
\end{aligned} \right\} \quad (13)$$

满足边界条件的位移解可取为

$$\begin{aligned}
u^0 &= u(t) \sin \frac{2m\pi x}{a} \cos \frac{n\pi y}{b} \\
v^0 &= v(t) \cos \frac{m\pi x}{a} \sin \frac{2n\pi y}{b} \\
w &= f(t) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}
\end{aligned}$$

和 $\xi = f_0 \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$ (m, n 奇整数)

代入到 (10), (11), (12) 式中, 并且以 $\sin \frac{2m\pi x}{a} \cos \frac{n\pi y}{b}$ 乘 (10) 式二边, 以 $\cos \frac{m\pi x}{a} \sin \frac{2n\pi y}{b}$ 乘 (11) 式, 以 $\cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}$ 乘 (12) 式, 再在整个中面上求积分, 并取 $m = n = 1$, 横向力 $q(x, y, t)$ 作用于原点的集中力, $q(0, 0, t) = p(t)$, 简记

$$\begin{aligned}
a_1 &= \frac{4\pi^2}{a^2} + d_1 \frac{\pi^2}{b^2}, \quad a_2 = \frac{4\pi^2}{b^2} + d_1 \frac{\pi^2}{a^2}, \quad a_3 = \frac{D}{\rho} \left(\frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right)^2 \\
b_1 &= \frac{64}{9ab} d_2, \quad b_3 = \frac{Eh}{1-\nu^2} \left(\frac{8}{3a} \left(\frac{\pi^2}{a^2} + \nu \frac{\pi^2}{b^2} \right) + \frac{\pi^2}{6ab^2} (1-\nu) \right) / \rho \\
c_1 &= \frac{8}{3} \left(\frac{1}{a} \left(\frac{\pi^2}{a^2} + d_1 \frac{\pi^2}{b^2} \right) - \frac{\pi^2}{2ab^2} d_2 \right), \quad c_2 = \frac{8}{3} \left(\frac{1}{b} \left(\frac{\pi^2}{b^2} + d_1 \frac{\pi^2}{a^2} \right) - \frac{\pi^2 d_2}{2a^2 b} \right) \\
c_3 &= \frac{Eh}{1-\nu^2} \cdot \frac{1}{\rho} \left(\frac{8}{3b} \left(\frac{\pi^2}{b^2} + \nu \frac{\pi^2}{a^2} \right) + \frac{\pi^2}{6a^2 b} (1-\nu) \right) \\
d_3 &= \frac{Eh}{1-\nu^2} \cdot \frac{\pi^4}{8\rho} \left(3 \left(\frac{1}{a^4} + \frac{2\nu}{a^2 b^2} + \frac{1}{b^4} \right) - \frac{1-\nu}{a^2 b^2} \right)
\end{aligned}$$

和 $\psi(\dot{f}(t), f(t), f_0) = f_0(x\dot{f}(t) + f(t)) + f(t)\left(\frac{1}{2}f(t) + x\dot{f}(t)\right)$

经过计算和整理可得

$$x\dot{u}(t) + u(t) = \frac{c_2b_1 - c_1a_2}{b_1^2 - a_1a_2} \psi(\dot{f}, f, f_0) = A_1\psi \quad (14)$$

$$x\dot{v}(t) + v(t) = \frac{c_1b_1 - c_2a_1}{b_1^2 - a_1a_2} \psi(\dot{f}, f, f_0) = B_1\psi \quad (15)$$

$$\ddot{f}(t) + a_3(x\dot{f}(t) + f(t)) + (f(t) + f_0)(b_3A_1 + c_3B_1 + d_3)\psi = \frac{4}{\rho}p(t) \quad (16)$$

从 (14), (15), (16) 式看到振幅 $u(t), v(t)$ 是依赖于 $f(t)$ 的非独立变量, 而振幅 $f(t)$ 的变化不依赖于 $u(t), v(t)$. 因此由 $f(t)$ 可决定 $u(t), v(t)$ 的变化, 真正独立的方程仅仅方程 (16), 把 (16) 式整理一下, 并记 $w_1 = f(t) + f_0$, $H_1 = A_1b_3 + B_1c_3 + d_3$, 则有

$$\ddot{w}_1 + x(a_3 + H_1w_1^2)\dot{w}_1 + \left(a_3 - \frac{H_1}{2}f_0^2\right)w_1 + \frac{H_1}{2}w_1^3 - a_3f_0 = \frac{4}{\rho}p(t) \quad (17)$$

这是一个二阶非线性微分方程, 所假定条件下微曲板的动力性质完全由它所决定. 记号 “ \bullet ” 是时间的微商. 在这一方程里, Kelvin-Voigt 模型表现材料粘弹性形变规律的粘性项, 在板的动力方程中起着粘性阻尼的作用. 由于考虑了板大挠度的非线性性质, 粘性阻尼与速度和位移呈非线性关系.

三、分析及数值结果

1. 考虑在静载荷下中心挠度与载荷的关系.

把静力条件, $\frac{df}{dt} = 0$, $\frac{d^2f}{dt^2} = 0$ 代入方程 (17), 记中心挠度 $w_0 = f + f_0$, 得到

$$p_0 = \frac{H_1\rho}{8}w_0^3 + \frac{\rho}{4}\left(a_3 - \frac{H_1}{2}f_0^2\right)w_0 - \frac{\rho a_3}{4}f_0 \quad (18)$$

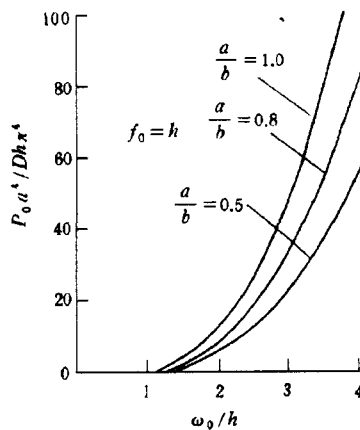


图 2 板中心挠度与静力载荷的关系

Fig.2 Central deflection versus static concentrated load for simply supported plate

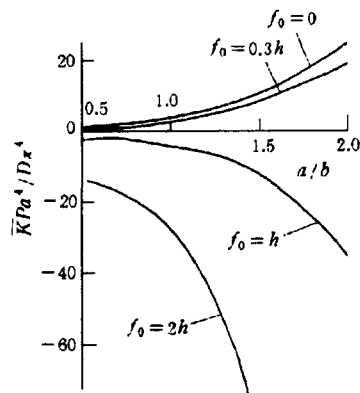


图 3 不同初挠度下线性刚度与长宽比的关系

Fig.3 Linear stiffness versus the aspect ratio of length to width for simply supported plate with various initial curvatures

根据前面的推导, 得知 p_0 是长宽比 a/b 和初始挠度 f_0 的函数. 图 2 示出在初挠度 $f_0 = h$ 时三种长宽比情况下板中心挠度与静力载荷的关系.

从方程 (17) 式中, 还可得到系统的线性刚度 \bar{K} 关于长宽比 a/b 和初挠度 f_0 的函数

$$\bar{K} = a_3 - \frac{H_1}{2} f_0^2 \quad (19)$$

图 3 表示在不同初挠度下线性刚度随长宽比的变化. 其中可以看到当 f_0 较大时, 系统出现负刚度. 这是大挠度带初挠度系统的显著特点.

2. 在周期外力作用下系统的动力性质

设 $p(t) = p_1 \cos \Omega t$, 并记

$$\begin{aligned} k_1 &= \left| a_3 - \frac{H_1}{2} f_0^2 \right|, & \alpha_0 &= \frac{H_1}{2}, & t &= \frac{\tau}{\sqrt{k_1}} \\ \delta &= \frac{x a_3}{\sqrt{k_1}}, & \beta &= 2x \sqrt{k_1}, & \gamma &= \frac{a_3 f_0}{k_1} \sqrt{\frac{\alpha_0}{k_1}} \\ Q &= \frac{4}{\rho} p_1 \cdot \frac{1}{k_1} \sqrt{\frac{\alpha_0}{k_1}}, & \theta &= \frac{\Omega}{\sqrt{k_1}}, & w_1 &= x_1 \sqrt{\frac{K_1}{\alpha_0}} \end{aligned}$$

则方程 (17) 可化成,

$$x_1'' + (\delta + \beta x_1^2) x_1' \pm x_1 + x_1^3 = \gamma + Q \cos \theta \tau \quad (20)$$

“,” 表示对 τ 的微商.

(a) 考察在正刚度时解的分歧.

当初挠度较小时, 刚度为正, 参数 $(\delta, \beta, \gamma, Q)$ 是小量, $\sigma = 1 - \theta^2$, 阻尼非负. 系统在激励频率下存在近似的周期解. 应用 KBM 平均法. 把正刚度情况的方程 (20) 改写为

$$\left. \begin{aligned} x_1' &= y_1 \\ y_1' &= -x_1 - x_1^3 - (\delta + \beta x_1^2) y_1 + \gamma + Q \cos \theta \tau \end{aligned} \right\} \quad (21)$$

采用变换, $u_1 = x_1 \cos \theta \tau - \frac{1}{\theta} y_1 \sin \theta \tau$, $v_1 = -x_1 \sin \theta \tau - \frac{1}{\theta} y_1 \cos \theta \tau$ 和反变换, $x_1 = u_1 \cos \theta \tau - v_1 \sin \theta \tau$, $y_1 = -\theta u_1 \sin \theta \tau - \theta v_1 \cos \theta \tau$, 则系统为

$$\left. \begin{aligned} u_1' &= \frac{1}{\theta} (\sigma x_1 + x_1^3 + (\delta + \beta x_1^2) y_1 - \gamma - Q \cos \theta \tau) \sin \theta \tau \\ v_1' &= \frac{1}{\theta} (\sigma x_1 + x_1^3 + (\delta + \beta x_1^2) y_1 - \gamma - Q \cos \theta \tau) \cos \theta \tau \end{aligned} \right\} \quad (22)$$

注意到这一变换把方程 (21) 中的解 $A \cos(\theta \tau + \phi_0)$ 变换为 (22) 中的定常解 $(A \cos \phi_0, A \sin \phi_0)$. 对 (22) 式在一周期内求平均

$$\begin{aligned} u_1' &= \frac{1}{2\theta} \left[-\sigma v_1 - \frac{3}{4} v_1 (u_1^2 + v_1^2) - \delta \theta u_1 - \frac{\theta u_1}{4} \beta (u_1^2 + v_1^2) \right] \\ v_1' &= -\frac{1}{2\theta} \left[-\sigma u_1 - \frac{3}{4} u_1 (u_1^2 + v_1^2) + \delta \theta v_1 + \frac{\beta}{4} \theta v_1 (u_1^2 + v_1^2) + Q \right] \end{aligned}$$

化为极坐标形式, $u_1 = r \cos \phi$, $v_1 = r \sin \phi$, 上式为

$$\begin{aligned} r' &= \frac{1}{2\theta} \left(-\delta\theta r - \frac{\theta\beta}{4} r^3 - Q \sin \phi \right) \\ r\phi' &= \frac{1}{2\theta} \left(\sigma r + \frac{3}{4} r^3 - Q \cos \phi \right) \end{aligned}$$

因此直接得到定常解, A 为振幅, ϕ_0 为相位.

$$\left(\delta\theta + \frac{\beta\theta}{4} A^2 \right)^2 A^2 + \left(\sigma + \frac{3}{4} A^2 \right)^2 A^2 = Q^2 \quad (23)$$

$$\operatorname{tg} \phi_0 = -\frac{\theta(4\delta + \beta A^2)}{4\sigma + 3A^2} \quad (24)$$

方程 (23) 的解已解出. 数值解如图 4, 固定 δ, β 和 Q 得到振幅随频率的曲线. 低频区域是稳态解, 随频率的增加在分歧点出现两个稳态解和一个不稳定鞍点, 再高频又回复于稳态解.

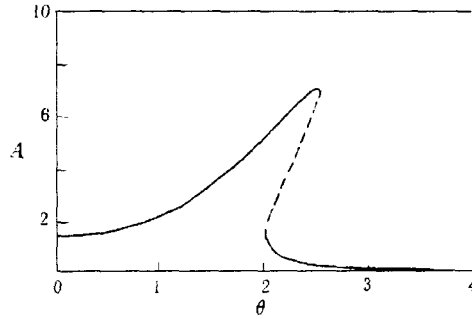


图 4 方程 (21) 的简谐解, $Q = 2.5$, $\delta = 0.2$, $\beta = 0.1$; —— 稳定解, - - - - 不稳定解

Fig.4 Harmonic solutions of equation (21) for $Q = 2.5$, $\delta = 0.2$, $\beta = 0.1$;

—— stable, - - - - unstable

分歧集:

方程 (23) 是以参数 θ, δ, β 和 Q 给出的幅值公式, 展开得

$$\frac{1}{16} (9 + \theta^2 \beta^2) A^6 + \frac{1}{2} (3\sigma + \beta\delta\theta^2) A^4 + (\sigma^2 + \delta^2\theta^2) A^2 - Q^2 = 0 \quad (25)$$

化成形如, $(A^2 + l_0)^3 + l_1(A^2 + l_0) + s = 0$

其中 $l_0 = \frac{8}{3} \cdot \frac{3\sigma + \beta\delta\theta^2}{9 + \theta^2\beta^2}$, $l_1 = \frac{16(\sigma^2 + \delta^2\theta^2)}{9 + \theta^2\beta^2} - 3l_0^2$, $s = -\frac{16Q^2}{9 + \theta^2\beta^2} - l_0^3 - l_0l_1$

系统出现分歧的条件是

$$4l_1^3 + 27s^2 = 0 \quad (26)$$

也即

$$\begin{aligned} & \frac{(9 + \theta^2\beta^2)^2}{16} Q^4 + Q^2 \cdot \frac{(3\sigma + \beta\delta\theta^2)}{3} \cdot \left[(\sigma^2 + \delta^2\theta^2)(9 + \theta^2\beta^2) - \frac{8}{9}(3\sigma + \beta\delta\theta^2)^2 \right] \\ & + \frac{4}{27} [(\sigma^2 + \delta^2\theta^2)^3(9 + \theta^2\beta^2) - (\sigma^2 + \delta^2\theta^2)^2(3\sigma + \beta\delta\theta^2)^2] = 0 \end{aligned} \quad (27)$$

由于 β, δ 都依赖于粘弹性系数 x . 图 5 示出在不同 x 情况下 Q 与频率 θ 的分歧集. 区域 I 存在稳态解, 区域 II 存在着二个稳态解和一个不稳定鞍点.

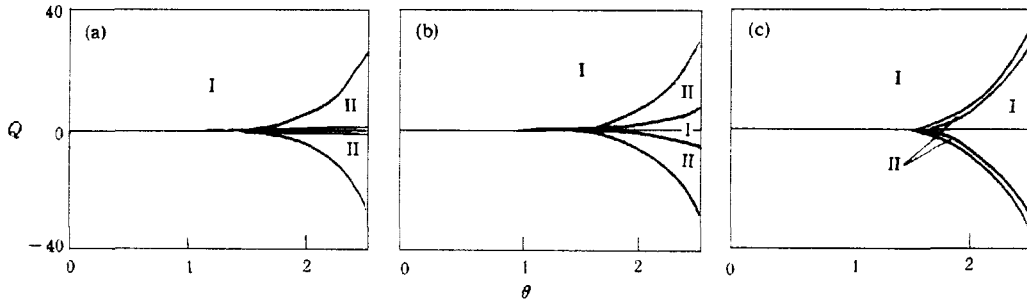


图 5 Q, θ 空间里的分歧集, $\delta = 2x, \beta = x$, (a) $x = 0.01$, (b) $x = 0.1$, (c) $x = 0.25$
 Fig.5 The bifurcation set in Q, θ space, $\delta = 2x, \beta = x$; (a) $x = 0.01$, (b) $x = 0.1$, (c) $x = 0.25$

(b) 在负刚度下系统的全局动力行为

设粘性和外力是 ϵ 小量. 负刚度时的方程 (20) 写成

$$\left. \begin{aligned} x_1' &= y_1 \\ y_1' &= x_1 - x_1^3 - \epsilon(\delta + \beta x_1^2)y_1 + \epsilon(\gamma + Q \cos \theta \tau) \end{aligned} \right\} \quad (28)$$

由于 δ, β 依赖于 x , 记 $\delta = x\delta_1, \beta = x\beta_1, \epsilon = 0$ 时未扰系统的 Hamilton 量为

$$H(x_1, y_1) = \frac{1}{2}y_1^2 - \frac{1}{2}x_1^2 + \frac{1}{4}x_1^4 = \bar{h}$$

当 $\bar{h} = 0$ 时, 存在一对同宿轨道; $(0,0)$ 点是方程 (28) 的双曲鞍点. $\bar{h} \neq 0$ 时存在以 \bar{h} 为参数的周期轨道, 表达式分别如下

$$\begin{cases} x_{10}(\tau) = \pm\sqrt{2}\operatorname{sech}\tau \\ y_{10}(\tau) = \mp\sqrt{2}\operatorname{sech}\tau \cdot \operatorname{th}\tau \end{cases}$$

和

$$\begin{cases} x_{1k}(\tau) = \frac{\sqrt{2}}{\sqrt{2-k^2}}\operatorname{dn}\left(\frac{\tau}{\sqrt{2-k^2}}\right) \\ y_{1k}(\tau) = -\frac{\sqrt{2}k^2}{2-k^2}\operatorname{sn}\left(\frac{\tau}{\sqrt{2-k^2}}\right)\operatorname{cn}\left(\frac{\tau}{\sqrt{2-k^2}}\right) \end{cases}$$

k 满足关系式, $\bar{h}_k = \frac{k^2-1}{(2-k^2)^2}$, 周期 $T(k) = 2\sqrt{2-k^2}K(k)$, 应用 Melnikov 方法 [8], 计算同宿轨道的 Melnikov 函数, 取上枝轨道

$$\begin{aligned} M(\tau_0) &= \int_{-\infty}^{\infty} y_{10}(\tau)(\gamma + Q \cos \theta(\tau + \tau_0) - (\delta + \beta x_{10}^2(\tau))y_{10}(\tau))d\tau \\ &= -\sqrt{2}\pi\theta Q \operatorname{sech}\left(\frac{\pi\theta}{2}\right) \sin \theta\tau_0 - \frac{4}{3}\delta - \frac{28}{15}\beta \end{aligned}$$

其中积分

$$\begin{aligned} \int_{-\infty}^{\infty} \operatorname{sech} \tau \cdot \operatorname{th} \tau \cos \theta(\tau + \tau_0) d\tau &= \pi \theta \operatorname{sech} \left(\frac{\pi \theta}{2} \right) \sin \theta \tau_0 \\ \int_{-\infty}^{\infty} \operatorname{sech}^2 \tau \operatorname{th}^2 \tau d\tau &= \frac{2}{3} \\ \int_{-\infty}^{\infty} \operatorname{sech}^4 \tau \cdot \operatorname{th}^2 \tau d\tau &= \frac{7}{15} \end{aligned}$$

$$\text{记 } R_{1\infty}(\theta) = \frac{2\sqrt{2}}{3\pi\theta \operatorname{sech} \left(\frac{\pi\theta}{2} \right)}, \quad R_{2\infty}(\theta) = \frac{14\sqrt{2}}{15\pi\theta \operatorname{sech} \left(\frac{\pi\theta}{2} \right)}$$

当横向力与粘弹性系数的比值 $\frac{Q}{x}$ 达到

$$\frac{Q}{x} > \delta_1 R_{1\infty}(\theta) + \beta_1 R_{2\infty}(\theta)$$

时, $M(\tau_0)$ 有简单零点, 系统将进入混沌状态.

对于满足共振条件 $T(k) = 2\sqrt{2-k^2}K(k) = \frac{2\pi}{\theta} \cdot \frac{m'}{n'} = \frac{m'T}{n'}$ 的次谐 Melnikov 函数为

$$\begin{aligned} M^{m'/n'}(\tau_0) &= \int_0^{m'T} y_{1k}(\tau)(\gamma + Q \cos \theta(\tau + \tau_0) - (\delta + \beta x_{1k}^2(\tau))y_{1k}(\tau)) d\tau \\ &= -QJ_3(m', n', \theta) \sin \theta \tau_0 - \delta J_1(m', n') - \beta J_2(m', n') \end{aligned}$$

其中

$$\begin{aligned} J_1(m', n') &= \frac{4m'\sqrt{2-k^2}}{3(2-k^2)^2} (2(k^2-1)K(k) + (2-k^2)E(k)) \\ J_2(m', n') &= \frac{8m'\sqrt{2-k^2}}{15k^2(2-k^2)^3} ((8-7k^2+4k^4-5k^6)K(k) - (8-3k^2+3k^4-10k^6)E(k)) \\ J_3(m', n', \theta) &= \begin{cases} 0 & n' \neq 1 \\ \sqrt{2}\pi\theta \operatorname{sech} \frac{\pi m' K'(k(m', 1))}{K(k(m', 1))} & n' = 1 \end{cases} \end{aligned}$$

这里, sn , cn 和 dn 为椭圆函数, k 为椭圆函数的模, $k'^2 = 1 - k^2$. $K(k)$, $K'(k)$ 和 $E(k)$ 是椭圆积分.

于是当

$$\frac{Q}{x} > \delta_1 \frac{J_1(m', 1)}{J_3(m', 1, \theta)} + \beta_1 \frac{J_2(m', 1)}{J_3(m', 1, \theta)} = \delta_1 R_1^{m'}(\theta) + \beta_1 R_2^{m'}(\theta)$$

时, $M^{m'}(\tau_0)$ 有简单零点, 系统产生次谱分枝; 当 $m' \rightarrow \infty$ 时, 如果 $\frac{Q}{x} > \delta_1 R_{1\infty}(\theta) + \beta_1 R_{2\infty}(\theta)$, 系统在鞍点 $(0,0)$ 的稳定和不稳定流形出现无限多次相交, 示意图 6 所示. 对于每次相交, 考虑鞍点不稳定流形 W^u 上的一小正方形 R , 经过 p 次向后迭代得到水平地逼近稳定流形的拉伸图像 $P^{-p}(R)$, 若经过 q 次向前迭代将得到沿不稳定流形拉伸的图像 $P^{+q}(R)$. 这样映射 P^{p+q} 表示为以长方带 $P^{-p}(R)$ 经过弯曲而交于

自身的映射，类似于马蹄映射。由于系统出现无限多次如上所述的 W^u 与 W^s 的相交，因此系统具有无限多个马蹄，每个马蹄以及所包含的所有周期为 $(p+q)$, $2(p+q)$, \dots , $k(p+q)$, \dots ，以及稠密的非周期轨道一起就构成了鞍点在 W^u 和 W^s 领域内复杂的吸引集，即意味着混沌态的出现。

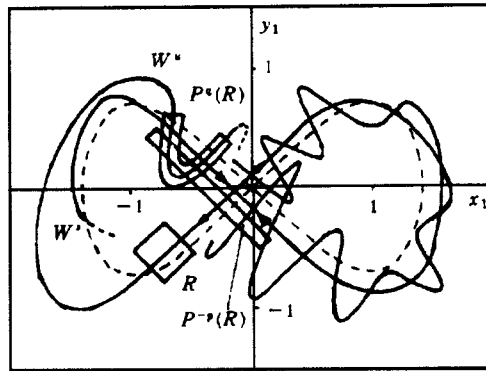


图 6 同宿轨道和马蹄的存在

Fig.6 The homoclinic orbit and the existence of a small horseshoe

参 考 文 献

- [1] Thompson JMT and Hunt GW. Dangers of Structural Optimization. *Engineering Optimization*, 1974, 1:99-110
- [2] Marguere K. Zur Theorie Der Gekrumnten Platte Grosser Formanderung. *Proc. 5th Intl. Cong. Appl. Mech.*, New York: John Wiley & Sons Inc., 1938: 93-101
- [3] Timoshenko S and Woinowdsky-Krieger S. *Theory of Plates and Shells*. New York: McGraw-Hill, 1959: 27
- [4] Coan JM. *J. Appl. Mech.* 1951, 18: 143
- [5] Chia Chuenyuan. *Nonlinear Analysis of Plates*. New York: McGraw-Hill. Inc. 1980: 96-107
- [6] Rushton KR. *Intl. J. Mech. Sci.*, 1970, 12: 1037-1051
- [7] Krätzig WB. Fundamentals of Numerical Algorithms for Static and Dynamic Instability Phenomena of Thin Shells. in: *Computational Mechanics of Nonlinear Response of Shells*, ed. by W.B. Krätzig, E. Oñate, Berlin: Springer-Verlag, 1990: 101-124
- [8] Guckenheimer J and Holmes P. *Non-Linear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*. Springer, Berlin, 1983

DYNAMIC BEHAVIOUR OF LARGE DEFLECTION OF RECTANGULAR THIN-PLATE WITH INITIAL CURVATURE

Dong Zhifa

(*Institute of Mechanics, Chinese Academy of Sciences, Beijing 100080, China*)

Abstract In this paper the equations of motion for the large deflection problem of a visco-elastic rectangular thin plate with initial curvature are expressed in terms of three displacements at the midplane. For a simply supported plate subjected to transverse load, the dynamic response of the plate under periodic load is studied, and the criterion of arising subharmonic bifurcation and the horseshoe (or chaotic) state is given by the bifurcation theory and the Melnikov's method.

Key words visco-elastic, large deflection, bifurcation set, subharmonic bifurcation, Melnikov's method