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复合材料夹层扁壳的有限挠度方程和线性稳定问题

刘国玺　王震鸣

(中国科学院力学研究所)

提要　本文对表板采用Kirchhoff假设，考虑夹心沿厚度方向的剪切变形，用最小势能原理推导了复合材料夹层扁壳的有限挠度方程和边界条件．作为一个特例，求解了矩形夹层圆柱壳在轴压(或侧压)作用下的线性稳定问题．

本文在文献〔1〕和〔2〕的基础上，对表板采用Kirchhoff假设，考虑夹心沿厚度方向的剪切变形，用最小势能原理推导了复合材料夹层扁壳的有限挠度方程和边界条件．它既能应用于复合材料夹层扁壳的各种情况(软夹心或硬夹心，表板对称或不对称)，又能解决混杂铺层的复合材料扁壳问题．它既可用来求解应力和变形问题，也可用来求解稳定问题．混杂铺层的复合材料扁壳在某些情况下可以用文献〔1〕中的多层扁壳理论处理，但是，如果外层材料的厚度不是很薄，或者内外层的刚度相差太远，则文献〔1〕的理论误差较大，而本文的理论误差较小．这组方程具有较大的普遍性和一般性．复合材料扁壳的各种方程，一般夹层扁壳的方程，甚至单层扁壳的方程都可以看作是它的特例．

一、基本关系式

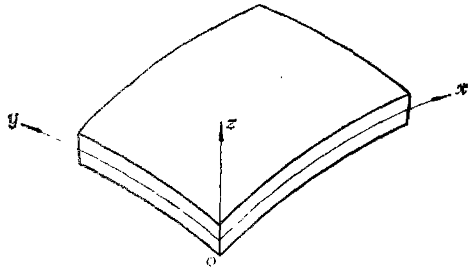

图 1　扁壳上的坐标系

夹层扁壳如图1所示，夹心和表板的厚度如图2所示，各层的顺序和坐标如图3所示．以u、v、w 表示壳体内任一点沿x、y、z方向的位移，以u_0、v_0、w_0表示它们在xoy坐标面上的值．以φ_x和φ_y表示变形过程中夹心段的法线由z轴向x和y轴的转角．夹心中任一点的位移可以表示为

$$\left.\begin{array}{ll} u=u_0+z\varphi_x & w=w_0 \\ v=v_0+z\varphi_y & \end{array}\right\} \quad (1)$$

表板i ($i=1$ 表示下表板，$i=2$ 表示上表板)中任一点的位移为

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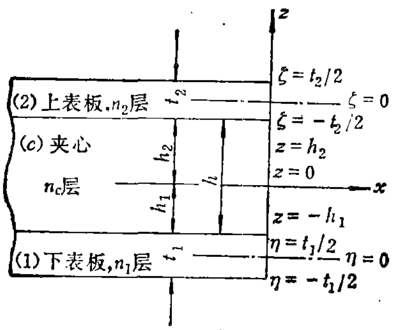

图 2　表板和夹心的几何参数

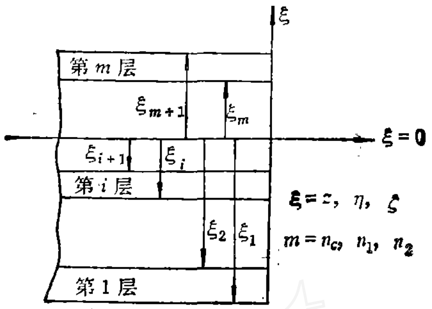

图 3　各层的顺序和坐标

$$\left.\begin{aligned}u&=u_0+(-1)^i h_i\varphi_x-[z-(-1)^i h_i]\frac{\partial w_0}{\partial x}\\ v&=v_0+(-1)^i h_i\varphi_y-[z-(-1)^i h_i]\frac{\partial w_0}{\partial y}\\ w&=w_0\end{aligned}\right\}\tag{2}$$

扁壳中的应变是

$$\left.\begin{aligned}\varepsilon_x&=\frac{\partial u}{\partial x}+\frac{w}{R_x}+\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\\ \varepsilon_y&=\frac{\partial v}{\partial y}+\frac{w}{R_y}+\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2\\ \gamma_{xy}&=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}+\frac{2w}{R_{xy}}+\frac{\partial w}{\partial x}\cdot\frac{\partial w}{\partial y}\end{aligned}\right\}\tag{3}$$

其中$1/R_x$，$1/R_y$和$1/R_{xy}$分别为xoy曲面在x、y方向的曲率和扭率。

复合材料的虎克定律是

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}\overline{Q}_{11}&\overline{Q}_{12}&\overline{Q}_{16}\\ \overline{Q}_{12}&\overline{Q}_{22}&\overline{Q}_{26}\\ \overline{Q}_{16}&\overline{Q}_{26}&\overline{Q}_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{Bmatrix}\tag{4}$$

其中$\overline{Q}_{ij}$ $(i,\ j=1,\ 2,\ 6)$是材料在x、y方向的刚度系数．把(1)、(2)、(3)式代入(4)式，再做相应的积分，则可得

$$\begin{Bmatrix}N\\ M^e\\ M^s\end{Bmatrix}=\begin{bmatrix}A&B&P\\ B&D&E\\ P&E&F\end{bmatrix}\begin{Bmatrix}\varepsilon\\ K\\ X\end{Bmatrix}\tag{5}$$

其中

$$\{N\}=\begin{Bmatrix}N_x\\ N_y\\ N_{xy}\end{Bmatrix}=\int_{-(h_1+t_1)}^{h_2+t_2}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix}dz\tag{6.1}$$

$$\{M^c\}=\begin{bmatrix}M_x^c\\M_y^c\\M_{xy}^c\end{bmatrix}=\int_{-h_1}^{h_2}\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}zdz-h_1\int_{-(h_1+t_1)}^{-h_1}\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}dz+h_2\int_{h_2}^{h_2+t_2}\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}dz \tag{6.2}$$

$$\{M^s\}=\begin{bmatrix}M_x^s\\M_y^s\\M_{xy}^s\end{bmatrix}=\int_{-(h_1+t_1)}^{-h_1}\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}(h_1+z)dz-\int_{h_2}^{h_2+t_2}\begin{bmatrix}\sigma_x\\\sigma_y\\\tau_{xy}\end{bmatrix}(h_2-z)dz \tag{6.3}$$

$$\{\varepsilon\}=\begin{Bmatrix}\dfrac{\partial u_0}{\partial x}+\dfrac{w_0}{R_x}+\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2\\ \dfrac{\partial v_0}{\partial y}+\dfrac{w_0}{R_y}+\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial y}\right)^2\\ \dfrac{\partial u_0}{\partial y}+\dfrac{\partial v_0}{\partial y}+\dfrac{2w_0}{R_{xy}}+\dfrac{\partial w_0}{\partial x}\cdot\dfrac{\partial w_0}{\partial y}\end{Bmatrix} \tag{7.1}$$

$$\{K\}=\begin{Bmatrix}\dfrac{\partial\varphi_x}{\partial x}\\ \dfrac{\partial\varphi_y}{\partial y}\\ \dfrac{\partial\varphi_x}{\partial y}+\dfrac{\partial\varphi_y}{\partial x}\end{Bmatrix} \tag{7.2}$$

$$\{\chi\}=\begin{Bmatrix}-\dfrac{\partial^2 w_0}{\partial x^2}\\ -\dfrac{\partial^2 w_0}{\partial y^2}\\ -2\dfrac{\partial^2 w_0}{\partial x^2 y}\end{Bmatrix} \tag{7.3}$$

$$\left.\begin{aligned}
[A]&=[A^{(c)}]+[A^{(1)}]+[A^{(2)}]\\
[B]&=[B^{(c)}]-h_1[A^{(1)}]+h_2[A^{(2)}]\\
[P]&=[B^{(1)}]+[B^{(2)}]-\frac{t_1}{2}[A^{(1)}]+\frac{t_2}{2}[A^{(2)}]\\
[D]&=[D^{(c)}]+h_1^2[A^{(1)}]+h_2^2[A^{(2)}]\\
[E]&=-h_1[B^{(1)}]+h_2[B^{(2)}]+\frac{h_1t_1}{2}[A^{(1)}+\frac{h_2t_2}{2}(A^{(2)}]\\
[F]&=[D^{(1)}]+[D^{(2)}]-t_1[B^{(1)}]+t_2[B^{(2)}]+\frac{t_1^2}{4}[A^{(1)}]+\frac{t_2^2}{4}[A^{(2)}]
\end{aligned}\right\} \tag{8}$$

(8) 式中的各矩阵都是 3×3 的方阵，其元素排列方式完全一样，现以[A]为例表示如下

$$[A]=\begin{bmatrix}A_{11}&A_{12}&A_{16}\\A_{12}&A_{22}&A_{26}\\A_{16}&A_{26}&A_{66}\end{bmatrix} \tag{9}$$

(8)式右端各矩阵的元素可以表示为如下的通式

$$\left.\begin{aligned} A_{pq}^{(\lambda)} &= \sum_{j=1}^{n_\lambda} (\overline{Q}_{pq}^{\lambda})_j (\xi_{j+1} - \xi_j) \\ B_{pq}^{(\lambda)} &= \sum_{j=1}^{n_\lambda} (\overline{Q}_{pq}^{\lambda})_j \frac{1}{2} (\xi_{j+1}^2 - \xi_j^2) \\ D_{pq}^{(\lambda)} &= \sum_{j=1}^{n_\lambda} (\overline{Q}_{pq}^{\lambda}) \frac{1}{3} (\xi_{j+1}^3 - \xi_j^3) \end{aligned}\right\} \tag{10}$$

$(p,\ q=1,\ 2,\ 6)$

其中 j 表示夹心、上表板、下表板中的第 j 层(见图3)．$\lambda=C$，1,2 分别表示夹心，下表板和上表板，并且

$\lambda=C$ 时　　$\xi=z$

$\lambda=1$ 时　　$\xi=\eta$

$\lambda=2$ 时　　$\xi=\zeta$

坐标 z、η、ξ 的取法见图2．$\overline{Q}_{pq}^{(c)}$，$\overline{Q}_{pq}^{(1)}$ 和 $\overline{Q}_{pq}^{(2)}$ 分别表示夹心，下表板和上表板的刚度系数．n_c，n_1 和 n_2 分别表示夹心，下表板和上表板的层数．

关于夹心中的剪力 $Q_x^{(c)}$，$Q_y^{(c)}$ 和夹心的剪应变 $\gamma_{xy}=\varphi_x+\dfrac{\partial w_0}{\partial x}$，$\gamma_{yz}=\varphi_y+\dfrac{\partial w_0}{\partial y}$ 之间的关系，我们采用文献〔1〕中的结果

$$\begin{Bmatrix} Q_y^{(c)} \\ Q_x^{(c)} \end{Bmatrix} = \begin{bmatrix} C_{44} & C_{45} \\ C_{45} & C_{55} \end{bmatrix} \begin{Bmatrix} \varphi_y + \dfrac{\partial w_0}{\partial y} \\ \varphi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix} \tag{11}$$

$$C_{ij} = \frac{5}{4}\sum_{k=1}^{n_c} (\overline{Q}_{ij}^{(c)})_k \left[z_{k+1} - z_k - \frac{4}{3} \cdot \frac{\left(z_{k+1} - \dfrac{H}{2} + H_1\right)^3 - \left(z_k - \dfrac{H}{2} + H_1\right)^3}{H^2} \right] \tag{12}$$

其中 C_{ij}（i，$j=4$，5）是剪切刚度系数，$H_1=h_1+t_1$，$H=h_1+h_2+t_1+t_2$.

二、用最小势能原理推导平衡方程和边界条件

最小势能原理是

$$\delta\pi=\delta(U-W)=0 \tag{13}$$

其中 π 是总势能，U 是变形能，W 是外力功.

$$U=\iint_{\Omega}\frac{1}{2}(\{N\}^T\{\varepsilon\}+\{M^L\}^T\{K\}+\{M^s\}^T\{\chi\} + [Q_y^c Q_x^c][\gamma_{yz}\gamma_{xz}]^T)\,d\Omega \tag{14}$$

$$W=\iint_{\Omega} q_z w_0\,d\Omega+\int_{\Gamma^\sigma}(\overline{N}_n u_{on}+\overline{N}_{nt}u_{0t}+\overline{M}_n^c\varphi_n + \overline{M}_{nt}^c\varphi_t' - \overline{M}_n^s\frac{\partial w_0}{\partial n}+\overline{Q}_n w_0)\,d\Gamma \tag{15}$$

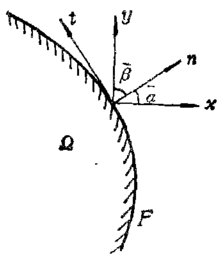

图 4 边界上的法线和切线

其中 Ω 是扁壳在 xoy 曲面上的面域，Γ_σ 是壳体边界 Γ 上给定边界力的部分．q_z 是 z 方向的面积载荷．n 是 Γ 的外法线，t 是 Γ 的切线(图 4)．u_{on}, u_{ot}, N_n, N_{nt}, M_n^c, M_{nt}^L, M_n^s, Q_n 是 ntz 坐标系中的位移和内力．短横表示各量在边界上的给定值．

将(5)和(7)式代入(14)和(15)式中，然后由(13)式可以得平衡方程和边界条件如下：

平衡方程

$$\left.\begin{aligned}&\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0 \qquad \frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0\\&\frac{\partial M_x^c}{\partial x}+\frac{\partial M_{xy}^c}{\partial y}=Q_x^c \qquad \frac{\partial M_{xy}^c}{\partial x}+\frac{\partial M_y^c}{\partial y}=Q_y^c\\&\frac{\partial Q_x^c}{\partial x}+\frac{\partial Q_y^c}{\partial y}+\frac{\partial^2 M_x^s}{\partial x^2}+2\frac{\partial^2 M_{xy}^s}{\partial x\partial y}+\frac{\partial^2 M_y^s}{\partial y^2}-\frac{N_x}{R_x}-\frac{N_y}{R_y}-\frac{2N_{xy}}{R_{xy}}\\&\quad+\frac{\partial}{\partial x}\left(N_x\frac{\partial w_0}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_y\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial x}\left(N_{xy}\frac{\partial w_0}{\partial y}\right)+\frac{\partial}{\partial y}\left(N_{xy}\frac{\partial w_0}{\partial x}\right)+q_z=0\end{aligned}\right\} \tag{16}$$

边界条件

在 Γ_ε（$\Gamma_\varepsilon=\Gamma-\Gamma_\sigma$，是给定位移的边界）上：

$$\left.\begin{aligned}&u_{0n}=\bar{u}_{0n},\ \ u_{0t}=\bar{u}_{0t},\ \ \varphi_n=\bar{\varphi}_n\\&\varphi_t=\bar{\varphi}_t,\ \ \frac{\partial w_0}{\partial n}=\frac{\partial \bar{w}_0}{\partial n},\ \ w_0=\bar{w}_0\end{aligned}\right\} \tag{17}$$

在 Γ_σ 上：

$$\begin{aligned}&N_n=\overline{N}_n,\ \ N_{nt}=\overline{N}_{nt},\ \ M_n^c=\overline{M}_n^c\\&M_{nt}^c=\overline{M}_{nt}^c,\ \ M_n^s=\overline{M}_n^s\\&N_n\frac{\partial w_0}{\partial n}+N_{nt}\frac{\partial w_0}{\partial t}+Q_n^c+\frac{\partial M_n^s}{\partial n}+2\frac{\partial M_{nt}^s}{\partial t}=\overline{Q}_n\end{aligned} \tag{18}$$

三 用广义位移表示的平衡方程

将(5)式代入(16)式，则得到用广义位移 u_0, v_0, φ_x, φ_y 和 W_o 表示的平衡方程．它可以写成

$$\{V_1\}+\{V_2\}+\{V_3\}+\{V_4\}+\{V_6\}=0 \tag{19}$$

其中

$$\{V_1\}=\begin{bmatrix} L_{11} & L_{12} & L_{13} & L_{14} & L_{15} \\ L_{12} & L_{22} & L_{23} & L_{24} & L_{25} \\ L_{13} & L_{23} & L_{33} & L_{34} & L_{35} \\ L_{14} & L_{24} & L_{34} & L_{44} & L_{45} \\ L_{15} & L_{25} & L_{35} & L_{45} & L_{55} \end{bmatrix}\begin{Bmatrix} u_0 \\ v_0 \\ \varphi_x \\ \varphi_y \\ w_0 \end{Bmatrix} \tag{20}$$

$$\{V_2\}=[G_1,\ G_2,\ G_3,\ G_4,\ G_5]^T w_0 \tag{21}$$

$$\{V_3\}=[I_1,\ I_2,\ I_3,\ I_4,\ I_5]^T w_0 \tag{22}$$

$$\{V_4\}=[0,\ 0,\ 0,\ 0,\ -NW]^T \tag{23}$$

$$\{V_5\}=[0,\ 0,\ 0,\ 0,\ -q_z]^T \tag{24}$$

$$\left.\begin{aligned}
&L_{11}=\overline{L}_1(A_{11},\ A_{16},\ A_{66}) \qquad L_{12}=\overline{L}_2(A_{16},\ A_{12},\ A_{66},\ A_{26})\\
&L_{13}=\overline{L}_1(B_{11},\ B_{16},\ B_{66}) \qquad L_{14}=\overline{L}_2(B_{16},\ B_{12},\ B_{66},\ B_{26})\\
&L_{15}=\overline{L}_3(A_{11},\ A_{12},\ A_{16})+\overline{L}_4(A_{16},\ A_{26},\ A_{66})\\
&\qquad -\frac{\partial}{\partial x}\overline{L}_1(P_{11},\ P_{16},\ P_{66})-\frac{\partial}{\partial y}\overline{L}_2(P_{16},\ P_{12},\ P_{66},\ P_{26})\\
&L_{22}=\overline{L}_1(A_{66},\ A_{26},\ A_{22}) \qquad L_{23}=L_{14}\\
&L_{24}=\overline{L}_1(B_{66},\ B_{26},\ B_{22})\\
&L_{25}=\overline{L}_3(A_{16},\ A_{26},\ A_{66})+\overline{L}_4(A_{12},\ A_{22},\ A_{26})\\
&\qquad -\frac{\partial}{\partial x}\overline{L}_2(P_{16},\ P_{12},\ P_{66},\ P_{26})-\frac{\partial}{\partial y}\overline{L}_1(P_{66},\ P_{26},\ P_{22})\\
&L_{33}=\overline{L}_1(D_{11},\ D_{16},\ D_{66})-C_{55} \qquad \overline{L_{34}}=L_2(D_{16},\ D_{12},\ D_{66},\ D_{26})-C_{45}\\
&L_{35}=-\left(C_{55}\frac{\partial}{\partial x}+C_{45}\frac{\partial}{\partial y}\right)+\overline{L}_3(B_{11},\ B_{12},\ B_{16})+\overline{L}_4(B_{16},\ B_{26},\ B_{66})\\
&\qquad -\frac{\partial}{\partial x}\overline{L}_1(E_{11},\ E_{16},\ E_{66})-\frac{\partial}{\partial y}\overline{L}_2(E_{16},\ E_{12},\ E_{66},\ E_{26})\\
&L_{44}=\overline{L}_1(D_{66},\ D_{26},\ D_{22})-C_{44}\\
&L_{45}=-\left(C_{45}\frac{\partial}{\partial x}+C_{44}\frac{\partial}{\partial y}\right)+\overline{L}_3(B_{16},\ B_{26},\ B_{66})+\overline{L}_4(B_{12},\ B_{22},\ B_{26})\\
&\qquad -\frac{\partial}{\partial x}\overline{L}_2(E_{16},\ E_{12},\ E_{66},\ E_{26})-\frac{\partial}{\partial y}\overline{L}_1(E_{66},\ E_{26},\ E_{22})\\
&L_{55}=-\overline{L}_1(C_{55},\ C_{45},\ C_{44})-P_{11}\overline{L}_5(R_x)-P_{12}\overline{L}(R_y)-2P_{16}\overline{L}_5(R_{xy})\\
&\qquad -P_{12}\overline{L}_6(R_x)-P_{22}\overline{L}_6(R_y)-2P_{66}\overline{L}_6(R_{xy})-2P_{16}\overline{L}_9(R_x)\\
&\qquad -2P_{26}\overline{L}_7(R_y)-4P_{66}\overline{L}_7(R_{xy})-\frac{1}{R_x}\overline{L}_1(P_{11},\ P_{16},\ P_{12})\\
&\qquad -\frac{1}{R_y}\overline{L}_1(P_{12},\ P_{26},\ P_{22})-\frac{2}{R_{xy}}\overline{L}_1(P_{16},\ P_{66},\ P_{26})+F_{11}\frac{\partial^4}{\partial x^4}\\
&\qquad +4F_{16}\frac{\partial^4}{\partial x^3\partial y}+(2F_{12}+4F_{66})\frac{\partial^4}{\partial x^2\partial y^2}+4F_{26}\frac{\partial^4}{\partial x\partial y^3}+F_{22}\frac{\partial^4}{\partial y^4}
\end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}
&G_1=L_{R1}(A_{11},\ A_{12},\ A_{16},\ A_{16},\ A_{26},\ A_{66})\\
&G_2=L_{R1}(A_{16},\ A_{26},\ A_{66},\ A_{12},\ A_{22},\ A_{26})\\
&G_3=L_{R1}(B_{11},\ B_{12},\ B_{16},\ B_{16},\ B_{26},\ B_{66})\\
&G_4=L_{R1}(B_{16},\ B_{26},\ B_{66},\ B_{12},\ B_{22},\ B_{26})\\
&G_5=\frac{1}{R_x}L_{R2}(A_{11},\ A_{12},\ A_{16})+\frac{1}{R_y}L_{R2}(A_{12},\ A_{22},\ A_{26})\\
&\qquad+\frac{2}{R_{xy}}L_{R2}(A_{16},\ A_{26},\ A_{66})
\end{aligned}\right\}\tag{26}$$

$$\left.\begin{aligned}
&I_1=\frac{\partial w_0}{\partial x}\overline{L}_1(A_{11},\ A_{16},\ A_{66})+\frac{\partial w_0}{\partial y}\overline{L}_2(A_{16},\ A_{12},\ A_{66},\ A_{26})\\
&I_2=\frac{\partial w_0}{\partial x}\overline{L}_2(A_{16},\ A_{12},\ A_{66},\ A_{26})+\frac{\partial w_0}{\partial y}\overline{L}_1(A_{66},\ A_{26},\ A_{22})\\
&I_3=\frac{\partial w_0}{\partial x}\overline{L}_1(B_{11},\ B_{16},\ B_{66})+\frac{\partial w_0}{\partial y}\overline{L}_2(B_{16},\ B_{12},\ B_{66},\ B_{26})\\
&I_4=\frac{\partial w_0}{\partial x}\overline{L}_2(B_{16},\ B_{12},\ B_{66},\ B_{26})+\frac{\partial w_0}{\partial y}\overline{L}_1(B_{66},\ B_{26},\ B_{22})\\
&I_5=-\frac{\partial^2 w_0}{\partial x^2}\overline{L}_1(P_{11},\ P_{16},\ P_{66})-\frac{\partial^2 w_0}{\partial y^2}\overline{L}_1(P_{66},\ P_{26},\ P_{22})\\
&\qquad-2\frac{\partial^2 w_0}{\partial x\partial y}\overline{L}_2(P_{16},\ P_{12},\ P_{66},\ P_{26})-\frac{\partial w_0}{\partial x}\overline{L}_8(P_{11},\ 3P_{16},\ P_{12}\\
&\qquad+2P_{66},\ P_{26})-\frac{\partial w_0}{\partial y}\overline{L}_8(P_{16},\ P_{12}+2P_{66},\ 3P_{16},\ P_{22})\\
&\qquad+\frac{1}{2}L_{R2}(A_{11},\ A_{12},\ A_{16})\frac{\partial w_0}{\partial x}\cdot\frac{\partial}{\partial x}\\
&\qquad+\frac{1}{2}L_{R2}(A_{12},\ A_{22},\ A_{26})\frac{\partial w_0}{\partial y}\cdot\frac{\partial}{\partial y}+L_{R2}(A_{16},\ A_{26},\ A_{66})\frac{\partial w_0}{\partial x}\cdot\frac{\partial}{\partial y}
\end{aligned}\right\}\tag{27}$$

$$\left.\begin{aligned}
&\overline{L}_1(\lambda_1,\ \lambda_2,\ \lambda_3)=\lambda_1\frac{\partial^2}{\partial x^2}+2\lambda_2\frac{\partial^2}{\partial x\partial y}+\lambda_3\frac{\partial^2}{\partial y^2}\\
&\overline{L}_2(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)=\lambda_1\frac{\partial^2}{\partial x^2}+(\lambda_2+\lambda_3)\frac{\partial^2}{\partial x\partial y}+\lambda_4\frac{\partial^2}{\partial y^2}\\
&\overline{L}_3(\lambda_1,\ \lambda_2,\ \lambda_3)=R_2(\lambda_1,\ \lambda_2,\ \lambda_3)\frac{\partial}{\partial x}\\
&\overline{L}_4(\lambda_1,\ \lambda_2,\ \lambda_3)=R_2(\lambda_1,\ \lambda_2,\ \lambda_3)\frac{\partial}{\partial y}\\
&\overline{L}_5(\lambda)=-\frac{\partial}{\partial x}\left(\frac{1}{\lambda^2}\frac{\partial\lambda}{\partial x}\right)-\frac{2}{\lambda^2}\frac{\partial\lambda}{\partial x}\frac{\partial}{\partial x}+\frac{1}{\lambda}\frac{\partial^2}{\partial x^2}\\
&\overline{L}_6(\lambda)=-\frac{\partial}{\partial y}\left(\frac{1}{\lambda^2}\cdot\frac{\partial\lambda}{\partial y}\right)-\frac{2}{\lambda^2}\frac{\partial\lambda}{\partial y}\frac{\partial}{\partial y}+\frac{1}{\lambda}\frac{\partial^2}{\partial y^2}\\
&\overline{L}_7(\lambda)=-\frac{\partial}{\partial y}\left(\frac{1}{\lambda^2}\frac{\partial\lambda}{\partial x}\right)-\frac{1}{\lambda^2}\left(\frac{\partial\lambda}{\partial x}\cdot\frac{\partial}{\partial y}+\frac{\partial\lambda}{\partial y}\cdot\frac{\partial}{\partial x}\right)+\frac{1}{\lambda}\frac{\partial^2}{\partial x\partial y}\\
&\overline{L}_8(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4)=\lambda_1\frac{\partial^3}{\partial x^3}+\lambda_2\frac{\partial^3}{\partial x^2\partial y}+\lambda_3\frac{\partial^3}{\partial x\partial y^2}+\lambda_4\frac{\partial^3}{\partial y^3}\\
&L_{R1}(\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6)=-\frac{\lambda_1}{R_x^2}\frac{\partial R_x}{\partial x}-\frac{\lambda_2}{R_y^2}\frac{\partial R_y}{\partial x}-\frac{2\lambda_3}{R_{xy}^2}\frac{\partial R_{xy}}{\partial x}\\
&\qquad-\frac{\lambda_4}{R_x^2}\cdot\frac{\partial R_x}{\partial y}-\frac{\lambda_5}{R_y^2}\cdot\frac{\partial R_y}{\partial y}-\frac{2\lambda_6}{R_{xy}^2}\frac{\partial R_{xy}}{\partial y}\\
&L_{R2}(\lambda_1,\ \lambda_2,\ \lambda_3)=\frac{\lambda_1}{R_x}+\frac{\lambda_2}{R_y}+\frac{2\lambda_3}{R_{xy}}
\end{aligned}\right\}\tag{28}$$

$$NW=\begin{Bmatrix}\dfrac{\partial^2 w_0}{\partial x^2}\\ \dfrac{\partial^2 w_0}{\partial y^2}\\ 2\dfrac{\partial^2 w_0}{\partial x\partial y}\end{Bmatrix}^T\cdot\begin{Bmatrix}A_{11}&A_{12}&A_{16}\\A_{12}&A_{22}&A_{26}\\A_{16}&A_{26}&A_{66}\\B_{11}&B_{12}&B_{16}\\B_{12}&B_{22}&B_{26}\\B_{16}&B_{26}&B_{66}\\P_{11}&P_{12}&P_{16}\\P_{12}&P_{22}&P_{26}\\P_{16}&P_{26}&P_{66}\end{Bmatrix}^T\cdot\begin{Bmatrix}\dfrac{\partial u_0}{\partial x}+\dfrac{w_0}{R_x}+\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2\\ \dfrac{\partial v_0}{\partial y}+\dfrac{w_0}{R_y}+\dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial y}\right)^2\\ \dfrac{\partial u_0}{\partial y}+\dfrac{\partial v_0}{\partial x}+\dfrac{2w_0}{R_{xy}}+\dfrac{\partial w_0}{\partial x}\cdot\dfrac{\partial w_0}{\partial y}\\ \dfrac{\partial \varphi_x}{\partial x}\\ \dfrac{\partial \varphi_y}{\partial y}\\ \dfrac{\partial \varphi_x}{\partial y}+\dfrac{\partial \varphi_y}{\partial x}\\ -\dfrac{\partial^2 w_0}{\partial x^2}\\ -\dfrac{\partial^2 w_0}{\partial y^2}\\ -2\dfrac{\partial^2 w_0}{\partial x\partial y}\end{Bmatrix} \tag{29}$$

对于小挠度弯曲问题

$$\{V_3\}=0\quad\{V_4\}=0$$

对于小挠度(线性)稳定问题

$$\left.\begin{aligned}&\{V_3\}=0\quad\{V_5\}=0\\&NW=N_{x0}\frac{\partial^2 w_0}{\partial x^2}+N_{y0}\frac{\partial^2 w_0}{\partial y^2}+2N_{xy0}\frac{\partial^2 w_0}{\partial x\partial y}\end{aligned}\right\} \tag{31}$$

(31)式中的 N_{x0}，N_{y0}，N_{xy0} 是失稳时扁壳的中面力．这时的五个广义位移 u_0，v_0，φ_x，φ_y，w_0 应该理解为失稳过程中的位移改变．

将(5)式代入(18)式，就可得到用广义位移表示的力的边界条件，这里从略．

四　正交各向异性复合材料夹层圆柱壳的线性稳定问题

设有一复合材料夹层矩形圆柱块壳，直边长为 a，曲边长为 b，半径为 R．壳体所在的曲面域为 Ω：$0\leqslant x\leqslant a$，$0\leqslant y\leqslant b$．那么有

$$\frac{1}{R_x}=0\qquad\frac{1}{R_y}=\frac{1}{R}\qquad\frac{1}{R_{xy}}=0 \tag{32}$$

壳体的四边都是简支的，其边界条件是

$$\text{在 } x=0,\ a \text{ 处：}\quad\left.\begin{aligned}&N_x=0\quad v_0=0\quad \varphi_y=0\\&w_0=0\quad M_x^l=0\quad M_x^s=0\end{aligned}\right\} \tag{33.1}$$

$$\text{在 } y=0,\ b \text{ 处：}\quad\left.\begin{aligned}&N_y=0\quad u_0=0\quad \varphi_x=0\\&w_0=0\quad M_y^l=0\quad M_y^s=0\end{aligned}\right\} \tag{33.2}$$

如果表板和夹心的各层材料都是正交各向异性的，那么

$$(A,\ B,\ P,\ D,\ E,\ F)_{16,26}=0 \tag{34.1}$$

$$C_{45}=0 \tag{34.2}$$

把(32)和(34)式代入(19)式，并考虑到(31)式，则得到正交各向异性复合材料夹层圆柱壳线性稳定问题的基本方程

$$\left.\begin{aligned}
&\left(A_{11}\frac{\partial^2}{\partial x^2}+A_{66}\frac{\partial^2}{\partial y^2}\right)u_0+(A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\partial y}+\left(B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}\right)\varphi_x\\
&\quad+(B_{12}+B_{66})\frac{\partial^2\varphi_y}{\partial x\partial y}\\
&\quad+\left\{\frac{A_{12}}{R}\frac{\partial}{\partial x}-P_{11}\frac{\partial^3}{\partial x^3}-(P_{12}+2P_{66})\frac{\partial^3}{\partial x\partial y^2}\right\}w_0=0\\
&(A_{12}+A_{66})\frac{\partial^2 u_0}{\partial x\partial y}+\left(A_{66}\frac{\partial^2}{\partial x^2}+A_{22}\frac{\partial^2}{\partial y}\right)v_0+(B_{12}+B_{66})\frac{\partial^2\varphi_x}{\partial x\partial y}\\
&\quad+\left(B_{66}\frac{\partial^2}{\partial x^2}+B_{22}\frac{\partial^2}{\partial y^2}\right)\varphi_y\\
&\quad+\left\{\frac{A_{22}}{R}\frac{\partial}{\partial y}-(P_{12}+2P_{66})\frac{\partial^3}{\partial x^2\partial y}-P_{22}\frac{\partial^3}{\partial y^3}\right\}w_0=0\\
&\left(B_{11}\frac{\partial^2}{\partial x^2}+B_{66}\frac{\partial^2}{\partial y^2}\right)u_0+(B_{12}+B_{66})\frac{\partial^2 v_0}{\partial x\partial y}+\left(D_{11}\frac{\partial^2}{\partial x^2}+D_{66}\frac{\partial^2}{\partial y^2}-C_{55}\right)\varphi_x\\
&\quad+(D_{12}+D_{66})\frac{\partial^2\varphi_y}{\partial x\partial y}+\left\{\frac{B_{12}}{R}\frac{\partial}{\partial x}-C_{55}\frac{\partial}{\partial x}-E_{11}\frac{\partial^3}{\partial x^3}\right.\\
&\quad\left.-(E_{12}+2E_{66})\frac{\partial^3}{\partial x\partial y^2}\right\}w_0=0\\
&(B_{12}+B_{66})\frac{\partial^2 u_0}{\partial x\partial y}+\left(B_{66}\frac{\partial^2}{\partial x^2}+B_{22}\frac{\partial^2}{\partial y^2}\right)v_0+(D_{12}+D_{66})\frac{\partial^2\varphi_x}{\partial x\partial y}\\
&\quad+\left(D_{66}\frac{\partial^2}{\partial x^2}+D_{22}\frac{\partial^2}{\partial y^2}-C_{44}\right)\varphi_y+\left\{\frac{B_{22}}{R}\frac{\partial}{\partial y}-C_{44}\frac{\partial}{\partial y}\right.\\
&\quad\left.-(E_{12}+2E_{66})\frac{\partial^3}{\partial x^2\partial y}-E_{22}\frac{\partial^3}{\partial y^3}\right\}w_0=0\\
&\left\{\frac{A_{12}}{R}\frac{\partial}{\partial x}-P_{11}\frac{\partial^3}{\partial x^3}-(P_{12}+2P_{66})\frac{\partial^3}{\partial x\partial y^2}\right\}u_0+\left\{\frac{A_{22}}{R}\frac{\partial}{\partial y}\right.\\
&\quad\left.-(P_{12}+2P_{66})\frac{\partial^3}{\partial x^2\partial y}-P_{22}\frac{\partial^3}{\partial y^3}\right\}v_0+\left\{\frac{B_{12}}{R}\frac{\partial}{\partial x}-C_{55}\frac{\partial}{\partial x}\right.\\
&\quad\left.-E_{11}\frac{\partial^3}{\partial x^2}-(E_{12}+2E_{66})\frac{\partial^3}{\partial x\partial y^2}\right\}\varphi_x\\
&\quad+\left\{\frac{B_{22}}{R}\frac{\partial}{\partial y}-C_{44}\frac{\partial}{\partial y}-(E_{12}+2E_{66})\frac{\partial^3}{\partial x^2\partial y}-E_{22}\frac{\partial^3}{\partial y^3}\right\}\varphi_y\\
&\quad+\left\{\frac{A_{22}}{R^2}-C_{55}\frac{\partial^2}{\partial x^2}-C_{44}\frac{\partial^2}{\partial y^2}-\frac{2P_{12}}{R}\frac{\partial^2}{\partial x^2}-\frac{2P_{22}}{R}\frac{\partial^2}{\partial y^2}\right.\\
&\quad+F_{11}\frac{\partial^4}{\partial x^4}+(2F_{12}+4F_{66})\frac{\partial^4}{\partial x^2\partial y^2}\partial^2 w_0\\
&\quad\left.+F_{22}\frac{\partial^4}{\partial y^4}\right\}w_0-N_{xo}\frac{\partial^2 w_0}{\partial x^2}-N_{yo}\frac{\partial^2 w_0}{\partial y^2}-2N_{xyo}\frac{\partial^2 w_0}{\partial x\partial y}=0
\end{aligned}\right\} \tag{35}$$

对于矩形圆柱块壳，设

$$\left.\begin{aligned}&u_0=u_{mn}\cos\alpha x\sin\beta y \qquad \varphi_x=\varphi_{xmn}\cos\alpha x\sin\beta y\\&v_0=v_{mn}\sin\alpha x\cos\beta y \qquad \varphi_y=\varphi_{ymn}\sin\alpha x\cos\beta y\\&w_0=w_{mn}\sin\alpha x\sin\beta y\end{aligned}\right\}\tag{36}$$

其中 $\alpha=\dfrac{m\pi}{a}$，$\beta=\dfrac{n\pi}{b}$，m，n 为正整数.

(36)式满足边界条件(33)式. 对于轴压或侧压的情况，$N_{xyo}=0$，这时把(36)式代入(35)式，则得到

$$\left.\begin{aligned}
&(A_{11}\alpha^2+A_{66}\beta^2)u_{mn}+(A_{12}+A_{66})\alpha\beta v_{mn}+(B_{11}\alpha^2+B_{66}\beta^2)\varphi_{xmn}\\
&\quad+(B_{12}+B_{66})\alpha\beta\varphi_{ymn}-\left[\frac{A_{12}}{R}\alpha+P_{11}\alpha^3+(P_{12}+2P_{66})\alpha\beta^2\right]w_{mn}=0\\
&(A_{12}+A_{66})\alpha\beta u_{mn}+(A_{66}\alpha^2+A_{22}\beta^2)v_{mn}+(B_{12}+B_{66})\alpha\beta\varphi_{xmn}\\
&\quad+(B_{66}\alpha^2+B_{22}\beta^2)\varphi_{ymn}-\left[\frac{A_{22}}{R}\beta+(P_{12}+2P_{66})\alpha^2\beta+P_{22}\beta^3\right]w_{mn}=0\\
&(B_{11}\alpha^2+B_{66}\beta^2)u_{mn}+(B_{12}+B_{66})\alpha\beta v_{mn}+(D_{11}\alpha^2+D_{66}\beta^2+C_{55})\varphi_{xmn}\\
&\quad+(D_{12}+D_{66})\alpha\beta\varphi_{ymn}-\left[\left(\frac{B_{12}}{R}-C_{55}\right)\alpha+E_{11}\alpha^3+(E_{12}+2E_{66})\alpha\beta^2\right]\\
&\quad w_{mn}=0\\
&(B_{12}+B_{66})\alpha\beta u_{mn}+(B_{66}\alpha^2+B_{22}\beta^2)v_{mn}+(D_{12}+D_{66})\alpha\beta\varphi_{xmn}\\
&\quad+(D_{66}\alpha^2+D_{22}\beta^2+C_{44})\varphi_{ymn}-\left[\left(\frac{B_{22}}{R}-C_{44}\right)\beta+(E_{12}+2E_{66})\alpha^2\beta\right.\\
&\quad\left.+E_{22}\beta^3\right]w_{mn}=0\\
&\left[\frac{A_{12}}{R}\alpha+P_{11}\alpha^3+(P_{12}+2P_{66})\alpha\beta^2\right]u_{mn}+\left[\frac{A_{22}}{R}\beta+(P_{12}+2P_{66})\alpha^2\beta\right.\\
&\quad\left.+P_{22}\beta^3\right]v_{mn}\\
&\quad+\left[\left(\frac{B_{12}}{R}-C_{55}\right)\alpha+E_{11}\alpha^3+(E_{12}+2E_{66})\alpha\beta^2\right]\varphi_{xmn}+\left[\left(\frac{B_{22}}{R}-C_{44}\right)\beta\right.\\
&\quad\left.+(E_{12}+2E_{66})\alpha^2\beta+E_{22}\beta^3\right]\varphi_{ymn}-\left[\frac{A_{22}}{R^2}+\frac{2P_{12}}{R}\alpha^2+\frac{2P_{22}}{R}\beta^2\right.\\
&\quad\left.+C_{55}\alpha^2+C_{44}\beta^2+F_{11}\alpha^4+2(F_{12}+2F_{66})\alpha^2\beta^2+F_{22}\beta^4\right]w_{mn}\\
&\quad+(N_{x0}\alpha^2+N_{y0}\beta^2)w_{mn}=0
\end{aligned}\right\}\tag{37}$$

令(37)式的系数行列式为零，则可求出失稳时的临界薄膜力 N_{x0} 或 N_{y0}. 为了完成这一步的计算工作，编了一个FORTRAN程序.

五　算　例

I. 各向同性软夹心圆柱壳　如果上、下表板是相同的各向同性材料，夹心为只能承受剪应力的各向同性材料，并且 $t_1=t_2=t$，$n_1=n_2=n_c=1$，$h_1+h_2=h$，那么本文的基本方程组退化为文献〔3〕的基本方程组. 这里不做理论上的证明，仅用三个算例加以说明. 算例中 E_f 为表板的弹性模量，$\gamma_f=0.3$ 为表板的波桑比，G_c 为夹心的剪切模量.

算例	〈1〉	〈2〉	〈3〉
E_f(kg/cm²)	0.7×10^6	0.7×10^6	0.1×10^8
G_c(kg/cm²)	0.1×10^4	0.1×10^3	0.3×10^6
a(cm)	95.089	190.031	789
b(cm)	47.545	38.062	789
R(cm)	75.686	132.84	2084
t(cm)	0.1	0.2	1.0
h(cm)	1.191	1.9082	17.257
N_{x0}(kg/cm，本文)	0.14353×10^4	0.25918×10^4	0.178288×10^6
N_{x0}(kg/cm)[3]	0.14350×10^4	0.25913×10^4	0.178281×10^6

Ⅱ．复合材料夹层圆柱壳 铺层情况如图5．表板为碳纤维-环氧复合材料，性能数据见算例Ⅲ．夹心是各向同性软夹心，剪切模量为G_c．

算例	a(cm)	b(cm)	R(cm)	G_c(kg/cm²)	N_{x0}(kg/cm)
〈1〉	50	50	200	10^3	0.53096×10^3
〈2〉	50	50	200	10^4	0.62096×10^3
〈3〉	50	50	200	10^5	0.63198×10^3

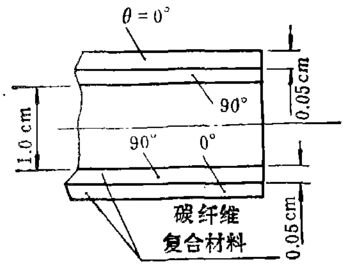

图 5

Ⅲ．混杂铺层复合材料圆柱壳 对于混杂铺层复合材料圆柱壳在轴压作用下的稳定问题，用三种方法计算了临界载荷N_{x0}．用不考虑厚度方向的剪切变形的理论算得的临界载荷记为$(N_{x0})_{kir}$；用文献〔1〕的复合材料多层壳理论计算的临界载荷记为$(N_{x0})_{mul}$；用本文的理论算得的临界载荷记为$(N_{x0})_{san}$．

铺层材料的性能如下：

钢：$E=2.1\times10^6$kg/cm²，$\gamma=0.3$

碳纤维-环氧复合材料(T300/5208)[1)]

$E_{11}=18.5\times10^5$kg/cm²　　$\gamma_{12}=0.28$

$E_{22}=1.05\times10^5$kg/cm²　　$G_{23}=4.08\times10^4$kg/cm²

$G_{12}=7.31\times10^4$kg/cm²　　($V_f=0.70$)

Kevlar 纤维-环氧复合材料[1)]

$E_{11}=7.75\times10^5$kg/cm²　　$\gamma_{12}=0.34$

$E_{22}=5.61\times10^5$kg/cm²　　$G_{23}=2.04\times10^4$kg/cm²

$G_{12}=2.35\times10^4$kg/cm²　　($V_f=0.60$)

1）这两组数据中的E_{11}，E_{22}，G_{12}，γ_{12}取自文献〔4〕P_{12}，G_{23}。的数据是根据横观各向同性估算的。

铺层情况如下图6所示.

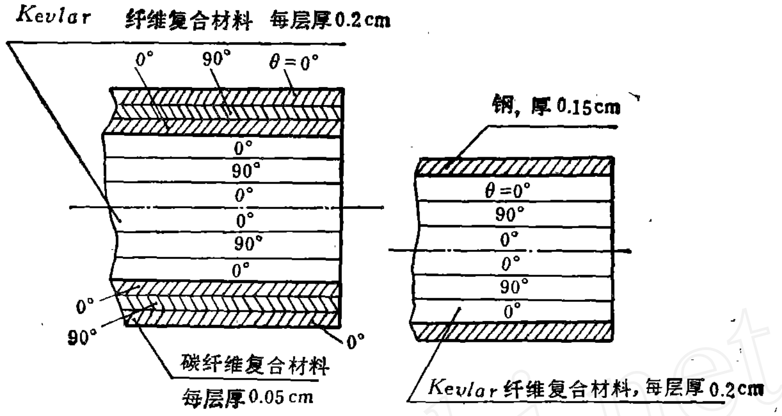

图 6

算例〈1〉的铺层　　算例$\left\{\begin{matrix}\langle 2\rangle\\ \langle 3\rangle\end{matrix}\right.$的铺层

三个算例的计算结果如下：

算　例	〈1〉	〈2〉	〈3〉
a(cm)	50	50	50
b(cm)	50	50	50
R(cm)	200	200	500
$(N_{x0})_{kir}$ (kg/cm)	0.20815×10^4	0.67928×10^4	0.57538×10^4
$(N_{x0})_{mul}$ (kg/cm)	0.20255×10^4	0.66116×10^4	0.55726×10^4
$(N_{x0})_{san}$ (kg/cm)	0.20236×10^4	0.63302×10^4	0.52911×10^4

六 讨 论

通过算例Ⅰ可以看到本文基本方程组的适用性和普遍性.

从算例Ⅱ可以看出，夹心的剪切刚度对临界载荷是有影响的. 夹心的剪切刚度愈大，临界载荷愈高. 当夹心剪切刚度大到一定程度之后，临界载荷就不再增加. 这表明，这时可采用Kirchhoff假设，不考虑厚度方向的剪切变形.

从算例Ⅲ可以看到，$(N_{x0})_{kir}>(N_{x0})_{mul}>(N_{x0})_{san}$. 这表明，认为整个截面有相同的剪切变形的理论(文献〔1〕)与不考虑剪切变形的理论相比，文献〔1〕较符合实际情况，较为精确；而本文认为变形后平截面成为折截面的理论比文献〔1〕的理论更加精确。从算例中还可以看出，对于混杂铺层复合材料扁壳来说，外层与内层的刚度相差愈大，$(N_{x0})_{mul}$与$(N_{x0})_{san}$的差别也愈大. 这表明，对于外层与内层的刚度相差较大的混杂铺

层复合材料扁壳，采用文献〔1〕的理论已不合适，而应该采用本文的理论．当然，如果外层和内层的刚度相差不大，或者外层的厚度很薄，则仍可采用文献〔1〕的理论．

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THE FINITE DEFLECTION EQUATIONS AND LINEAR STABILITY PROBLEM OF SHALLOW SANDWICH SHELLS OF COMPOSITE MATERIALS

Liu Guo-xi Wang Zhen-ming

(Institute of Mechanics, Academia Sinica)

Abstract

In this paper, by using Kirchhoff hypothesis for face sheets and assuming that the core can bear in-plane stresses and contribute to shear deformation in thickness direction, the finite deflection equations and their boundary conditions of a shallow sandwich shell made of composite materials are derived through the minimum potential principle. As an example, a linear buckling problem of sandwich rectangular cylindrical shell under axial or lateral load is solved.