On the initial unloading slope in indentation of elastic-plastic solids by an indenter with an axisymmetric smooth profile

Che-Min Cheng (Zheng Zhemin)[a)]
Institute of Mechanics, Chinese Academy of Sciences, Beijing 100 080, People's Republic of China

Yang-Tse Cheng[b)]
Physics and Physical Chemistry Department, General Motors Research and Development Center, Warren, Michigan 48090

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A simple relationship between the initial unloading slope, the contact area, and the elastic modulus is derived for indentation in elastic-plastic solids by an indenter with an arbitrary axisymmetric smooth profile. Although the same expression was known to hold for elastic solids, the new derivation shows that it is also true for elastic-plastic solids with or without work hardening and residual stress. These results should provide a sound basis for the use of the relationship for mechanical property determination using indentation techniques. © *1997 American Institute of Physics.* [S0003-6951(97)04544-0]

Indentation experiments have been performed for nearly one hundred years for measuring the hardness of materials.[1] Recent years have seen increased interest in indentation because of the significant improvement in indentation equipment and the need for measuring the mechanical property of materials on small scales. With the improvement in indentation instruments, it is now possible to monitor, with high precision and accuracy, both the load (F) and displacement (h) of an indenter during indentation experiments in the respective micro-Newtons and nanometer ranges[2–4] (Fig. 1). In addition to hardness, the elastic modulus may be deduced from the indentation load versus displacement curves for unloading.

The basis for obtaining the elastic modulus relies on the theory for elastic contacts. The relationships between load and displacement for several shapes of rigid indenters contacting with an elastic half-space are known, including that for spheres,[5] flat punches,[5] and conical punches.[6] More generally, Sneddon[7] has derived expressions for load and displacement for elastic contacts between a rigid, axisymmetric punch with an arbitrary smooth profile and an elastic half-space. Using Sneddon's results, Pharr *et al.*[8] have recently derived an expression relating the slope (dF/dh), projected contact area (A), Young's modulus (E), and Poisson's ratio (ν) at any point on the load versus displacement curves:

$$\frac{dF}{dh}=\frac{2E}{\sqrt{\pi}(1-\nu^2)}\sqrt{A}. \tag{1a}$$

If the slope, dF/dh can be evaluated at a point on the load versus displacement curve (loading and unloading coincide for purely elastic contacts) and the corresponding projected contact area A is known, Eq. (1a) can be used to evaluate the quantity $E/(1-\nu^2)$. In the case that the indenter itself has finite elastic constants, E_i and ν_i, the reduced modulus, E_r, conventionally defined as

$$\frac{1}{E_r}=\frac{1-\nu^2}{E}+\frac{1-\nu_i^2}{E_i}$$

may be calculated from

$$\frac{dF}{dh}=\frac{2E_r}{\sqrt{\pi}}\sqrt{A}. \tag{1b}$$

Equation (1) has also been applied to indentation experiments where plastic deformation clearly occurs during loading. Doerner and Nix[9] suggested that if the area in contact remains constant during initial unloading, the elastic behavior may be modeled as that of a blunt punch indenting an elastic solid and, consequently, Eq. (1) can be used to determine E_r. Oliver *et al.*[10] pointed out that Eq. (1) can be used to determine E_r using the initial unloading slope and the initial projected contact area even when the contact area between the indenter and the solid changes continuously as the indenter is withdrawn and the indenter does not behave like a flat punch.

While Sneddon's solution for elastic contacts has been used to derive Eq. (1), there has not been general discussions on the validity of applying Eq. (1) to analyzing the initial

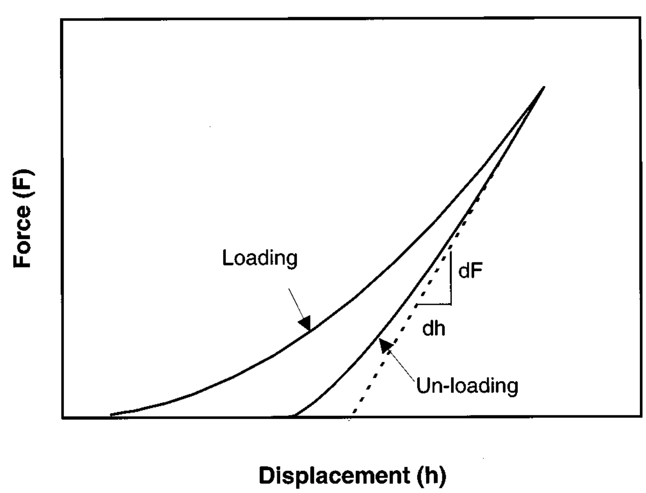

FIG. 1. Typical indentation load-displacement curve.

[a)]Electronic mail: zhengzm@LNM.imech.ac.cn
[b)]Electronic mail: Yang_Cheng@notes.gmr.com

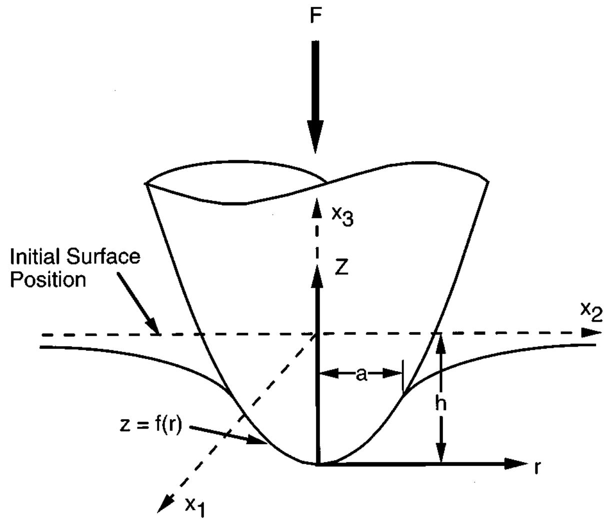

FIG. 2. Illustration of surface deformation by an axisymmetric indenter.

unloading slopes when plastic deformation occurs during the loading part of indentation experiments. Such discussions seem necessary since both experiments and simulations have demonstrated the dependence of initial unloading slope and contact area on the plasticity as well as elasticity of materials. For example, the unloading behavior is significantly different for aluminum and fused silica, which have about the same Young's modulus and very different yield strength values.[10] Furthermore, it has been recognized that work hardening behavior and residual stresses can also affect contact area and load versus displacement curves.[11,12] Therefore, a verification of Eq. (1) when plasticity and residual stresses are involved appears necessary.

We show in this letter that Eq. (1) holds for general elastic-plastic solids with or without strain hardening and residual stresses. Furthermore, the equation can be obtained without invoking Sneddon's results.[7] In doing so we also show possible limitations of this relationship.

We first consider the stress and deformation field at the maximum load. We denote the corresponding stress field by σ_{ij}^0, the strain field by ϵ_{ij}^0, and the displacement field by u_i^0, where $i,j=1,2,3$. The stress components satisfy the equilibrium equations $\sigma_{ij,j}^0=0$. Furthermore, based on the infinitesimal theory of continuum mechanics, the following boundary conditions are satisfied on the undeformed surface of the specimen ($x_3=0$) being indented:

$$u_3^0=h_m-f(r),\ \ \text{for}\ \ r=\sqrt{x_1^2+x_2^2}\leqslant a_m=\sqrt{A_m/\pi}, \tag{2a}$$

$$\sigma_{33}^0=0,\ \ \text{for}\ \ r>a_m, \tag{2b}$$

$$\sigma_{31}^0=\sigma_{32}^0=0,\ \ \text{for all}\ \ r>0, \tag{2c}$$

where a_m and A_m are the respective contact radius and the projected contact area at the maximum indenter displacement, h_m. The shape function of the rigid indenter $f(r)$ is assumed to be smooth (Fig. 2). The boundary condition in Eq. (2c) assumes that there is no friction between the indenter and the specimen. In addition, the stress field approaches zero as $\sqrt{x_1^2+x_2^2+x_3^2}\rightarrow\infty$.

Next, we let an infinitesimal unloading take place so that the depth of penetration measured with reference to the undeformed surface decreases from h_m to h. We denote the change in stress components by σ_{ij} and the change in displacement by u_i. Then, since at initial unloading all additional deformations are elastic and the total stress is the $\sigma_{ij}+\sigma_{ij}^0$, σ_{ij} must satisfy the equations of equilibrium and are related to changes of strain expressed in terms of $(u_{i,j}+u_{j,i})/2$ by Hooke's law with the same elastic constants everywhere. The following boundary conditions at $x_3=0$ must be satisfied:

$$u_3=h-h_m,\quad \text{for}\ \ r=\sqrt{x_1^2+x_2^2}\leqslant a=\sqrt{A/\pi}, \tag{3a}$$

$$\sigma_{33}=-\sigma_{33}^0,\ \ \text{for}\ \ a<r=\sqrt{x_1^2+x_2^2}\leqslant a_m=\sqrt{A_m/\pi}, \tag{3b}$$

$$\sigma_{33}=0,\ \ \text{for}\ \ r>a_m, \tag{3c}$$

$$\sigma_{31}=\sigma_{32}=0,\ \ \text{for all}\ \ r>0. \tag{3d}$$

This stress field again approaches zero as $\sqrt{x_1^2+x_2^2+x_3^2}\rightarrow\infty$. We note that according to (3a) the displacement u_3 of the specimen at the contact surface is constant so that (3a) by itself is identical to the boundary condition that would have to be satisfied by a rigid circular flat punch.

For an indenter with smooth profile, the stress component σ_{33}^0 at $r=a_m$ is continuous. Since σ_{33}^0 is zero just outside the contact area, we have, in the limit of $u_3=h-h_m\rightarrow dh$, $\sigma_{33}^0=0$ in (3b) as required by continuity. The other conditions then show that the initial unloading problem is now reduced to the indentation by a rigid, flat ended circular flat punch of constant radius. According to Ref. 13, to produce a uniform displacement dh under the rigid, flat ended circular flat punch, a decrease in contact pressure in the following form is required,

$$dp=-(1-r^2/a_m^2)^{-1/2}dp_0, \tag{4}$$

where dp_0 is the infinitesimal change in the average contact pressure in response to dh. Integrating Eq. (4) over the contact area, we obtain the decrease in load,

$$dF=-2\pi a_m^2\ dp_0. \tag{5}$$

The corresponding decrease in penetration depth, according to Ref. 13, is given by

$$dh=-\frac{\pi(1-\nu^2)a_m}{E}\ dp_0. \tag{6}$$

Equation (1a) is obtained from Eqs. (5) and (6), since $A_m=\pi a_m^2$.

Equation (1a) is true irrespective of the shape function of the indenter as long as it is axisymmetric and does not cause stress discontinuity in σ_{33}^0 at a_m. This is assured if the profile of the indenter is smooth. The usual substitution of $E/(1-\nu^2)$ by the reduced modulus E_r can obviously be made in the case of an elastic indenter, leading to Eq. (1b).

In the above derivation elastic unloading initiates from a general elastic/plastic state. The effect of the rules of plastic deformation imposed during loading with or without strain hardening and the effect of the residual stress field appear only implicitly in the value of A_m. Also, strain hardening does not change the elastic constants during unloading. Consequently, Eq. (1a) is valid for any strain hardening elastic-plastic materials and when residual stress is present prior to indentation experiments. Recent finite element calculations seem to support these conclusions. For example, Laursen and Simo[11] showed an excellent agreement between the reduced modulus calculated using Eq. (1a) and the Young's modulus and Poisson's ratio used in the finite element calculations for a rigid, conical indenter indenting into aluminum and silicon. Work hardening was assumed in their aluminum model and silicon was assumed to be elastic-perfectly plastic. In a recent finite element calculation for an aluminum alloy by Bolshakov *et al.*,[12] both work hardening and initial stress were included. Furthermore, the calculation showed that the contact area, A_m, can be strongly influenced by the initial stress. Nonetheless, a good agreement was noted between the elastic modulus calculated using Eq. (1a) and that used in the finite element model, provided that A_m used in Eq. (1a) is that given by the finite element calculations. The present analysis is consistent with these observations. It should be noted, however, that Eq. (1) is derived using the infinitesimal theory of mechanics, whereas finite element calculations may account for nonlinear effects, including large strain and moving contact boundaries.

If the profile of the indenter is smooth but not axisymmetric, Eq. (4) no longer applies, although by continuity the normal stress at the contact line is still zero. If the indenter is a pyramid, the requirement on the continuity of normal stress may not be met because of possible stress concentrations at sharp edges of the indenter. In these cases, the validity of Eq. (1a) requires further study.

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