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THE ANNULAR CRACK IN A NONHOMOGENEOUS MATRIX SURROUNDING A FIBER UNDER TORSIONAL LOADING—II. THE CRACK GOING THROUGH THE BIMATERIAL INTERFACE INTO THE FIBER

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Abstract—The axisymmetric problem of an elastic fiber perfectly bonded to a nonhomogeneous elastic matrix which contains an annular crack going through the interface into the fiber under axially symmetric shear stress is considered. The nature of the stress singularity is studied. It is shown that at the irregular point on the interface, whether the shear modulus is continuous or discontinuous the stresses are bounded. The problem is formulated in terms of a singular integral equation and can be solved by a regular method. The stress intensity factors and crack surface displacement are given.

1. INTRODUCTION

A CRACK in the neighborhood of a bimaterial interface may propagate towards the interface. Upon reaching the interface, the further propagation of the crack may intersect the interface into the adjacent material as a through crack or debond along the interface. Reference [1] presents a tentative fracture criterion which may be used in predicting the mode of the fracture propagation. References [1–5] have studied the plane crack problems that go through the interface of two bounded homogeneous layers. The antiplane shear crack problem in bonded nonhomogeneous plates was studied in ref. [6]. The problem of a fiber–matrix cylinder with an annular crack going through the interface has not been considered yet.

In this paper, the axially symmetric problem of a fiber bounded to a nonhomogeneous matrix which contains an annular crack going through the interface is studied. The medium is assumed to be under simple axially symmetric shear stress. The mixed boundary value problem is reduced to a singular integral equation with a Cauchy-type kernel. By using a Gauss–Chebyshev quadrature formula, the singular integral equation is reduced to a system of algebraic equations.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

We consider an infinitely long elastic fiber of radius a, perfectly bonded to a nonhomogeneous matrix, which contains an annular crack $b < r < c$, $z = 0$; $b < a < c < \infty$ going through the interface. The crack is subjected to axisymmetric shear stress. Due to the anti-symmetry with respect to the plane $z = 0$, only the semi-infinite domain $z \geqslant 0$ is considered.

Assuming the fiber and matrix having the elastic properties

$$\mu_1 = \text{const}, \quad 0 < r < a; \quad \mu_2 = \mu_0 r^m, \quad r > a$$

and μ_0 is a constrant $m > -1$, the axially symmetric torsional problem may be formulated as follows:

$$\frac{\partial^2 u_\theta^1}{\partial z^2} + \frac{\partial^2 u_\theta^1}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta^1}{\partial r} - \frac{u_\theta^1}{r^2} = 0, \quad 0 < r < a, \tag{1}$$

$$\frac{\partial u_\theta^2}{\partial z^2} + \frac{\partial^2 u_\theta^2}{\partial r^2} + \frac{(1+m)}{r}\frac{\partial u_\theta^2}{\partial r} - \frac{(1+m)}{r^2} u_\theta^2 = 0, \quad a < r < \infty, \tag{2}$$

where the superscripts or subscripts 1 and 2 refer to the fiber and matrix, respectively.

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Using Hankel and Fourier transforms, the displacements and then the stresses can be expressed as

$$u_\theta^1(r,z)=\int_0^\infty G(\xi)J_1(r\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\frac{2}{\pi}\int_0^\infty A(t)I_1(rt)\sin(tz)\,\mathrm{d}t \tag{3}$$

$$\sigma_{r\theta}^1(r,z)/\mu_1=-\int_0^\infty \xi G(\xi)J_2(r\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\frac{2}{\pi}\int_0^\infty tA(t)I_2(rt)\sin(tz)\,\mathrm{d}t \tag{4}$$

$$\sigma_{\theta z}^1(r,z)/\mu_1=-\int_0^\infty \xi G(\xi)J_1(r\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\frac{2}{\pi}\int_0^\infty tA(t)I_1(rt)\cos(tz)\,\mathrm{d}t \tag{5}$$

$$u_\theta^2(r,z)=r^{-m/2}\left[\int_0^\infty H(\xi)J_{1+m/2}(r\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\frac{2}{\pi}\int_0^\infty B(t)K_{1+m/2}(rt)\sin(tz)\,\mathrm{d}t\right]+\frac{C}{r^{1+m}} \tag{6}$$

$$\sigma_{r\theta}^2(r,z)/\mu_0=r^{m/2}\left[-\int_0^\infty \xi H(\xi)J_{2+m/2}(r\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi-\frac{2}{\pi}\int_0^\infty tB(t)K_{2+m/2}(rt)\sin(tz)\,\mathrm{d}t\right]-(2+m)\frac{C}{r^2} \tag{7}$$

$$\sigma_{\theta z}^2(r,z)/\mu_0=r^{m/2}\left[-\int_0^\infty \xi H(\xi)J_{1+m/2}(r\xi)\,\mathrm{e}^{-\xi z}\,\mathrm{d}\xi+\frac{2}{\pi}\int_0^\infty tB(t)K_{1+m/2}(rt)\cos(tz)\,\mathrm{d}t\right], \tag{8}$$

where G, H, A and B are functions yet to be determined, C is a constant, $J_\nu(\,)$ is the first kind of Bessel function, and $I_\nu(\,)$ and $K_\nu(\,)$ are the modified Bessel functions of the first and second kind.

The boundary conditions at the interface are given by

$$u_\theta^1(a,z)=u_\theta^2(a,z),\quad 0\leqslant z<\infty \tag{9}$$

$$\theta_{r\theta}^1(a,z)=\sigma_{r\theta}^2(a,z),\quad 0\leqslant z<\infty. \tag{10}$$

The conditions on the crack plane $z=0$ are

$$\sigma_{\theta z}^1(r,0)=-p_1(r),\quad a<r<c \tag{11}$$

$$\sigma_{\theta z}^2(r,0)=-p_2(r),\quad b<r<a \tag{12}$$

$$u_\theta^1(r,0)=0,\quad 0\leqslant r\leqslant b \tag{13}$$

$$u_\theta^2(r,0)=0,\quad c\leqslant r<\infty. \tag{14}$$

According to these boundary conditions the displacements and stresses can be determined.

3. THE ANALYTICAL SOLUTION

To reduce the problem to integral equations we first introduce two new unknown functions as follows

$$\varphi_1(r)=\frac{1}{r}\frac{\partial}{\partial r}[ru_\theta^1(r,0)],\quad b<r<a \tag{15}$$

$$\varphi_2(r)=\frac{1}{r^{1+m}}\frac{\partial}{\partial r}[r^{1+m}u_\theta^2(r,0)],\quad a<r<c \tag{16}$$

and hence from eqs (3), (6), (13)–(16) it can be shown that

$$\varphi_1(r)=\int_0^\infty \xi G(\xi)J_0(r\xi)\,\mathrm{d}\xi,\quad b<r<a \tag{17}$$

$$\varphi_2(r)=r^{-m/2}\int_0^\infty \xi H(\xi)J_{m/2}(r\xi)\,\mathrm{d}\xi,\quad a<r<c \tag{18}$$

$$G(\xi)=\int_b^a r\varphi_1(r)J_0(r\xi)\,\mathrm{d}r \tag{19}$$

$$H(\xi) = \int_a^c r^{1+m/2}\varphi_2(r)J_{m/2}(r\xi)\,\mathrm{d}r. \tag{20}$$

It should be observed that at $z = 0$, the following condition must be satisfied

$$u_\theta^2(r, 0) = 0, \quad c \leqslant r < \infty; \quad u_\theta^1(a, 0) = u_\theta^2(a, 0).$$

We find

$$C = -\int_a^c r^{1+m}\varphi_2(r)\,\mathrm{d}r \tag{21}$$

$$a^m \int_b^a r\varphi_i(r)\,\mathrm{d}rt = C. \tag{22}$$

Equations (21) and (22) show that

$$a^m \int_b^a r\varphi_1(r)\,\mathrm{d}r + \int_a^c r^{1+m}\varphi_2(r)\,\mathrm{d}r = 0. \tag{23}$$

By substituting the boundary conditions (9) and (10) into eqs (3), (4), (6) and (7), then inverting the Fourier sine transforms, we obtain the following equations for A and B in terms of φ_1 and φ_2.

$$A(t)I_1(at) - B(t)a^{-m/2}K_{1+m/2}(at) = R_1(t) \tag{24}$$

$$\mu_1 A(t)I_2(at) + \mu_0 B(t)a^{m/2}K_{2+m/2}(at) = R_2(t), \tag{25}$$

where the functions $R_1(t)$ and $R_2(t)$ are given by

$$R_1(t) = K_1(at)\int_b^a rI_0(rt)\varphi_1(r)\,\mathrm{d}r + a^{m/2}I_{1+m/2}(at)\int_a^c r^{1+m/2}K_{m/2}(rt)\varphi_2(r)\,\mathrm{d}r \tag{26}$$

$$R_2(t) = -\mu_1 K_1(at)\int_b^a rI_0(rt)\varphi_1(r)\,\mathrm{d}r + \mu_2 a^{-m/2}I_{2+m/2}(at)\int_a^c r^{1+m/2}K_{m/2}(rt)\varphi_2(r)\,\mathrm{d}r,$$
$$+\frac{[2\mu_1 - (2+m)\mu_0 a^m]}{(at)^2}\int_b^a r\varphi_1(r)\,\mathrm{d}r. \tag{27}$$

Solving eqs (24) and (25), A and B can be expressed as

$$A(t) = \frac{A_1(t)}{\Delta(t)}\int_b^a rI_0(rt)\varphi_1(r)\,\mathrm{d}r + \frac{A_2(t)}{\Delta(t)}\int_a^c r^{1+m/2}K_{m/2}(rt)\varphi_2(r)\,\mathrm{d}r + \frac{A_3(t)}{\Delta(t)}\int_b^a r\varphi_1(r)\,\mathrm{d}r \tag{28}$$

$$B(t) = \frac{B(t)}{\Delta(t)}\int_b^a rI_0(rt)\varphi_1(r)\,\mathrm{d}r + \frac{B_2(t)}{\Delta(t)}\int_a^c r^{1+m/2}K_{m/2}(rt)\varphi_2(r)\,\mathrm{d}r + \frac{B_3(t)}{\Delta(t)}\int_b^a r\varphi_1(r)\,\mathrm{d}r, \tag{29}$$

where

$$\Delta(t) = \mu_0 a^{m/2}I_1(at)K_{2+m/2}(at) + \mu_1 a^{-m/2}I_2(at)K_{1+m/2}(at) \tag{30}$$

$$A_1(t) = \mu_0 a^{m/2}K_1(at)K_{2+m/2}(at) - \mu_1 a^{-m/2}K_2(at)K_{1+m/2}(at)$$
$$A_2(t) = \mu_0 I_{1+m/2}(at)K_{2+m/2}(at) + \mu_0 I_{2+m/2}(at)K_{1+m/2}(at)$$
$$A_3(t) = [2\mu_1 - (2+m)\mu_0 a^m]a^{-m/2}K_{1+m/2}(at)/(at)^2 \tag{31a–c}$$

$$B_1(t) = -\mu_1 I_1(at)K_2(at) - \mu_1 I_2(at)K_1(at)$$
$$B_2(t) = \mu_0 a^{m/2}I_1(at)I_{2+m/2}(at) - \mu_1 a^{-m/2}I_2(at)I_{1+m/2}(at)$$
$$B_3(t) = [2\mu_1 - (2+m)\mu_0 a^m]I_1(at)/(at)^2. \tag{32a–c}$$

By substituting eq. (5) into eq. (11) and eq. (8) into eq. (12) we have

$$-\int_0^\infty \xi G(\xi)J_1(r\xi)\,\mathrm{d}\xi + \frac{2}{\pi}\int_0^\infty tA(t)I_1(rt)\,\mathrm{d}t = -\frac{p_1(r)}{\mu_1}, \quad b < r < a \tag{33}$$

$$r^{-m/2}\left[-\int_0^\infty \xi H(\xi)J_{1+m/2}(r\xi)\,d\xi+\frac{2}{\pi}\int_0^\infty tB(t)K_{1+m/2}(rt)\,dt\right]=-\frac{p_2(r)}{\mu_0 r^m},\quad a<r<c. \tag{34}$$

Substituting eqs (19) and (28) into eq. (33), and eqs (20) and (29) into eq. (34), and after some lengthy manipulations, we obtain

$$\int_b^a I_1(r,s)\varphi_1(s)\,ds+\frac{2}{\pi}\int_b^a I_2(r,s)\varphi_1(s)\,ds+\frac{2}{\pi}\int_a^c I_3(r,s)\varphi_2(s)\,ds=-\frac{p_1(r)}{\mu_1},\quad b<r<c \tag{35}$$

$$\int_a^c L_1(r,s)\varphi_2(s)\,ds+\frac{2}{\pi}\int_a^c L_2(r,s)\varphi_2(s)\,ds+\frac{2}{\pi}\int_b^a L_3(r,s)\varphi_1(s)\,ds=-\frac{p_2(r)}{\mu_0 r^m},\quad a<r<c, \tag{36}$$

where

$$I_1(r,s)=\begin{cases}\dfrac{2}{\pi}\dfrac{s}{s^2-r^2}E\left(\dfrac{s}{r}\right), & s<r\\[2mm] \dfrac{2}{\pi}\left[\dfrac{s^2}{s^2-r^2}E\left(\dfrac{r}{s}\right)-K\left(\dfrac{r}{s}\right)\right]\dfrac{1}{r}, & s>r\end{cases} \tag{37}$$

$$L_1(r,s)=\begin{cases}-\dfrac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\gamma)}\left(\dfrac{s}{r}\right)^{1+m}\left[\dfrac{m}{r}\,{}_2F_1\left(\alpha,\beta,\gamma,\dfrac{s^2}{r^2}\right)+\dfrac{r}{r^2-s^2}\,{}_2F_1\left(\alpha-1,\beta,\gamma,\dfrac{s^2}{r^2}\right)\right], & s>r\\[2mm] \dfrac{\Gamma(\alpha)}{\Gamma(\beta)\Gamma(\gamma)}\dfrac{1}{r}\left[-{}_2F_1\left(\alpha,\beta,\gamma,\dfrac{r^2}{s^2}\right)+\dfrac{s^2}{s^2-r^2}\,{}_2F_1\left(\alpha-1,\beta,\gamma,\dfrac{r^2}{s^2}\right)\right], & s>r\\[2mm] \left(\alpha=\dfrac{1}{2}+\dfrac{m}{2},\beta=\dfrac{1}{2},\gamma=1+\dfrac{m}{2}\right)\end{cases} \tag{38}$$

$$I_2(r,s)=\int_0^\infty \bar I_2(r,s,t)\,dt,\quad \bar I_2(r,s,t)=t\frac{A_1(t)}{A(t)}sI_0(st)I_1(rt)+t\frac{A_3(t)}{A(t)}sI_1(rt) \tag{39a, b}$$

$$I_3(r,s)=\int_0^\infty \bar I_3(r,s,t)\,dt,\quad \bar I_3(r,s,t)=t\frac{A_2(t)}{A(t)}s^{1+m/2}K_{m/2}(st)I_1(rt) \tag{40a, b}$$

$$L_2(r,s)=\int_0^\infty \bar L_2(r,s,t)\,dt,\quad \bar L_2(r,s,t)=r^{-m/2}t\frac{B_2(t)}{\Delta(t)}s^{1+m/2}K_{m/2}(st)K_{1+m/2}(rt) \tag{41a, b}$$

$$L_3(r,s)=\int_0^\infty \bar L_3(r,s,t)\,dt,\quad \bar L_3(r,s,t)=r^{-m/2}t\frac{B_1(t)}{\Delta(t)}sI_0(st)K_{1+m/2}(rt)+r^{-m/2}t\frac{B_3(t)}{\Delta(t)}sK_{1+m/2}(rt); \tag{42a, b}$$

$K()$ and $E()$ being the complete elliptic integrals of the first and second kind, ${}_2F_1()$ being a hypergeometric function.

It can be seen that $I_1(r,s)$ and $L_1(r,s)$ have Cauchy-type singularities. The singular nature of other terms must be investigated. Since the functions $\bar I_2$, $\bar I_3$, $\bar L_2$ and $\bar L_3$ are integrable at $t\to 0$ and bounded elsewhere, any possible divergence in $\bar I_2$, $\bar I_3$, $\bar L_2$ and $\bar L_3$ must be due to the asymptotic behavior of I_2, I_3, L_2 and L_3 as $t\to\infty$. By analysing the asymptotic behavior, we have

$$I_i(r,s)=I_{if}(r,s)+I_{is}(r,s),\quad (i=1,2) \tag{43}$$

$$L_i(r,s)=L_{if}(r,s)+L_{is}(r,s),\quad (i=1,2), \tag{44}$$

where

$$I_{2f}(r,s)=\int_0^\infty\left[\bar I_2(r,s,t)-\frac{\lambda}{2}\left(\frac{s}{r}\right)^{1/2}e^{-t(2a-s-r)}\right]dt,\quad I_{2s}(r,s)=\frac{\lambda}{2}\left(\frac{s}{r}\right)^{1/2}/(2a-s-r) \tag{45a, b}$$

$$I_{3f}(r,s)=\int_0^\infty\left[\bar I_3(r,s,t)-\frac{\lambda_2}{2}\left(\frac{s}{a}\right)^{m/2}\left(\frac{s}{r}\right)^{1/2}e^{-t(s-r)}\right]dt,\quad I_{3s}(r,s)=\frac{\lambda_2}{2}\left(\frac{s}{a}\right)^{m/2}\left(\frac{s}{r}\right)^{1/2}/(s-r) \tag{46a, b}$$

$$L_{2f}(r,s)=\int_0^\infty\left[\bar{L}_2(r,s,t)-\frac{\lambda}{2}\left(\frac{s}{r}\right)^{1/2+m/2}e^{-t(s+r-2a)}\right]dt,\quad L_{2s}(r,s)=\frac{\lambda}{2}\left(\frac{s}{r}\right)^{2/m+1/2}/(s+r-2a) \tag{47a, b}$$

$$L_{3f}(r,s)=\int_0^\infty\left[\bar{L}_3(r,s,t)+\frac{\lambda_1}{2}\left(\frac{a}{r}\right)^{m/2}\left(\frac{s}{r}\right)^{1/2}e^{-t(r-s)}\right]dt,\quad L_{3s}(r,s)=-\frac{\lambda_1}{2}\left(\frac{a}{r}\right)^{m/2}\left(\frac{s}{r}\right)^{1/2}/(r-s), \tag{48a, b}$$

where

$$\lambda=\frac{\mu_2(a)-\mu_1}{\mu_2(a)+\mu_1},\quad \lambda_1=\frac{2\mu_1}{\mu_2(a)+\mu_1},\quad \lambda_2=\frac{2\mu_2(a)}{\mu_2(a)+\mu_1}.$$

By separating the Cauchy-type singularities from $I_1(r,s)$ and $L_1(r,s)$ we obtain

$$I_1(r,s)=\frac{1}{\pi(s-r)}+\frac{1}{\pi}I_{1f}(r,s),\quad I_{1f}(r,s)=\pi I_1(r,s)-\frac{1}{s-r} \tag{49a, b}$$

$$L_1(r,s)=\frac{1}{\pi(s-r)}+\frac{1}{\pi}L_{1f}(r,s),\quad L_{1f}(r,s)=\pi L_1(r,s)-\frac{1}{s-r}. \tag{50a, b}$$

By substituting eqs (43)–(50) into eqs (35) and (36), we obtain the following system of singular integral equations:

$$\frac{1}{\pi}\int_b^a\frac{\varphi_1(s)}{s-r}ds+\frac{1}{\pi}\int_b^a 2L_{2s}(r,s)\varphi_1(s)\,ds+\frac{1}{\pi}\int_a^c 2I_{3s}(r,s)\varphi_2(s)\,ds$$
$$+\frac{1}{\pi}\int_b^a[I_{1f}(r,s)+2I_{2f}(r,s)]\varphi_1(s)\,ds+\frac{1}{\pi}\int_a^c 2I_{3f}(r,s)\varphi_2(s)\,ds=-\frac{P_1(r)}{\mu_1},\quad b<r<a \tag{51}$$

$$\frac{1}{\pi}\int_a^c\frac{\varphi_2(s)}{s-r}ds+\frac{1}{\pi}\int_a^c 2L_{2s}(r,s)\varphi_2(s)\,ds+\frac{1}{\pi}\int_b^a 2L_{3s}(r,s)\varphi_1(s)\,ds$$
$$+\frac{1}{\pi}\int_a^c[L_{1f}(r,s)+2L_{2f}(r,s)]\varphi_2(s)\,ds+\frac{1}{\pi}\int_b^a 2L_{3f}(r,s)\varphi_1(s)\,ds=-\frac{p_2(r)}{\mu_0 r^m},\quad a<r<c. \tag{52}$$

In eqs (51) and (52), the first three terms have Cauchy-type singularities, the others are Fredholm kernel integral terms.

4. SOLUTION PROCEDURE AND RESULTS

To examine the behavior of φ_1 and φ_2 around the irregular points, we assume that the unknown functions may be expressed as

$$\varphi_1(s)=\frac{g_1(s)}{(s-b)^\alpha(a-s)^\beta} \tag{53}$$

$$\varphi_2(s)=\frac{g_2(s)}{(s-a)^\beta(c-s)^\gamma} \tag{54}$$

$$0<\mathrm{Re}(\alpha,\beta,\gamma)<1.0,$$

where α, β, γ are the powers of singularity at the three irregular points. The functions g_1 and g_2 satisfy the Hölder condition in the closed intervals $b\leqslant s\leqslant a$ and $a\leqslant s\leqslant c$, respectively. Following the technique described in refs [1, 4], we obtain the following system of equations:

$$\frac{g_1(b)}{(a-b)^\beta}\,\mathrm{ctg}\pi\alpha=0$$

$$\frac{g_2(c)}{(c-a)^\beta}\,\mathrm{ctg}\pi\gamma=0$$

$$(-\cos\pi\beta+\lambda)\frac{g_1(a)}{(a-b)^\alpha}+\lambda_2\frac{g_2(a)}{(c-a)^\gamma}=0$$

$$-\lambda_1\frac{g_1(a)}{(a-b)^\alpha}+(\lambda+\cos\pi\beta)\frac{g_2(a)}{(c-a)^\gamma}=0, \tag{55a–d}$$

$g_1(a)$, $g_1(b)$, $g_2(a)$ and $g_2(c)$ are non-zero constants. Thus, eqs (55a–d) give the following characteristic equations:

$$c\,\mathrm{tg}\,\pi\alpha = c\,\mathrm{tg}\,\pi\gamma = 0 \tag{56}$$

$$\cos^2 \pi\beta = 1 \tag{57}$$

$$\mu_1 \varphi_1(a) = \mu_2(a)\varphi_2(a). \tag{58}$$

By eq. (56) the acceptable roots are $\alpha = \gamma = 0.5$, which is the well-known result. By eq. (57) it is shown that $\beta = 0$. This result indicates that at the irregular point $r = a$ the unknown functions φ_1 and φ_2, and hence the stresses will have no power singularity. The possibility of a weaker (i.e. logarithmic) singularity must also be investigated. Following the procedure described by ref. [2] or the technique in ref. [5], it can be shown that φ_1 and φ_2 have no singularity at $r = a$ and stress $\sigma_{\theta z}(r, 0)$ is bounded at this point.

From eq. (58) it is clear that $\varphi_1(a)$ and $\varphi_2(a)$ are not independent and are related. By defining

$$\varphi(r) = \begin{cases} \mu_1 \varphi_1(r), & b < r < a \\ \mu_2(a)\varphi_2(r), & a < r < c \end{cases} \tag{59}$$

eqs (51) and (52) may be expressed as

$$\frac{1}{\pi}\int_b^c \frac{\varphi(s)}{s - r}\,\mathrm{d}s + \frac{1}{\pi}\int_b^c K(r, s)\varphi(s) = -p(r), \quad b < r < c, \tag{60}$$

where $K(r, s)$ and $p(r)$ are known bounded functions given by

$$p(r) = \begin{cases} p_1(r), & b < r < a \\ \left(\dfrac{a}{r}\right)^m p_2(r), & a < r < c \end{cases} \tag{61}$$

$$K(r, s) = \begin{cases} I_{1f}(r, s) + 2I_2(r, s), & b < (r, s) < a \\ 2I_3(r, s)\mu_1/\mu_2(a), & b < r < a, a < s < c \\ 2L_3(r, s)\mu_2(a)/\mu_1, & a < r < c, b < s < a \\ L_{1f}(r, s) + 2L_2(r, s), & a < (r, s) < c. \end{cases} \tag{62}$$

The single-valued condition (23) becomes

$$\frac{a^m}{\mu_1}\int_b^a r\varphi(r)\,\mathrm{d}r + \frac{1}{\mu_2(a)}\int_a^c r^{1+m}\varphi(r)\,\mathrm{d}r = 0. \tag{63}$$

It may be expressed further as

$$\int_b^c r \cdot h(r)\varphi(r)\,\mathrm{d}r = 0, \tag{64}$$

where

$$h(r) = \begin{cases} 1, & b < r < a \\ \left(\dfrac{r}{a}\right)^m \dfrac{\mu_1}{\mu_2(a)}, & a < r < c. \end{cases} \tag{65}$$

Thus, the going through crack problem reduces to a Cauchy-type singularity integral equation and can be solved using a regular method. First normalizing the equation by defining

$$\begin{aligned} s &= a_1\tau + b_1, \quad r = a_1\rho + b_1 \\ \varphi(s) &= F(\tau)(1 - \tau^2)^{-1/2} \\ p(r) &= Q(\rho), \quad L(\rho, \tau) = a_1 K(r, s) \\ s \cdot h(s) &= g(\tau), \end{aligned} \tag{66}$$

where $a_1 = (c - b)/2$, $b_1 = (c + b)/2$, then eqs (60) and (64) become

$$\frac{1}{\pi}\int_{-1}^{1}\left[\frac{1}{\tau - \rho} + L(\rho, \tau)\right]\frac{F(\tau)}{(1 - \tau^2)^{1/2}}\,\mathrm{d}\tau = -Q(\rho), \quad -1 < \rho < 1 \tag{67}$$

$$\int_{-1}^{1} g(\tau)\frac{F(\tau)}{(1-\tau^2)^{1/2}}\,d\tau = 0. \tag{68}$$

The function $F(\tau)$ can be obtained numerically by means of a Gauss–Chebyshev type quadrature formula [7].

The stress intensity factors are defined by

$$K(b) = \lim_{r\to b}\sqrt{2(b-r)}\,\sigma^1_{\theta z}(r,0) \tag{69}$$

$$K(c) = \lim_{r\to c}\sqrt{2(r-c)}\,\sigma^2_{\theta z}(r,0). \tag{70}$$

They can be expressed further as:

$$K(b) = \lim_{r\to b}\sqrt{2(r-b)}\,\varphi(r) = \sqrt{a_1}\,F(-1) \tag{71}$$

$$K(c) = -\left(\frac{r}{a}\right)^m \lim_{r\to c}\sqrt{2(c-r)}\,\varphi(r) = -\left(\frac{c}{a}\right)^m\sqrt{a_1}\,F(+1), \tag{72}$$

where $a_1 = (c-b)/2$.

When the crack penetrates through the fiber, i.e. $b = 0$, we need to extend the functions and kernels which appear in the singular integral equations (51) and (52) that are defined for positive variable values only into the symmetric negative range. For this circular crack problem, the stress intensity factor at the crack tips is defined by eq. (70).

The half crack circumferential displacements u^1_θ and u^2_θ are related to the density functions φ_1 and φ_2 through eqs (15) and (16). Noting that $u^1_\theta = 0$ for $r \leqslant b$ and $u^2_\theta = 0$ for $r \geqslant c$, after evaluating φ_1 and φ_2, the half displacements may be obtained from

$$u^1_\theta(r,0) = \frac{1}{r}\int_b^r s\varphi_1(s)\,ds, \quad b < r \lessdot a \tag{73}$$

$$u^2_\theta(r,0) = -\frac{1}{r^{1+m}}\int_r^c s^{1+m}\varphi_2(s)\,ds, \quad a < r < c. \tag{74}$$

Equations (60), (64), (71)–(74) show the effect of nonhomogeneity on the stress intensity factors and crack surface displacement.

It is shown that at the going through interface crack tips, the stresses are square-roots and not affected by the nonhomogeneity of the shear modulus, whether the shear modulus at the interface is continuous or not. It is also shown that at the irregular points on the interface the stresses are not singular and are bounded.

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