A PHYSICAL LIMIT OF WEIBULL MODULUS IN ROCK FRACTURE

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Generally speaking, there are numerous microcracks with various length scales in rocks. As the volume of a rock specimen increases, its strength obviously decreases. The strength of rocks is very diverse and exhibits a clear size effect. Therefore, a statistical description for rock fracture is necessary [1]. So far, the Weibull statistical distribution of fracture, which is based on a weakest link model, is one of the most frequently used theories [2]. In this theory, one supposes that there are a lot of defects with different strength in a specimen and that fracture occurs when a weakest defect is unstable.

According to the weakest link model, the fracture probability of a rock specimen is given by

$$P_f = 1 - [1 - p(\sigma)]^N \cong 1 - \exp[-\alpha V p(\sigma)] \quad (1)$$

where N is the number of microcracks, α is the number density of microcracks, V is the volume of the specimen, and $p(\sigma)$ is the fracture probability of a microcrack when loaded at stress σ . Weibull found that the following statistical distribution can fit the experimental data very well in most cases

$$p_f = 1 - \exp\left[-k_1 V \left(\frac{\sigma}{\sigma_0}\right)^m\right] \quad (2)$$

where k_1 and σ_0 are constants, and m is the Weibull modulus.

It is worth noting that m is an important parameter in the Weibull distribution. As we have known, m is a measure of the spread of the distribution. The broader the distribution is, the smaller the Weibull modulus m . Thus, the Weibull modulus is also called the shape factor. In the practical application, m is only an empirical parameter which is determined by fitting experimental data. The physical meaning of the Weibull modulus, especially the range of its value, is not clear.

In the last decade, fractal geometry has been widely used in physics and mechanics [3,4]. Since Mandelbrot et al. [5] discovered that the fracture surfaces of metals exhibit fractal character, a lot of experimental results have shown that both fracture surfaces and microcrack or fragmentation distribution in rocks can be described by the concept of fractal dimensions [6,7]. Therefore, we suppose that the microcrack distribution in a rock specimen yields the following relation

$$N(c) \sim c^{-D} \quad (3)$$

where $N(c)$ represents the number of cracks with size greater than c , and D is the fractal dimension.

According to the Griffith energy criterion, the critical stress σ at fracture is

$$\sigma \sim c_f^{-1/2} \quad (4)$$

where c_f is the critical crack size. Here, we consider that the fracture probability of a microcrack $p\{\sigma \geq \sigma_c\}$ is equal to $p\{c \geq c_f\}$, that is

$$p(\sigma) = \int_{c_f}^{\infty} dN(c) \sim c_f^{-D} \quad (5)$$

Combining (5) with (4), we obtain

$$p(\sigma) \sim \sigma^{2D} \quad (6)$$

Inserting (6) into (1), we have

$$p_f = 1 - \exp[-k_2 V \sigma^{2D}] \quad (7)$$

where k_2 is a constant.

It is easy to see that the probability of a rock specimen with fractal microcrack distribution yields the Weibull statistics. Comparing (7) with (2), we can determine that the relation between Weibull modulus and fractal dimension [8,9] is

$$m = 2D \quad (8)$$

According to fractal geometry, if we use a cube lattice with edge length c to cover the rock specimen, the number of cubes needed to cover the microcracks is also given by (3). Here, the fractal dimension D is the measure of the space

occupied by microcracks or damage content. Obviously, the value range of the fractal dimension D is 0-3, and the limit of the Weibull modulus is $m < 6$. To our knowledge, m values obtained in most rock fracture experiments lie in this limit.

On the other hand, based on the self-consistent method, Budiansky and O'Connell [10] showed that the presence of microcracks in an elastic solid reduces the elastic modulus to zero i.e. fracture when

$$\int_0^{\infty} c^3 n(c) dc > \frac{9}{16} \quad (9)$$

where $n(c)$ is the density of microcrack distribution [11].

From (3) and (8), we have

$$n(c) = \frac{dN(c)}{dc} \sim c^{-D-1} \sim c^{-m/2-1} \quad (10)$$

Inserting (10) into the left of (9), we obtain

$$\int_0^{\infty} c^3 n(c) dc \sim \int_0^{\infty} c^{-(m-4)/2} dc \quad (11)$$

In order to keep the integral on the left of (9) or (11) to be finite, the following condition must be satisfied

$$\begin{aligned} & -(m-4)/2 > -1 \\ \text{that is} & \\ & m < 6 \end{aligned} \quad (12)$$

which is consistent with the above result.

In conclusion, by invoking the concept of fractals, we can give the Weibull distribution of rock fracture a clear physical explanation. At the same time, a physical limit of the Weibull modulus is naturally obtained by the relation between fractal dimension and Weibull modulus. The expected limit value is the same as that obtained by the self consistent method. This result may provide a reference for the experimental data fitting.

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