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CONVECTIVE COUPLED MAP FOR SIMULATING SPATIOTEMPORAL CHAOS IN FLOWS

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ABSTRACT: A coupled map lattices with convective nonlinearity or, for short, Convective Coupled Map (CCM) is proposed in this paper to simulate spatiotemporal chaos in fluid flows. It is found that the parameter region of spatiotemporal chaos can be determined by the maximal Liapunov exponent of its complexity time series. This simple model implies a similar physical mechanism for turbulence such that the route to spatiotemporal chaos in fluid flows can be envisaged.

KEY WORDS: convective nonlinearity, coupled map lattices, spatiotemporal chaos, turbulence

I. INTRODUCTION

The Coupled Map Lattices[1] (CML) has become a fruitful approach to the intuitive understanding of spatiotemporal complexity in spatial distributed systems, like turbulence in fluid flows. The CML is advantageous not only in numerical simulation due to its explicit spatial and temporal discretization, but also in direct application of the low-dimensional chaotic theory to circumstances of higher dimensions. For example, Liapunov exponents of complexity time series are suggested to characterize spatiotemporal chaos[2]. Therefore, a variety of CML models have been constructed to investigate the mechanisms of the corresponding physical phenomena. The construction of a CML model generally proceeds in the following steps[2]:

(1) Take a (set of) field variable(s) at each site.

(2) Decompose a complicated phenomenon into independent physical processes (e.g., convection, reaction, diffusion, and so on).

(3) Replace each process by possibly simplest parallel dynamics at each site, that is, the next state on each site depends on nonlinear evolution of the present state itself and of suitable chosen neighbours due to coupling.

(4) Carry out such dynamical process in succession for a lattice chain.

One of the well-known CML models is the Coupled Logistic Map Lattices (CLML) consisting of a local nonlinear function and a coupled term among the nearest neighbour sites. According to the above steps of constructing CML, we take local nonlinear function

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as $f(u_k^n)$, where $f(x) = 1 - ax^2$ $(0 \le a \le 2)$ is the Logistic map , and the coupled term as a diffusive process

$$\frac{\varepsilon}{2}[f(u_{k+1}^n) + f(u_{k-1}^n) - 2f(u_k^n)] \propto \frac{\partial^2 f}{\partial u^2}$$

which, more or less, is something like what happens in a reaction-diffusion process. Combining these two physical mechanisms together, we can write CLML in the form of

$$\left.\begin{array}{ll} u_k^{n+1} = (1-\varepsilon)f(u_k^n) + \varepsilon/2 \cdot [f(u_{k+1}^n) + f(u_{k-1}^n)] & \\ u_0^n = u_N^n \qquad u_1^n = u_{N+1}^n & \end{array}\right\} \tag{1.1}$$

where n is a discrete time step and i is a sequential number for any site ($i = 1, 2, \cdots, N$, where N represents system size).

Although CLML looks rather simple, there indeed exist much rich and complicated spatiotemporal patterns in it beyond our imagination. Kaneko has found the following two routes from spatiotemporal order to spatiotemporal chaos (weak turbulence) in CLML by numerical simulation and visualization of their patterns. Qualitatively, these two routes can be stated as follows[2]:

(5) The first route (for large ε and increasing a):
Pattern Frozen $\Rightarrow$ Pattern Selection $\Rightarrow$ Pattern Intermittence $\Rightarrow$ Spatiotemporal chaos

(6) The second route (for small ε and increasing a):
Pattern Frozen $\Rightarrow$ Defect $\Rightarrow$ Diffusion of Defect $\Rightarrow$ Spatiotemporal Chaos
from which it is evident that CLML will eventually exhibit spatiotemporal chaos with increasing a even for large ε.

Let us come back to fluid mechanics: turbulent state may exist only for large Reynolds' number, that is, either for large inertia force or for small viscosity. Consequently, spatiotemporal chaos should correspond to larger nonlinearity a and less diffusivity ε if we actually attempt to find some similarities between CML and turbulence. However, the behaviours of CLML, where spatiotemporal chaos (weak turbulence) may exist even for large diffusivity, happen to be contrary to those of turbulence. Therefore, CLML is not suitable for modelling turbulence. Other CML models such as boundary layer coupled map lattices[3] and turbulence coupled map[4] seem to possess the same shortcoming.

The reason for this lies in the lack of convective nonlinearity. The above mentioned CML models contain only local nonlinearity and not any coupled nonlinear term, that is, they have no convective terms. It is commonly known that the convective term is the main source of nonlinearity in hydrodynamical equations like the Burgers equation. Based on this equation, a new model of CML involving convective term is constructed in the present paper (see II). Nevertheless, our objective is to find an appropriate nonlinear coupling term in CML rather than to design a difference scheme of Burgers equation. In section III, we show that CCM do exhibit the mechanism of turbulence to a certain extent.

II. A CML MODEL INVOLVING CONVECTIVE MECHANISM

As is well known, the Burgers equation includes both the convective and diffusive terms. We consider this equation with periodic boundary

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

where u stands for a velocity component, ν the kinetic viscosity, t time, and x one dimensional Cartesian coordinate. As usual, we discretize the spatial derivative by upwind scheme

$$\mathrm{d}u_i/\mathrm{d}t + (u_i - u_{i-1})u_i/h = \nu(u_{i+1} + u_{i-1} - 2u_i)/h^2 \qquad (i = 1, 2, \cdots, N)$$

where h is a space step, and define

$$U = (u_1, \cdots, u_N)^{\mathrm{T}}$$

$$g(u_i, u_{i-1}) = -(u_i - u_{i-1})u_i/h$$

$$G(U) = [g(u_1, u_0), \cdots, g(u_N, u_{N-1})]$$

$$E = \frac{\nu}{h^2}\begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & -2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ddots & \ddots & 1 \\ 1 & \cdots & 0 & \cdots & 1 & -2 \end{bmatrix}$$

then the discretized Burgers equation can be written as

$$\mathrm{d}U/\mathrm{d}t = G(U) + EU \tag{2.1}$$

Introducing the transformation

$$U = \exp\{(t - t_n)E\}V \tag{2.2}$$

where $t_n = nT$ $(n = 0, 1, 2, \cdots)$ with T as the time step, and substituting Eq.(2.2) into Eq.(2.1), we obtain

$$\mathrm{d}V/\mathrm{d}t = \exp\{-(t - t_n)E\}G\{\exp[(t - t_n)E]V\} \tag{2.3}$$

Noting E is a quasi-tridiagonal matrix, E^k is a $(2k + 1)$ quasi-diagonal matrix. Since the effect of the ith's nearest neighbour lattices on the ith lattice is only considered, U in Eq.(2.2) can be reduced as

$$U = V + (t - t_n)EV \tag{2.4}$$

According to the rule of coupling among the nearest neighbour lattices and Eq.(2.4), we discretize Eq.(2.3) at time $t_n = nT$

$$\begin{aligned} v_i^{n+1} &= v_i^n - T \cdot g(v_i^n, v_{i-1}^n) \\ &= v_i^n(1 - \frac{T}{h}v_i^n) + \frac{T}{h}v_i^n v_{i-1}^n \\ &\equiv p(v_i^n, v_i^{n-1}) \end{aligned} \tag{2.5}$$

Substituting $t = nT$ and $t = (n + 1)T$ respectively into Eq.(2.4), and combining Eq.(2.5) lead to

$$u_i^{n+1} = p(u_i^n, u_{i-1}^n) + \nu\frac{T}{h}\Delta p(u_i^n, u_{i-1}^n)$$

where Δ is a discretized Laplace operator. Let $u_i^n = \frac{h}{T} z_i^n$, then

$$z_i^{n+1} = \left[z_i^n(1-z_i^n) + z_i^n z_{i-1}^n\right] + \frac{T}{h}\nu \cdot \Delta\left[z_i^n(1-z_i^n) + z_i^n z_{i-1}^n\right] \tag{2.6}$$

We see from Eq.(2.6) that a CML model, which can properly simulate turbulence, should include the following two physical mechanisms:

(1) The local nonlinearity and coupled nonlinearity due to the convective term: the former comes from the Logistic map $f(x) = \lambda x(1-x)$ with a nonlinear parameter λ, whereas the latter is a nonlinear coupled term like $z_i^n z_{i-1}^n$.

(2) The diffusion of nonlinear terms due to the viscosity: we only consider the diffusion of the local nonlinear term like $\Delta f(x^i)$ and ignore the diffusion of coupled nonlinearity because of its strong smoothing effect.

Combining the above two mechanisms together, we can construct a CML model involving convective mechanism

$$\left.\begin{array}{ll} x_i^{n+1} = (1-\alpha-\beta)f(x_i^n) + \frac{\alpha}{2}[f(x_{i-1}^n) + f(x_{i+1}^n)] + \beta x_i^n x_{i-1}^n & \\ x_0^n = x_N^n \qquad\qquad x_1^n = x_{N+1}^n & \end{array}\right\} \tag{2.7}$$

where $\alpha, \beta \in [0, 0.4]$. In order to compare it with CLML, we would rather choose the local function $f(x)$ in an alternative form of Logistic map: $f(x) = 1 - ax^2$, in which a is a nonlinear parameter. For the sake of convenience in numerical simulation and theoretical analysis, we further assume $\alpha = \beta = \varepsilon$, and then Eq.(2.7) is transformed to a simpler form

$$\left.\begin{array}{ll} x_i^{n+1} = (1-2\varepsilon)f(x_i^n) + \frac{\varepsilon}{2}[f(x_{i+1}^n) + f(x_{i-1}^n)] + \varepsilon x_i^n x_{i-1}^n & \\ x_0^n = x_N^n \qquad\qquad x_1^n = x_{N+1}^n & \end{array}\right\} \tag{2.8}$$

The above model with both nonlinear convection and diffusion is called Convective Coupled Map (CCM). It is necessary to emphasize that CCM is proposed for investigating mechanism of spatiotemporal complexity in fluid flows, but not for simulating Burgers equation. Actually, CCM is not a suitable difference scheme for Burgers equation.

III. CHARACTERIZING SPATIOTEMPORAL CHAOS FOR CCM

In the present section, the temporal and spatiotemporal chaos is characterized by the maximal Liapunov exponent and the maximal Liapunov exponent of complexity time series respectively. The corresponding parameter regions can also be demarcated by the discussion on the critical one of them.

Let us begin with the maximal Liapunov exponent of CCM. Assume

$$\delta X_n = (\delta x_1^n, \cdots, \delta x_N^n)$$

to be a small perturbation, which satisfies the following linear equations

$$\delta X_{n+1} = A(n)\delta X_n = \prod_{j=1}^{n} A(j)\delta X_1$$

where $A(n)$ is the Jacobi matrix of Eq.(2.8) at $X_n = (x_1^n, \cdots, x_N^n)$. Denote the maximal eigenvalue of $\prod_{j=1}^{n} A(j)$ by ρ_n. Then, the maximal Liapunov exponent turn out to be

$\lambda = \lim_{n\to\infty} \frac{1}{n} \log |\rho_n|$, which is calculated by the Resband algorithm[5]. The main point of this method is the way to estimate the magnification of normalized perturbation vectors:

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \|\alpha_j\|$$

where $\alpha_j = \|\delta X_j\|$. Dividing the ranges of the nonlinear parameter a from 1.5 to 2 by a step 0.025 and of the diffusion parameter ε from 0 to 0.4 by a step 0.02, a topography map of the Liapunov exponents versus parameter pair (a, ε) and the contours of zero Liapunov exponent are shown in Fig.1 and Fig.2. It can be found from Fig.2 that the Liapunov exponents are positive to the left of the line vertically cutting across a-ε plane. And they are negative to the right of it. Evidently, there exist a few chaotic windows within the regions enclosed by these contours. As a result, we can conclude that the exponents are positive for small ε, whereas they vanish or even become negative as ε increases.

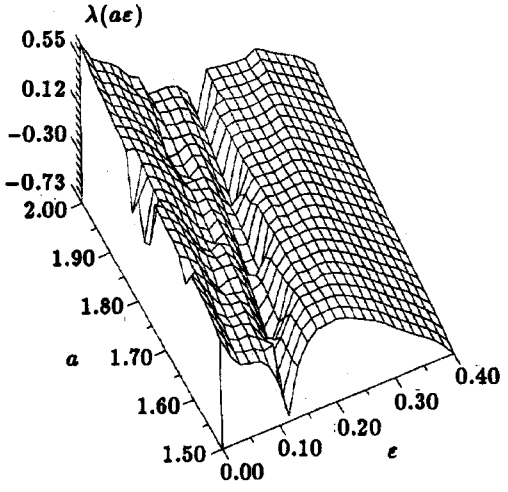

Fig.1 Surface plot of L-exponent

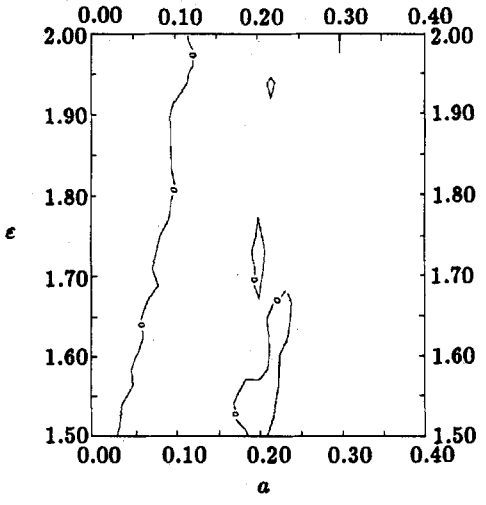

Fig.2 Isogram plot of L-exponent

Now we are further concerned with the spatiotemporal chaos. As random-like evolution of spatial patterns, spatiotemporal chaos is for the first time defined as sensitivity to initial pattern, thus leading to unpredictability of evolution of spatial patterns. To depict a pattern for spatially extended systems quantitatively, we should introduce a concept of complexity borrowed from Ref.[5] as follows:

Let S be a symbolic string consisting of 0 or 1 with a certain length $l(S)$ and substring set $v(S)$. Q is an alternative string. A new string R is produced by connecting them, such that $R = SQ$. If $Q \in v(SQ\pi) = v(R\pi)$, such a string connection is called a copy, otherwise, an insert, in which $R\pi$ represents an operation of truncating the last symbol in R. Furthermore, the number of inserted substrings required to generate R is represented by $LZ(R)$, i.e. so called LZ complexity. Then, the complexity of R is defined as $C(R) = LZ(R)/[l(R)\log_2 l(R)]$, whose denominator is LZ complexity of random symbolic string consisting of 0 or 1 with the same length $l(R)$. Finally, we find that the complexity of a string R is a relative complication of R to LZ complexity of the random symbolic string

with the same length $l(R)$. The larger is $C(R)$, the more complicated is the symbolic string R.

It is time for us to start characterizing spatiotemporal chaos for CCM by the maximal Liapunov exponent of a complexity time series. The procedure of its computation is divided into the following steps[6]:

(1) A special string $X_n = (x_1^n, \cdots, x_N^n)$ produced at each iteration step is coarse-grained and turned into a symbolic string $S_n = (s_1^n, \cdots, s_N^n)$, i.e. if $x_k^n \geq 0.5$, $s_k^n = 1$; otherwise, $s_k^n = 0$. In this way, we obtain a symbolic string $S_n = (s_1^n, \cdots, s_N^n)$, in which elements s_j^n are either 0 or 1. Physically, this kind of procedure represents the kink-antikink structures of the largest scale in CCM.

(2) Using the above mentioned techniques, we calculate the complexity $C_n = C(S_n)$ of a symbolic string, and the corresponding complexity time series C_n for pattern evolution.

(3) The maximal Liapunov exponent of complexity time series for CCM is obtained by Taken's embedding and Wolf's approach[7].

If the maximal Liapunov exponent of complexity time series for CCM is positive, it means nothing but spatiotemporal chaos, that is, CCM is sensitive to initial pattern. It should be noted that the positive maximal Liapunov exponent instead of that of a complexity time series does not imply spatiotemporal chaos. As a matter of fact, it merely means a temporal chaos.

By now, we have obtained the maximal Liapunov exponent of complexity time series and contours of zero Liapunov exponent (see Fig.3, Fig.4), from which we can see that if

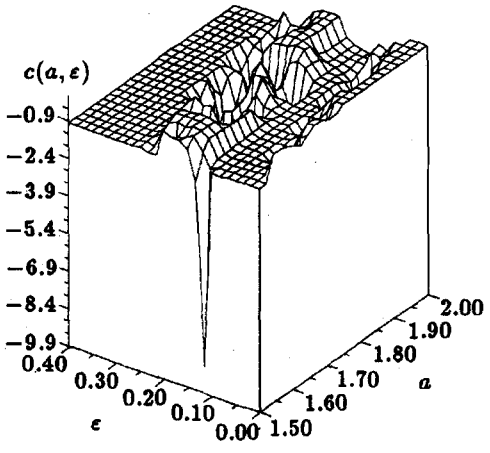

Fig.3 Surface plot of L-exponent for complexity series

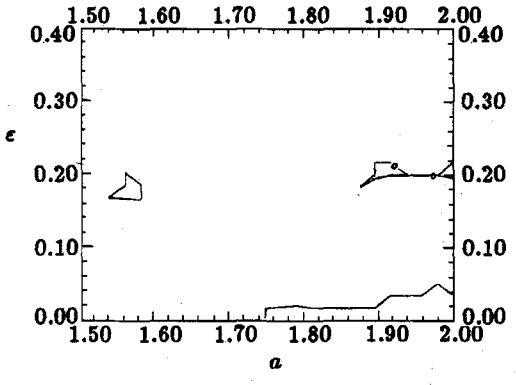

Fig.4 Isogram plot of L-exponent for complexity series

ε is small, they gradually increase from negative to zero and eventually become positive as a is growing. On the other hand, they ultimately become negative as ε is growing no matter what a is. Hence, the spatial pattern of CCM evolves from spatiotemporal order to spatiotemporal chaos as a is increasing for less ε. In this case the corresponding maximal Liapunov exponents of complexity time series vary from negative to positive. However, if ε is large enough, the spatiotemporal pattern will become regular or even an equilibrium state of homogeneity in space. For this case, the corresponding maximal Liapunov exponent is negative and the maximal Liapunov exponent of complexity time series is also negative.

IV. CONCLUSION

According to the foregoing discussion, all models of CML available so far are incapable of modelling turbulence mechanism. The reason for the situation is that the convective term, which nonlinearity in the Navier-Stokes equation comes from, has not been included in those CML models. In contrast, CCM proposed in the present paper includes the convective term indeed, and exhibits some behaviours of turbulence in fluid flows at least in the following two aspects:

(1) As a measurement of spatiotemporal chaos, dependence of the maximal Liapunov exponent for complexity time series upon nonlinearity a and diffusive coefficient ε to a certain extent reflects dependence of turbulence upon the Reynolds' number.

(2) There is an analogue of spatiotemporal pattern obtained by numerical simulation in CCM to those in fluid flows, namely, competition, intermittence and spatiotemporal chaos in realistic fluid flows may occur in CCM as well.

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