Parameter-transformation relations for travelling wave solutions of Kdv-Burgers equation*

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Burgers [1] suggested that the main properties of free-turbulence in the boundless area without basic flow might be understood with the aid of the following equation, which was much simpler than those of fluid dynamics,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial x^2}=0, \tag{1}$$

where u and v represent the turbulence velocity and the viscosity, respectively. However, it was later found not to be an adequate model equation of turbulence. Gao[2] suggested that the Kdv-Burgers equation,

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-v\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^3 u}{\partial x^3}=0, \tag{2}$$

which is the same as Burgers's equation except for containing an additional dispersion term ($\gamma>0$ being positive dispersion, and $\gamma<0$ negative dispersion), could be regarded as the "normal equation" of turbulence. Guan and Gao[3] proved that a unique bounded nontrivial travelling wave solution $u(x-ct)$ of eq. (2) exists, provided that A, an integral constant (see (5)) and c, the wave propagating velocity, satisfy the condition

$$c^2+2A>0, \tag{3}$$

and that if another condition,

$$v^2<4|\gamma|\sqrt{c^2+2A}, \tag{4}$$

is satisfied besides (3), then $u(x-ct)$ is a saddle-focus heteroclinic travelling wave solution, whose qualitative profiles are shown in figs. 1(a) and 2(a) corresponding to $\gamma>0$ and $\gamma<0$, respectively (note that $\xi=x-ct$). Guan and Gao[3] described the properties of the wave shape in fig. 1(a) ($\gamma>0$, $c>0$) by Theorem 3.2: as ξ decreases, the maximum of u, u_{max} decreases while the minimum of u, u_{min} increases, and both tend to the same value when $\xi\to 0$.

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We will give the parameter-transformation relations for bounded nontrivial travelling wave solutions, and show that the saddle-focus heteroclinic travelling wave solutions cannot represent the cascading down process of an eddy in turbulence.

1 Parameter-transformation relations for bounded nontrivial travelling wave solutions

Substituting $x=\xi-ct$ into eq. (2), and integrating it with respect to ξ, we obtain

$$\gamma \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} - v\frac{\mathrm{d}u}{\mathrm{d}\xi} + \frac{1}{2}u^2 - cu = A, \tag{5}$$

where A is an integral constant. When condition (3) is satisfied, according to the existence and uniqueness of bounded nontrivial travelling wave solution[3], the bounded nontrivial solution of (5), i.e. the bounded nontrivial traveling wave solution of (2), is determined by parameters γ, A and c, so it can be written as $u(\gamma, c, A, \xi)$.

Now define $\bar{u}$ and $\bar{\bar{u}}$ by writing

$$\bar{u}(\gamma, c, A, \xi) = u(\gamma, -c, A, \xi) + 2c, \tag{6}$$

and

$$\bar{\bar{u}}(\gamma, c, A, \xi) = -u(-\gamma, -c, A, -\xi). \tag{7}$$

It is readily seen that $\bar{u}$ and $\bar{\bar{u}}$ are also bounded nontrivial solutions of (5). Hence, the uniqueness of the solution gives

$$\bar{\bar{u}}(\gamma, -c, A, \xi) = \bar{u}(\gamma, c, A, \xi) = u(\gamma, c, A, \xi).$$

Substituting (6) and (7) into the above equation yields

$$u(\gamma, -c, A, \xi) = u(\gamma, c, A, \xi) - 2c, \tag{8}$$

and

$$u(\gamma, c, A, \xi) = -u(-\gamma, -c, A, -\xi). \tag{9}$$

Since the transformations $\gamma \to -\gamma$ and $c \to -c$ do not violate conditions (3) and (4), $u(-\gamma, c, A, \xi)$, $u(-\gamma, -c, A, \xi)$ and $u(-\gamma, -c, A, \xi)$ are all saddle-focus heteroclinic travelling wave solutions with different parameters, provided that $u(\gamma, c, A, \xi)$ is a saddle-focus heteroclinic travelling wave solution. It can be seen from eq. (8) that two saddle-focus heteroclinic travelling waves, propagating inversely at the same speed with the same other parameters, differ only by a translation quantity of $-2c$ along the u axis. The graphs of $u(-\gamma, -c, A, \xi)$ and $u(\gamma, c, A, \xi)$ are reverse about the center O, as can be seen in equation (9).

Translating down the wave shape of $u(\gamma, c, A, \xi)(\gamma>0, c>0$, (3) and (4) are satisfied) shown in fig. 1(a) by $-2c$ along the u axis gives fig. 1(b), which represents $u(\gamma, -c, A, \xi)$. Reversing fig. 1(b) about the center O yields fig. 2(a), representing $u(-\gamma, c, A, \xi)$. Reversing fig. 1(a) about the center gives fig. 2(b), representing $u(-\gamma, -c, A, \xi)$. If all the figures are represented by $u(\gamma, c, A, \xi)$, the two figs. 1(a), (b), and 2(a), (b) correspond respectively to four cases ($\gamma>0$, $c>0$), ($\gamma>0$, $c<0$), ($\gamma<0$, $c>0$), and ($\gamma<0$, $c<0$) with the same A and the same absolute values of γ and c.

2 Restriction of the parameter-transformation relations to the physical explanations of these solutions

One important consequence of the parameter-transformation relations (8) and (9) is that we cannot arbitrarily explain the physical meaning of u.

For instance, if we want to explain figs. 1(a) and 2(a) as cascading down and energy inversion processes of an eddy in turbulence (called eddy behavior below in short), and to make a further conclusion that the interaction between dissipation and positive dispersion ($\gamma>0$) results in turbulent cascading down and that between dissipation and negative dispersion ($\gamma<0$) results in turbulent energy inversion, then figs. 1(b) and 2(b) determined by transformation relations (8) and (9) would give the opposite conclusion that the interaction between dissipation and positive dispersion causes energy inversion and that between dissipation and negative dispersion causes cascading down. This contradiction excludes the possibility of explaining the dynamics of u as eddy behavior.

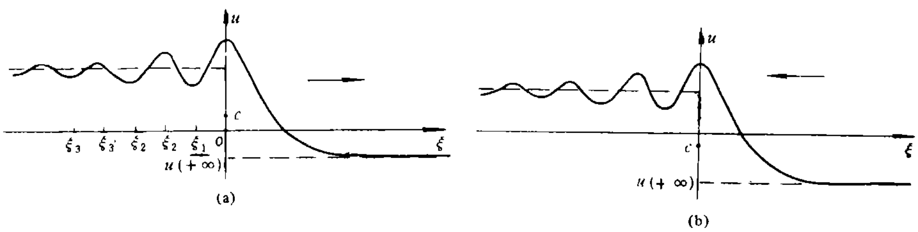

Fig. 1 (a). $\gamma>0$, $c>0$; (b) $\gamma>0$, $c<0$.

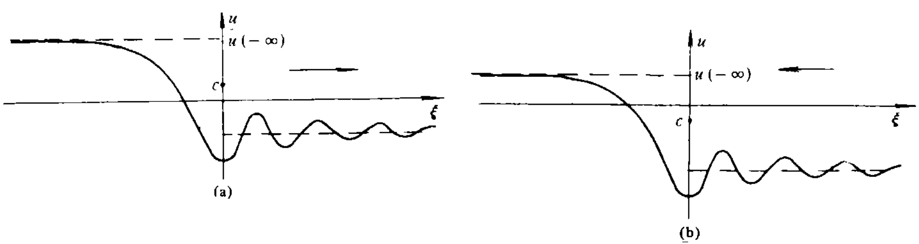

Fig. 2(a). $\gamma<0$, $c>0$; (b) $\gamma<0$, $c>0$.

Of course, this exclusion can also be drawn from the fact that the eddy behavior is concerned with the generation and disappearance of over one-dimensional structures, the stochastic relative phases between eddies of different scales, and the energy transportation in the spectrum space, while figs. 1 and 2 describe a constant over one-dimensional structure translating in the space, having definite phase relations between the wave packets of different scales, and containing no energy transportation phenomena in the spectrum space. Furthermore, the eddy behavior is a sort of certain phenomenon in turbulence, but as shown in fig. 3 of ref. [4], the saddle-focus heteroclinic travelling wave solution is an isolated bounded nontrivial travelling wave solution without attractiveness so it happens with probability zero (though A and c

can change continuously). Therefore, there are hardly any properties of eddy behavior which can be reasonably understood in terms of saddle-focus heteroclinic travelling wave solutions.

3 Notes on the expression of travelling wave solution

Xiong[5] found a kind of analytical solutions for saddle-node heteroclinic travelling waves. However, the exact or satisfactorily approximate saddle-focus heteroclinic travelling wave solution has not been found yet. By the way, the approximation of the two expressions of saddle-focus heteroclinic travelling wave solutions given in ref. [4] needs to be improved. It is readily seen by considering the limit case.

For example, consider formula (7) given in ref. [4]. Letting $v \to 0$, we have

$$\lim_{v \to 0} u = u_1 + \frac{1}{2}(u_1 - u_2)\cos\sqrt{\frac{u_1 - u_2}{2\gamma}}\,\xi, \quad (-\infty < \xi < 0),$$

which is an oscillation with constant amplitude. Write $\underline{\xi}_1$ as the value of ξ corresponding to the first minimum from the right. Then we have

$$\lim_{v \to 0} \underline{\xi}_1 = \pi\sqrt{\frac{\gamma}{2(u_1 - u_2)}}. \tag{10}$$

However, it can be proved[3] that the exact solution should approach the bell-shaped solitary wave as $v \to 0$, and so

$$\lim_{v \to 0} \underline{\xi}_1 = -\infty. \tag{11}$$

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