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# Unloading characteristics of anti-plane shear crack in softening materials

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**Abstract**

The local characteristics of the anti-plane shear stress and strain field are determined for a material where the stress increases linearly with strain up to a limit and then softens nonlinearly. Two unloading models are considered such that the unloading path always returns to the origin while the other assumes the unloading modulus to be that of the initial shear modulus. As the applied shear increases, an unloading zone is found to prevail between a zone in which the material softens and another zone in which the material is linear-elastic even though the crack does not propagate. The divisions of these zones are displayed graphically.

## 1. Introduction

Crack behavior in nonlinear materials has received much attention in the past. Constitutive relations of many different forms were introduced attempting to simulate the response of real materials. Depending on the mathematical model, the crack-tip stress field characteristics could vary in complexity. The anti-plane shear of a crack in a hardening material under small scale yielding has been considered [1]. But the method does not apply to materials with softening behavior. The state of affairs near an anti-plane shear crack in a softening material with local loading was considered in [2]. One portion of the solution was elliptic in character and the other hyperbolic. This work further considers unloading along a path with modulus equal to that of the elastic portion. The results are compared with those for changing moduli where the unloading path always returns to the original state of zero stress and strain.

In-plane extension of cracks of softening materials have been solved in [3] using the method established in [4]. As the material softens, the unloading modulus decreases with increasing damage. Such a model was also considered in [5].

## 2. Anti-plane shear with softening

Hodograph transformation was used in [6] to obtain a solution for the anti-plane shear crack

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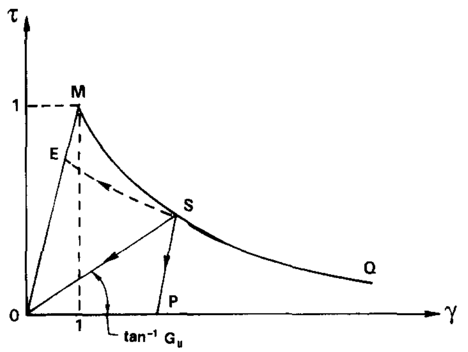


Fig. 1. Anti-plane shear stress versus shear strain for a material with softening.

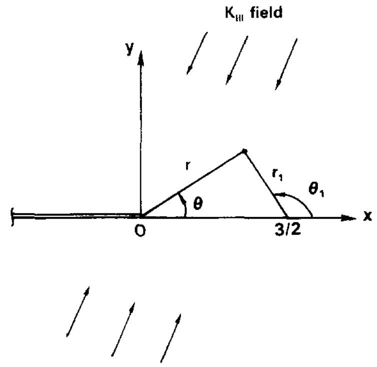


Fig. 3. Local coordinates referred to a semi-finite crack.

problem of a softening material whose shear stress $\tau$ versus shear strain $\gamma$ behavior is shown in Fig. 1. Note that E is on the elastic portion from O to M while S is the softening portion from M to Q. These two regions, i.e., elastic and softening with border OHA and OHA and AD (circular arc) in Fig. 2. While the antiplane displacement $w$ and normal shear stress $\tau_n$ are continuous, the normal shear strain $\gamma_n$ and tangential shear stress $\tau_s$ are discontinuous across OHA in Fig. 2. The solution in [6] is mathematically admissible, but may not be physically plausible. Referring to Fig. 2, a material particle P in the softening zone could return to its elastic state with increased loading. Moreover, when the particle P is swept by expanding of the softening zone boundary, its stress and strain state jumps from S to E in Fig. 1. These two situations are direct consequence of non-linear elastic constitutive relation where loading and unloading take place on the same curve. Such an assumption will not be invoked in this work. Two different unloading schemes will be used for the elastic-softening material that contains an anti-plane semi-infinite shear crack (Fig. 3).

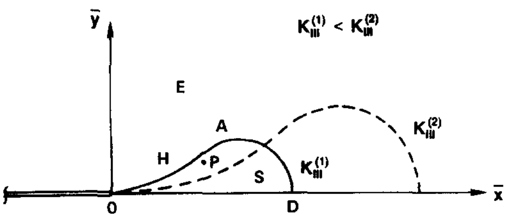


Fig. 2. Linear-elastic and softening boundary and self-similarity character.

## 3. Constitutive models

The constitutive relations in [6] are of the form

$$\tau_j = \left(\frac{\tau}{\gamma}\right)\gamma_j = \begin{cases} \gamma_j, & \text{OM}, \\ \gamma_j/\gamma^{3/2}, & \text{MSQ}, \end{cases} \tag{1}$$

where $j = r, \theta$. In Eqs. (1), OM is the linear elastic path and MSQ the softening path in Fig. 1. The corresponding $\tau$ versus $\gamma$ relations are

$$\tau = \begin{cases} \gamma, & \text{OM}, \\ 1/\sqrt{\gamma}, & \text{MSQ}. \end{cases} \tag{2}$$

For small deformation, $\gamma_j$ ($j = r, \theta$) are related to the anti-plane displacement $w$ as

$$\gamma_r = \frac{\partial w}{\partial r}, \qquad \gamma_\theta = \frac{1}{r}\frac{\partial w}{\partial \theta}. \tag{3}$$

Assume that the stresses and strains are continuous on the softening–unloading boundary $\Gamma_s$. That is, only continuous fields will be considered. Let the subscript t denote the instance when a particle is swept by $\Gamma_s$. Hence, $t-0$ and $t+0$ correspond to the instances when $\Gamma_s$ just before reaching and after passing the particle. The softening side of $\Gamma_s$ is $t-0$ and the unloading side

$t+0$. It follows that the continuity conditions on $\Gamma_s$ are:

$$\left(\tau^{t+0}, \tau_j^{t+0}\right)=\left(\tau^{t}, \tau_j^{t}\right)=\left(\tau^{t-0}, \tau_j^{t-0}\right),$$
$$\left(\gamma^{t+0}, \gamma_j^{t+0}\right)=\left(\gamma^{t}, \gamma_j^{t}\right)=\left(\gamma^{t-0}, \gamma_j^{t-0}\right). \tag{4}$$

The combination of Eqs. (2) and (4) yields

$$\tau^{t+0}=\left(\gamma^{t+0}\right)^{-1/2}$$
$$\tau_j^{t+0}=\left(\gamma^{t+0}\right)^{-3/2}\gamma_j^{t+0}. \tag{5}$$

### 3.1. Unloading Model I

Assume that the material unloads from S to O along a straight line as shown in Fig. 1. The unloading modulus $G_u$ is seen to decrease as S travels from M to Q. The unloading constitutive relation is given by

$$\tau_j=G_u\gamma_j, \tag{6}$$

in which

$$G_u=\tau^{t+0}/\gamma^{t+0}=\left(\gamma^{t+0}\right)^{-3/2} \tag{7}$$

### 3.2. Unloading Model II

Suppose that the unloading modulus is always equal to the initial elastic shear modulus, i.e., $G=1$. It follows then

$$\Delta\tau_j=\Delta\gamma_j, \qquad j=r, \theta. \tag{8}$$

The continuity conditions in Eqs. (4) and (5) can be invoked to yield

$$\tau_j=\gamma_j+\gamma_j^{t+0}\left[\left(\gamma^{t+0}\right)^{-3/2}-1\right], \qquad j=r, \theta. \tag{9}$$

The unloading path SP is shown in Fig. 1; it is not a straight line parallel with the elastic segment from O to M. In contrast to Model I, permanent or plastic deformation prevails when $\tau$ decreases to zero while $\gamma$ does not. For Model II, Eq. (9) can be written as

$$\tau^2=\gamma^2+\left(\gamma^{t+0}\right)^2-\left(\gamma^{t+0}\right)^2 + 2\sum_{j=r,\theta}\left(\tau_j^{t+0}\tau_j-\gamma_j^{t+0}\gamma_j\right). \tag{10}$$

## 4. Solution of linear elastic and softening zone

Let $w_e$ and $w_s$ be, respectively, the anti-plane displacement in the linear-elastic region E and softening region S. These two solutions take different forms.

### 4.1. Anti-plane displacements

In the linear-elastic region, the anti-plane displacement is given by [6]

$$w_e=\sqrt{2r_1}\,\sin(\theta_1/2), \tag{11}$$

in which $r_1$ and $\theta_1$ are the polar coordinates with origin at $x=3/2$ and $y=0$ as given in Fig. 3. Replacing $r_1$ and $\theta_1$, respectively, by $r$ and $\theta$ and expanding Eq. (11) at $r=0$, there results

$$w_e=\sqrt{3}-\frac{1}{\sqrt{3}}r\cos\theta+O(r^2). \tag{12}$$

In the softening region, the anti-plane displacement is

$$w_s=\frac{1}{r}f(\theta), \tag{13}$$

in which

$$f(\theta)=\frac{1}{\sqrt{2}}\frac{\left[3\cos\theta+\psi(\theta)\right]^{3/2}}{\left[\cos\theta+\psi(\theta)\right]^{1/2}}\sin\theta. \tag{14}$$

The function $\psi(\theta)$ stands for

$$\psi(\theta)=\left(9\cos^2\theta-8\right)^{1/2}. \tag{15}$$

### 4.2. Elastic / unloading / softening boundary

When unloading occurs, the asymptotic expression in Eqs. (11) and (13) remain valid asymptotically for the linear-elastic and softening region, respectively. The boundary $\Gamma_e$ is between the linear-elastic zone E and unloading zone U as shown in Fig. 4. It is apparent that unloading cannot take place in the linear-elastic zone without entering into the regime of softening. Non-dimensional variables have been used to invoke self-similarity which reveals that $\Gamma_e$ must be a straight line originating from the crack tip.

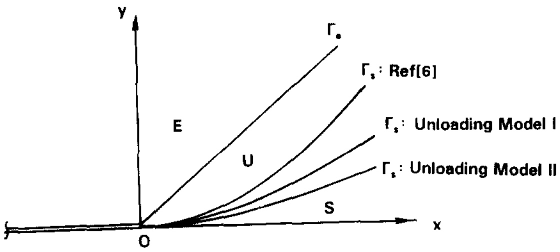


Fig. 4. Divisions of elastic, unloading and softening zones near crack tip.

Without loss in generality, the boundary $\Gamma_s$ between the unloading and softening zone can be expressed as

$$r = g(\theta). \tag{16}$$

Since $\Gamma_s$ passes through the crack tip, the condition

$$g(0) = 0 \tag{17}$$

follows. Strain continuity on $\Gamma_s$ can be imposed on $w_s$ as

$$\gamma_r^{t+0} = \gamma_r^{t-0} = \left(\frac{\partial w_s}{\partial r}\right)_{r=g(\theta)} = -f(\theta)/g^2(\theta),$$
$$\gamma_\theta^{t+0} = \gamma_\theta^{t-0} = \left(\frac{1}{r}\frac{\partial w_s}{\partial \theta}\right)_{r=g(\theta)} = \frac{\partial f}{\partial \theta}g^2(\theta). \tag{18}$$

The resultant strain $\gamma$ becomes

$$\gamma^{t+0} = \gamma^{t-0} = \phi(\theta)/g^2(\theta), \tag{19}$$

in which $\phi(\theta)$ stands for

$$\phi(\theta) = \left[f^2(\theta) + (\mathrm{d}g/\mathrm{d}\theta)^2\right]^{1/2}. \tag{20}$$

Only the variable $\theta$ is involved to account for strains at the start of unloading. Since $w_s$ in Eq. (13) must be finite at the crack tip and $f(\theta) = 0$ for $\theta = 0$, the interface $\Gamma_s$ must be tangent to the $x$-axis at $r = 0$, i.e.,

$$\lim_{\theta \to 0}\left[g(\theta) \Big/ \frac{\mathrm{d}g}{\mathrm{d}\theta}\right] = 0. \tag{21}$$

The function $f(\theta)$ in Eq. (14) and its first derivative $\mathrm{d}f/\mathrm{d}\theta$ can be expanded at $\theta = 0$ as follows:

$$f(\theta) = 4\theta\left[1 - \tfrac{7}{6}\theta^2 + O(\theta^4)\right],$$
$$\frac{\mathrm{d}f}{\mathrm{d}\theta} = 4\left[1 - \tfrac{7}{2}\theta^2 + O(\theta^4)\right]. \tag{22}$$

Now, Eqs. (22) may be inserted into Eq. (20) to yield $\phi(\theta)$ at $\theta = 0$:

$$\phi(\theta) = 4\left[1 - 3\theta^2 + O(\theta^4)\right]. \tag{23}$$

## 5. Solution in unloaded region

Let $w_u$ be the anti-plane displacement in the unloaded region. Expand $w_s$ near the crack tip $r = 0$ in a series:

$$w_u = w_0 + \sum_{m=1}^{\infty} r^m u_m(\theta). \tag{24}$$

Satisfaction of the equations of equilibrium given by

$$\frac{\partial}{\partial r}(r\tau_r) + \frac{\partial \tau_\theta}{\partial \theta} = 0 \tag{25}$$

can be made by making use of stress–strain and strain–displacement relations.

### 5.1. Variable unloading moduli (Model I)

Substituting the second of Eqs. (18) into Eq. (7), there results

$$G_u = g^3(\theta)\phi^{-3/2}(\theta). \tag{26}$$

Keep in mind that $G_u$ depends on $\theta$. Eq. (6) can be used in conjunction with Eqs. (3) for $w = w_u$ to express Eq. (25) as

$$G_u\frac{\partial^2 w_u}{\partial \theta^2} + \frac{\mathrm{d}G_u}{\mathrm{d}\theta}\frac{\partial w_u}{\partial \theta} + G_u\left[r^2\frac{\partial^2 w_u}{\partial r^2} + r\frac{\partial w_u}{\partial r}\right] = 0. \tag{27}$$

Making use Eqs. (24) and (26), Eq. (27) solves for $u_m(\theta)$ in the unloading zone:

$$\frac{\mathrm{d}^2 u_m}{\mathrm{d}\theta^2} + p(\theta)\frac{\mathrm{d}u_m}{\mathrm{d}\theta} + m^2 u_m = 0. \tag{28}$$

Consider the contraction

$$p(\theta) = \frac{dG_u}{d\theta}/G_u = p_1(\theta) + p_2(\theta) \tag{29}$$

such that

$$p_1(\theta) = 3\left(\frac{dg}{d\theta}/g\right), \quad p_2(\theta) = \frac{d\phi}{d\theta}/\phi. \tag{30}$$

Eq. (28) is a second order homogeneous differential equation. Since $p_1(\theta)$ depends on $g(\theta)$ of the unknown interface $\Gamma_s$, the solution is made more difficult. Hence, only a local solution near $\theta = 0$ is obtained. To begin with, let the asymptotic expression of $g(\theta)$ satisfying Eqs. (17) and (21) be of the form

$$r = g(\theta) = A\theta^n\left[1 + A_1\theta + A_2\theta^2 + O(\theta^3)\right],$$
$$n > 0, \tag{31}$$

where $A$, $A_1$ and $A_2$ are constants to be determined. To find $n$, substitute the first of Eqs. (22) and (31) into Eq. (13); the displacement on the softening side of $\Gamma_s$ becomes

$$w_s\,|_{\Gamma_s} = \frac{4}{A}\theta^{-n+1}[1 + O(\theta)]. \tag{32}$$

The displacement $w_u$ on the unloading side of $\Gamma_s$ can be obtained by substituting Eq. (31) into Eq. (24). This gives

$$w_u\,|_{\Gamma_s} = w_0 + \sum_{m=1}^{\infty} A^m\theta^{nm}[1 + O(\theta)]u_m(\theta). \tag{33}$$

Finiteness of the displacement implies that $w \sim O(1)$ must be the dominant term regardless of the form of $u_m(\theta)$. The displacements $w_s$ and $w_u$ must be continuous across $\Gamma_s$; their dominant terms must have the same order. Therefore, $n = 1$ and hence Eq. (31) can be written as

$$r = g(\theta) = A\theta\left[1 + A_1\theta + A_2\theta^2 + O(\theta^3)\right]. \tag{34}$$

Substituting Eq. (23) and Eq. (34) into Eq. (26), the asymptotic expression of the unloading modulus is given by

$$G_u(\theta) = \frac{1}{8}A^3\theta^3\left[1 + 3A_1\theta + 3\left(A_1^2 + A_2 + \tfrac{3}{2}\right)\theta^2 + O(\theta^3)\right]. \tag{35}$$

With the aid of Eqs. (34) and (23), Eqs. (30) become

$$p_1(\theta) = 3\theta^{-1}\left[1 + A_1\theta + \left(2A_2 - A_1^2\right)\theta^2 + O(\theta^3)\right],$$
$$p_2(\theta) = 9\theta\left[1 + O(\theta^2)\right] \tag{36}$$

such that Eq. (29) takes the form

$$p(\theta) = 3\theta^{-1}\left[1 + A_1\theta + \left(2A_2 - A_1^2 + 3\right)\theta^2 + O(\theta^3)\right]. \tag{37}$$

The two linearly independent solutions [1] $u_m^{(1)}$ and $u_m^{(2)}$ in Eq. (28) can be obtained as

$$u_m^{(1)}(\theta) = 1 - \tfrac{1}{8}m^2\theta^2 + O(\theta^2),$$
$$u_m^{(2)}(\theta) = \theta^{-2}\left[1 - 6A_1, \theta - \left(12A_1^2 - 6A_2 + \frac{m^2}{2} - 9\right)\theta^2 \log\theta + O(\theta^2)\right]. \tag{38}$$

The general solution of Eq. (28) is

$$\begin{aligned} u_m(\theta) &= a_m u_m^{(1)}(\theta) + b_m u_m^{(2)}(\theta) \\ &= a_m\left[1 - \tfrac{1}{8}m^2\theta^2 + O(\theta^2)\right] \\ &\quad + b_m\theta^{-2}\left[1 - 6A_1\theta - \left(12A_1^2 - 6A_2 + \tfrac{1}{2}m^2 - 9\right)\theta^2 \log\theta + O(\theta^2)\right], \end{aligned} \tag{39}$$

where $a_m$ and $b_m$ are constants to be determined. Differentiating Eq. (39), the result is

$$\begin{aligned} \frac{du_m}{d\theta} &= a_m\left[-\tfrac{1}{4}m^2\theta + O(\theta)\right] \\ &\quad - 2b_m\theta^{-3}\left[1 - 3A_1\theta + \left(6A_1^2 - 3A_2 + \frac{m^2}{4} - \frac{9}{2}\right)\theta^2 + O(\theta^2)\right]. \end{aligned} \tag{40}$$

Note that, if $b_m \neq 0$, terms with $a_m$ are of higher order and can be neglected in Eqs. (39) and (40).

[1] The second of Eqs. (55) in [2] for $w_m$ should be replaced by the second of Eqs. (38).

In view of the continuity conditions of $w_e$ and $w_u$ on $\Gamma_e$, set $r \to 0$ in Eqs. (12) and (24), there renders

$$w_0 = \sqrt{3}. \tag{41}$$

Eqs. (34) and (39) may now be inserted into Eq. (24). This leads to

$$w_u|_{\Gamma_s} = Ab_1\theta^{-1} + (w_0 + A^2b_2 - 5AA_1b_1) + Ab_1\left(12A_1^2 - 6A_2 - \frac{17}{2}\right)\theta \log\theta + O(\theta). \tag{42}$$

Putting the first of Eqs. (22) and (34) into Eq. (13), it is found that

$$w_s|_{\Gamma_s} = \frac{4}{A} + O(\theta). \tag{43}$$

Comparing the coefficients of $\theta^1$ and $\theta^0$ terms in the above two equations and referring to Eq. (41), two relations for $b_1$ and $b_2$ are found:

$$b_1 = 0, \qquad A^3b_2 = 4 - \sqrt{3}A. \tag{44}$$

Making use of the second of Eqs. (22), (40) and (44), the first derivatives of $w_u$ and $w_s$ with respect to $\theta$ on $\Gamma_s$ are derived:

$$\left.\frac{\partial w_u}{\partial\theta}\right|_{\Gamma s} = -2A^2b_2\theta^{-1} + 2A^2(A_1b_2 - Ab_3) - \left[2A^2b_2\left(A_1^2 - A_2 - \tfrac{7}{2}\right) + 2A^4b_4\right]\theta + O(\theta), \tag{45}$$

$$\left.\frac{\partial w_s}{\partial\theta}\right|_{\Gamma_s} = \frac{4}{A}\theta^{-1} - \frac{4A_1}{A} + \frac{4}{A}\left(A_1^2 - A_2 - \tfrac{7}{2}\right)\theta + O(\theta^2).$$

Comparison of the coefficients of $\theta^{-1}$, $\theta$ and constant terms yield

$$A^3b_2 = -2,$$
$$A^4b_3 = A^3b_2A_1 + 2A_1, \tag{46}$$
$$A^5b_4 = -(A^3b_2 + 2)\left(A_1^2 - A_2 - \frac{7}{2}\right).$$

The results in Eqs. (46) combined with those in Eqs. (44) can be applied to obtain

$$A = 2\sqrt{3}, \qquad b_2 = -\frac{1}{12\sqrt{3}}, \qquad b_3 = b_4 = 0. \tag{47}$$

With the results in Eqs. (41), (44) and (47), the local displacement for the unloading zone is

$$w_u = \sqrt{3} + a_1\left[1 - \tfrac{1}{8}\theta^2 + O(\theta^2)\right] - \frac{1}{12\sqrt{3}}\theta^{-2}[1 + O(\theta)]r^2 + a_3\left[1 - \tfrac{9}{8}\theta^2 + O(\theta^2)\right]r^3 + a_4\left[1 - 2\theta^2 + O(\theta^2)\right]r^4 + O(r^5). \tag{48}$$

The shape of the interface $\Gamma_s$ can be obtained from Eqs. (34) and (47) as

$$r = \frac{6}{\sqrt{3}}\theta[1 + O(\theta)]. \tag{49}$$

It follows from Eqs. (35) and (47) that the asymptotic expression of the unloading modulus can be written as

$$G_u(\theta) = 3\sqrt{3}\,\theta^3[1 + O(\theta)]. \tag{50}$$

Eqs. (48) and (50) can be used to obtain the local strains and stresses. When both $r$ and $\theta$ are small of the same order $O(\epsilon)$ and $a_1$, $a_3$ and $a_4$ are of the same order $O(1)$, the displacement takes the form

$$w_u = \sqrt{3} - \frac{1}{12\sqrt{3}}\left(\frac{r}{\theta}\right)^2 + O(\epsilon). \tag{51}$$

The strains are

$$\begin{Bmatrix}\gamma_r\\ \gamma_\theta\end{Bmatrix} = \frac{r}{6\sqrt{3}}\begin{Bmatrix}-\theta^{-2}\\ \theta^{-3}\end{Bmatrix} + \begin{Bmatrix}O(1)\\ O(\epsilon^{-1})\end{Bmatrix} \tag{52}$$

and stresses are

$$\begin{Bmatrix}\tau_r\\ \tau_\theta\end{Bmatrix} = \frac{r}{2}\begin{Bmatrix}-\theta^{-1}\\ 1\end{Bmatrix} + \begin{Bmatrix}O(\epsilon^3)\\ O(\epsilon^2)\end{Bmatrix}. \tag{53}$$

### 5.2. *Constant unloading modulus (Model II)*

Substituting the constitutive relation in Eq. (9) into the equilibrium equation (25) gives

$$\left(r\frac{\partial}{\partial r} + 1\right)\left\{\gamma_r + \gamma_r^{t+0}\left[\left(\gamma^{t+0}\right)^{-3/2} - 1\right]\right\} + \frac{\partial}{\partial\theta}\left\{\gamma_\theta + \gamma_\theta^{t+0}\left[\left(\gamma^{t+0}\right)^{-3/2} - 1\right]\right\} = 0.$$

Eqs. (18) show that $\gamma_r^{t+0}$ and $\gamma_\theta^{t+0}$ depend only on $\theta$. Hence, the above equation can be applied to obtain a second order partial differential equation for $w_u$:

$$\frac{\partial}{\partial r}\left(r\frac{\partial w_u}{\partial r}\right)+\frac{1}{2}\frac{\partial^2 w_u}{\partial\theta^2}=F(\theta), \tag{54}$$

in which

$$F(\theta)=\gamma_r^{t+0}\left[1-\left(\gamma^{t+0}\right)^{-3/2}\right] + \frac{\partial}{\partial\theta}\left\{\gamma_\theta^{t+0}\left[1-\left(\gamma^{t+0}\right)^{-3/2}\right]\right\}. \tag{55}$$

If $w_u^a$ and $w_u^c$ denote the particular solution, and the general solution of Eq. (54), respectively, then

$$w_u=w_u^a+w_u^c, \tag{56}$$

such that

$$\frac{\partial}{\partial r}\left(r\frac{\partial w_u^a}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 w_u^a}{\partial\theta^2}=F(\theta),$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial w_u^c}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 w_u^c}{\partial\theta^2}=0. \tag{57}$$

The product solution

$$w_u^a=rf_1(\theta) \tag{58}$$

when substituted into the first of Eqs. (57) gives

$$\frac{d^2 f_1}{d\theta^2}+f_1(\theta)=F(\theta). \tag{59}$$

Eqs. (18) and (19) may be put into Eq. (55) so that

$$F(\theta)=f(\theta)g(\theta)[\phi(\theta)]^{-3/2}-\frac{f(\theta)}{g^2(\theta)} - \frac{d}{d\theta}\left\{\frac{df}{d\theta}g(\theta)[\phi(\theta)]^{-3/2}\right\} + \frac{d}{d\theta}\left[\frac{1}{g^2(\theta)}\frac{df}{d\theta}\right]. \tag{60}$$

It can be estimated from Eqs. (22), (23) and (34) that Eq. (60) with terms up to the order of $O(\theta^{-2})$ can be retained to render

$$F(\theta)=-8A^{-2}\theta^{-3}\left[1-A_1\theta+O(\theta^2)\right]. \tag{61}$$

Eq. (59) simplifies to

$$\frac{d^2 f_1}{d\theta^2}+f_1(\theta)+8A^{-2}\theta^{-3}\left[1-A_1\theta+O(\theta^2)\right]=O, \tag{62}$$

the solution of which is

$$f_1(\theta)=-4A^{-2}\theta^{-1}\left[1+2A_1\theta\log\theta + O(\theta\log\theta)\right]. \tag{63}$$

The particular solution in Eq. (58) follows

$$w_u^a=-4A^{-2}r\theta^{-1}\left[1+2A_1\theta\log\theta + O(\theta\log\theta)\right]. \tag{64}$$

The differential operators in the second of Eqs. (57) is Laplacian so that $w_u^c$ can be assumed as follows

$$w_u^c=w_0^c+\sum_{m=1}^{\infty}r^m(U_m\sin m\theta+V_m\cos m\theta), \tag{65}$$

where $w_0^c$, $U_m$ and $V_m$ are coefficients to be determined by the continuity conditions between the different zones. Substitute Eqs. (64) and (65) into Eq. (56) and observe the order of the $r$-terms in $w_u^c$ and $w_u^a$, the total displacement in the unloading zone can be written as

$$w_u=w_0^c-4A^{-2}r\theta^{-1}\left[1+2A_1\theta\log\theta + O(\theta\log\theta)\right]+\sum_{m=2}^{\infty}r^m\left[U_m\sin m\theta + V_m\cos m\theta\right]. \tag{66}$$

On $\Gamma_e$, the continuity conditions of $w_e$ in Eq. (12) and $w_u$ in Eq. (66) require $w_0^c=\sqrt{3}$. Eq. (34) can be used to eliminate $r$ in $w_u$ and hence

$$w_u|_{\Gamma_s}=\sqrt{3}-4A^{-1}\left[1+2A_1\theta\log\theta + O(\theta\log\theta)\right]. \tag{67}$$

Comparing the coefficients of $\theta^0$ and $\theta\log\theta$ in Eqs. (43) and (67), $A$ and $A_1$ are found:

$$A=8\sqrt{3},\qquad A_1=0. \tag{68}$$

The continuity condition of $\partial w/\partial\theta$ on $\Gamma_s$ is therefore satisfied. The displacement of unloading

Mode II can be obtained by substituting Eq. (68) into Eq. (66), i.e.,

$$w_u = \sqrt{3} - \frac{3}{16}\frac{r}{\theta}[1 + O(\theta \log \theta)] + O(r^2). \tag{69}$$

The equation for $\Gamma_s$ is

$$r = \frac{8}{\sqrt{3}}\theta\left[1 + O(\theta^2)\right]. \tag{70}$$

The strains in unloading zone are given as

$$\begin{Bmatrix} \gamma_r \\ \gamma_\theta \end{Bmatrix} = \frac{3}{16}\begin{Bmatrix} -\theta^{-1} + O(\log \theta) \\ \theta^{-2} + O(\theta^{-1}) \end{Bmatrix} + O(r). \tag{71}$$

## 6. Concluding remarks

The foregoing results show that there prevails an unloading zone between the softening region and linear-elastic region. The asymptotic expression of the softening zone boundary in [6] can be written as

$$r = \frac{4}{\sqrt{3}}\theta, \qquad \text{as} \quad r \to 0, \tag{72}$$

which is plotted in Fig. 4 together with the softening zone boundaries described by Eqs. (49) and (70). Unloading tends to reduce the tangent of $\Gamma_s$ at the crack tip. In addition, softening zone boundary $\Gamma_s$ for the unloading Model II is closer to the $x$-axis than that of the unloading Model I.

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