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A NEW METHOD FOR EVALUATION OF STRESS INTENSITIES FOR INTERFACE CRACKS

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Abstract—A new method is presented for calculating the values of K_I and K_{II} in the elasticity solution at the tip of an interface crack. The method is based on an evaluation of the J-integral by the virtual crack extension method. Expressions for calculating K_I and K_{II} by using the displacements and the stiffness derivative of the finite element solution and asymptotic crack tip displacements are derived. The method is shown to produce very accurate solutions even with coarse element mesh.

INTRODUCTION

AS THE DEMANDS made upon the strength and durability of today's engineering materials increase, the advantage of combining the properties of two or more materials into a single member have become apparent. For these material systems it is the low fracture toughness that limits their use in engineering and structural components. The need to understand, quantify and improve the toughness of composite materials has renewed interest in the elastic interface crack problem. The presence of cracks on bimaterial interfaces presents special analytical problems not encountered in cracked homogeneous bodies. In single material bodies, stress intensification arises solely from a geometric discontinuity, the crack. The bimaterial body, in contrast, produces stress intensification from a geometric discontinuity and material discontinuity. These discontinuities induce K_I and K_{II} intensification for single mode loading. This coupling of stress intensification was first demonstrated by Williams [1]. He also showed that the stresses behave in an oscillatory manner as the crack tip is approached. Further analysis of crack bimaterial bodies by Cherpanov [2], England [3], Erdogan [4] and Rice and Sih [5] has given yield stress intensity factors for some simple geometries and loading.

Because of the complexity of these analyses, numerical procedures are a necessity when stress intensities are desired for more general configuration and loading. Lin and Mar [6] and Van der Zande and Grootenboer [7] used special hybrid crack tip finite elements to obtain K_I and K_{II} for the interface crack. In this case, standard finite element code cannot be used directly. Complicated computational procedures are required. Smelser [8] presented a method for obtaining the stress intensity factor for bimaterial bodies using numerical crack flank displacement data. The method is able to resolve the magnitude of the stress intensity factor from the finite element method with accuracy; the resolution of the angle is not satisfactory when the angles are small. Wang and Yan [9] developed a technique for obtaining K_I and K_{II} in bimaterial fracture by using the M-integral of Chen and Shield [10]. Matos *et al.* [11] presented a numerical method for obtaining the values K_I and K_{II} in the elasticity solution at the tip of an interface crack. The basis of the method is an evaluation of the J-integral by the virtual crack extension method. Individual stress intensities can then be obtained from further calculation of J perturbed by small increments of the stress intensity factor. Numerical examples have shown that values of K_I and K_{II} depend on the increments ΔK_I and ΔK_{II}.

In this paper, new expressions for obtaining K_I and K_{II} by using the method of Matos *et al.* are presented. It can show that the results of K_I and K_{II} are independent of the increments ΔK_I and ΔK_{II}. The new expressions can be implemented with very little programming effort by adding a subroutine to any existing finite element code.

FORMULATION

The crack tip geometry is shown in Fig. 1. Local polar and Cartesian coordinate systems have been introduced at crack tip. The traction at a distance r ahead of the crack tip takes the form [12],

$$(\sigma_{yy} + i\sigma_{xx})_{\theta=0} = kr^{i\varepsilon}/\sqrt{(2\pi r)} \tag{1}$$

The ε in eq. (1) is the bimaterial constant.

$$\varepsilon = (1/2\pi)\ln[(\kappa_1\mu_2 + \mu_1)/(\kappa_2\mu_1 + \mu_2)]. \tag{2}$$

Subscripts 1 and 2 refer to the materials in $y > 0$ and $y < 0$, respectively, as in Fig. 1, $\kappa = 3 - 4\nu$ for plane strain and $\kappa = (3 - \nu)/(1 + \nu)$ for plane stress, ν = Poisson ratio, and μ = shear modulus.

The fracture analysis of interface cracks has been complicated by the oscillatory feature of the near-tip deformation field. For an absolute characterization of the interface stress intensity factor, Rice [13] introduced a scaling length $\hat{r}$ so eq. (1) may be rewritten as

$$(\sigma_{yy} + i\sigma_{xx})_{\theta=0} = K(r/\hat{r})^{i\varepsilon}/\sqrt{(2\pi r)}, \tag{3}$$

where the $K = k\hat{r}^{i\varepsilon}$ shall be defined as the stress intensity factor for the interface crack. As pointed out by Rice, the scaling length $\hat{r}$ may be chosen arbitrarily as long as it is held fixed when specimens of a given material pair but with different loading and geometry conditions are considered. Different values of $\hat{r}$ will not alter the magnitude of K but will change its phase angle. Since the oscillation index ε is typically very small, the variable quantity $(r/\hat{r})^{i\varepsilon} = \exp[i\varepsilon \ln(r/\hat{r})]$ has a very weak variation with r. Thus it may define the mode I and II interface stress intensity factors K_I and K_{II} as

$$K = K_I + iK_{II} = K\hat{r}^{i\varepsilon}. \tag{4}$$

It should be understood that the definition of eq. (4) rigorously reduces to that of the classical mode K_I, K_{II} stress intensity factors only when $\varepsilon = 0$, but for simplicity we will use the same stress intensity factor notations (K_I, K_{II} and $K = K_I + iK_{II}$) for both homogeneous and interface cracks. The J-integral [14] is defined as

$$J = \int_\Gamma W\,dy - n_i\sigma_{ij}\frac{\partial u_j}{\partial x}\,ds, \tag{5}$$

where Γ is any contour from the bottom crack surface around the tip to the top surface, W is the strain energy density, σ_{ij} the stress tensor, u_i the displacement vector, and n_i the outward unit normal to the contour. The conservation integral J has been extended by Smelser and Gurtin [15] to bimaterial bodies proved that the crack surfaces are free from traction and the interface is a straight line. The J-integral is related to stress intensity factors of an interfacial crack by

$$J = (K_I^2 + K_{II}^2)/H, \tag{6}$$

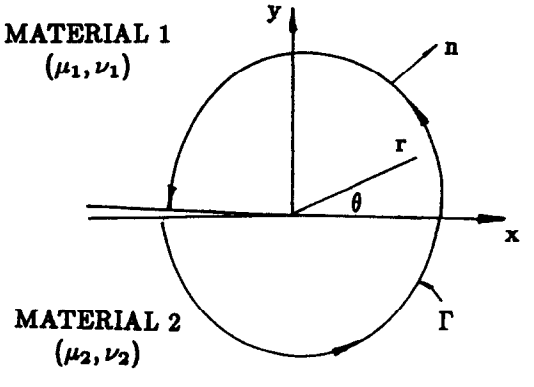

Fig. 1. Coordinates and typical contour used to evaluate the J-integral.

Table 1. Material property ratios for numerical study

Case no.	E_1/E_2	ν_1/ν_2
1	100/10	0.3/0.3
2	220/10	0.3/0.35
3	1000/10	0.3/0.3

Table 2. Stress intensities and phase angle in degrees

	Case 1			Case 2			Case 3		
	ref. [5]	Author	Err.%	ref. [5]	Author	Err.%	ref. [5]	Author	Err.%
K_I	1.7903	1.8110	1.16	1.7916	1.8107	1.07	1.7987	1.8165	0.99
K_{II}	−0.2216	−0.2193	1.04	−0.2240	−0.2265	1.11	−0.2625	−0.2652	1.03
ψ	−6.8982	−6.9057	0.11	−7.1270	−7.1304	0.05	−8.3038	−8.3066	0.03

where

$$H = 16\cosh^2(\pi\varepsilon)/(c_1 + c_2),$$

$$c_i = (\kappa_i + 1)/\mu_i.$$

Following Parks [16], the energy release rate definition of the J-integral can be used to develop a numerical method for its evaluation. As a result

$$J = -(\partial U/\partial a)_F = -(1/2)\{\mathbf{u}_n\}^T(\partial[S]/\partial a)\{\mathbf{u}_n\}. \tag{7}$$

Here, U is the potential energy of the body and differentiation with respect to crack length a is carried out at fixed load. The vector $\{\mathbf{u}_n\}$ contains as elements the nodal degree of freedom for a finite element calculation and $[S]$ is the stiffness matrix for the crack problem. In the virtual crack extension method [16] the crack problem is first solved to obtain the vector $\{\mathbf{u}_n\}$. Then a small virtual crack extension is caused in the plane of the crack and a new value of the stiffness computed. Usually the crack is extended by rigidly moving a core of elements around the tip and distorting only one ring such as the shaded one shown in Fig. 2. That is, all elements outside the distorted ring are also held rigid. As a result, the computations of changed stiffness required are limited and the multiplication involved in eq. (7) involves small vectors and matrices.

Now consider a bimaterial problem A for which it is desired to compute K_I and K_{II}. First, solve the problem by the finite element method to find $\{\mathbf{u}_n\}$ and $\partial[S]/\partial a$. Then add to displacements $\{\Delta\mathbf{u}_n^I\}$ for a problem in the same geometry for which $K_{II} = 0$ and $K_I = \Delta K_I$. This set of displacements can represent any problem desired and it should be noted that the field is actually needed only for the part associated with the distorted ring of elements. In view of this, the asymptotic crack tip displacements can be used everywhere as a suitable field. That is

$$\Delta\mathbf{u}_i^I = \frac{\Delta K_I\sqrt{r}}{4\mu_i\lambda_0^2\sqrt{(2\pi)\cosh(\pi\varepsilon)}}\left\{\delta_i\left[\frac{1}{2}\kappa_i\cos\left(\alpha - \frac{\theta}{2}\right) + \varepsilon\kappa_i\sin\left(\alpha - \frac{\theta}{2}\right) + 2\lambda_0^2\sin\theta\sin\left(\alpha + \frac{\theta}{2}\right)\right]\right.$$
$$\left. - \frac{1}{\delta_i}\left[\frac{1}{2}\cos\left(\alpha + \frac{\theta}{2}\right) + \varepsilon\sin\left(\alpha + \frac{\theta}{2}\right)\right]\right\} \tag{8a}$$

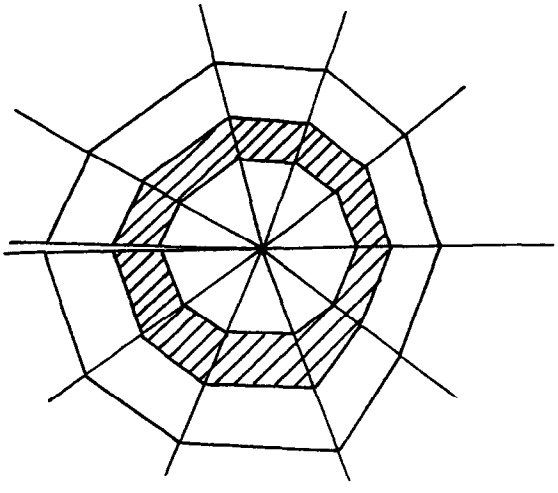

Fig. 2. A typical ring of elements to be distorted in the J-integral.

$$\Delta \mathbf{v}_i^{\mathrm{I}} = \frac{\Delta K_{\mathrm{I}}\sqrt{r}}{4\mu_i\lambda_0^2\sqrt{(2\pi)\cosh(\pi\varepsilon)}}\left\{\delta_i\left[-\frac{1}{2}\kappa_i\sin\left(\alpha-\frac{\theta}{2}\right)+\varepsilon\kappa_i\cos\left(\alpha-\frac{\theta}{2}\right)-2\lambda_0^2\sin\theta\cos\left(\alpha+\frac{\theta}{2}\right)\right]\right.$$

$$\left.+\frac{1}{\delta_i}\left[\frac{1}{2}\sin\left(\alpha+\frac{\theta}{2}\right)-\varepsilon\cos\left(\alpha+\frac{\theta}{2}\right)\right]\right\} \tag{8b}$$

where

$$\alpha = \varepsilon\ln(r/\hat{r}), \tag{9a}$$

$$\delta_1 = \mathrm{e}^{-(\pi-\theta)\varepsilon}, \quad \delta_2 = \mathrm{e}^{(\pi+\theta)\varepsilon}, \tag{9b}$$

$$\lambda_0^2 = 0.25 + \varepsilon^2. \tag{9c}$$

The vector $\{\Delta\mathbf{u}_i^{\mathrm{I}}\}$ is obtained by evaluating eqs (8a) and (8b) at the required nodes. For the superimposed state we have

$$[(K_{\mathrm{I}}+\Delta K_{\mathrm{I}})^2 + K_{\mathrm{II}}^2]/H = -(1/2)\{\mathbf{u}_n+\Delta\mathbf{u}_n^{\mathrm{I}}\}^{\mathrm{T}}(\partial[S]/\partial a)\cdot\{\mathbf{u}_n+\Delta\mathbf{u}_n^{\mathrm{I}}\}. \tag{10}$$

For the problem A and additional state we have

$$(K_{\mathrm{I}}^2+K_{\mathrm{II}}^2)/H = -(1/2)\{\mathbf{u}_n\}^{\mathrm{T}}(\partial[S]/\partial a)\{\mathbf{u}_n\}, \tag{11}$$

$$\Delta K_{\mathrm{I}}^2/H = -(1/2)\{\Delta\mathbf{u}_n^{\mathrm{I}}\}^{\mathrm{T}}(\partial[S]/\partial a)\{\Delta\mathbf{u}_n^{\mathrm{I}}\}. \tag{12}$$

Subtracting eqs (11) and (12) from eq. (10), one obtains the following expression

$$K_{\mathrm{I}} = -\frac{H}{2\Delta K_{\mathrm{I}}}\{\Delta\mathbf{u}_n^{\mathrm{I}}\}^{\mathrm{T}}\left(\frac{\partial[S]}{\partial a}\right)\{\mathbf{u}_n\}. \tag{13}$$

It is obvious that K_{I} is independent of the ΔK_{I} because $\{\Delta\mathbf{u}_n^{\mathrm{I}}\}$ is proportional to ΔK_{I}.

The procedure can then be repeated for an added vector such that $K_{\mathrm{I}} = 0$ and $K_{\mathrm{II}} = \Delta K_{\mathrm{II}}$, if desired.

These displacements are such that

$$\Delta\mathbf{u}_i^{\mathrm{II}} = \frac{\Delta K_{\mathrm{II}}\sqrt{r}}{4\mu_i\lambda_0^2\sqrt{(2\pi)\cosh(\pi\varepsilon)}}\left\{\delta_i\left[-\frac{1}{2}\kappa_i\sin\left(\alpha-\frac{\theta}{2}\right)+\varepsilon\kappa_i\cos\left(\alpha-\frac{\theta}{2}\right)+2\lambda_0^2\sin\theta\cos\left(\alpha+\frac{\theta}{2}\right)\right]\right.$$

$$\left.+\frac{1}{\delta_i}\left[\frac{1}{2}\sin\left(\alpha+\frac{\theta}{2}\right)-\varepsilon\cos\left(\alpha+\frac{\theta}{2}\right)\right]\right\} \tag{14a}$$

$$\Delta\mathbf{v}_i^{\mathrm{II}} = \frac{\Delta K_{\mathrm{II}}\sqrt{r}}{4\mu_i\lambda_0^2\sqrt{(2\pi)\cosh(\pi\varepsilon)}}\left\{\delta_i\left[-\frac{1}{2}\kappa_i\cos\left(\alpha-\frac{\theta}{2}\right)-\varepsilon\kappa_i\sin\left(\alpha+\frac{\theta}{2}\right)+2\lambda_0^2\sin\theta\sin\left(\alpha+\frac{\theta}{2}\right)\right]\right.$$

$$\left.+\frac{1}{\delta_i}\left[\frac{1}{2}\cos\left(\alpha+\frac{\theta}{2}\right)+\varepsilon\sin\left(\alpha+\frac{\theta}{2}\right)\right]\right\}. \tag{14b}$$

Then it follows that

$$K_{\mathrm{II}} = -\frac{H}{2\Delta K_{\mathrm{II}}}\{\Delta\mathbf{u}_n^{\mathrm{II}}\}^{\mathrm{T}}\left(\frac{\partial[S]}{\partial a}\right)\{\mathbf{u}_n\}. \tag{15}$$

K_{II} is independent of ΔK_{II}.

NUMERICAL STUDY

To study the applicability of the proposed method for calculating stress intensities in bimaterial fracture, a bimaterial plane stress plate that contains an interface crack of two length units and is subjected to unit biaxial tensile stress was investigated (Fig. 3). For simplicity, units are omitted implying that an appropriate normalization has been carried out. The scaling length $\hat{r}$ is one length unit. Because of the symmetry of the geometry and the load, only half the model was analysed. One of the meshes used in this analysis to obtain the finite element elasticity solution

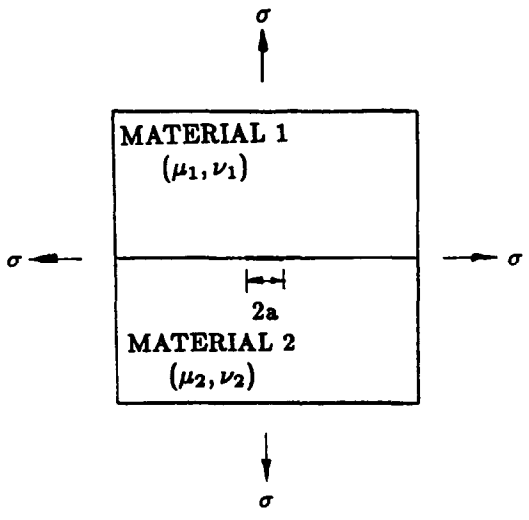

Fig. 3. Bimaterial cracked plate subjected to biaxial tension.

in a plate of 20 × 20 length units is shown in Fig. 4. It has 517 nodes and 160 isoparametric 8-noded elements with 64 distributed in the square focused mesh of 2 × 2 length units surrounding the crack tip. The collapsed singular quarter point elements molded the crack region. A 2 × 2 Gaussian quadrature integration scheme is employed to evaluate the element stiffness matrices. The theoretical expressions for stress intensity factors K_I and K_{II} for infinite body were given by Rice and Sih [5] as

$$K_I = \sigma\sqrt{(\pi a)}(\cos\beta + 2\varepsilon \sin\beta) \tag{16}$$

$$K_{II} = \sigma\sqrt{(\pi a)}(\sin\beta - 2\varepsilon \cos\beta), \tag{17}$$

where

$$\beta = \varepsilon \ln(2a/\hat{r}). \tag{18}$$

Table 1 lists the material properties ratios for three cases. A comparison between results by using eqs (7) and (9) and the analytical solution is shown in Table 2. Numerical study has shown

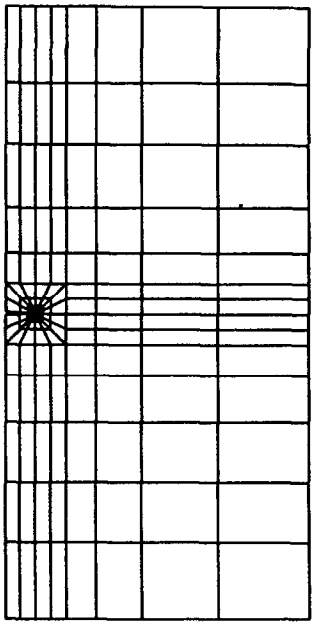

Fig. 4. The finite element mesh.

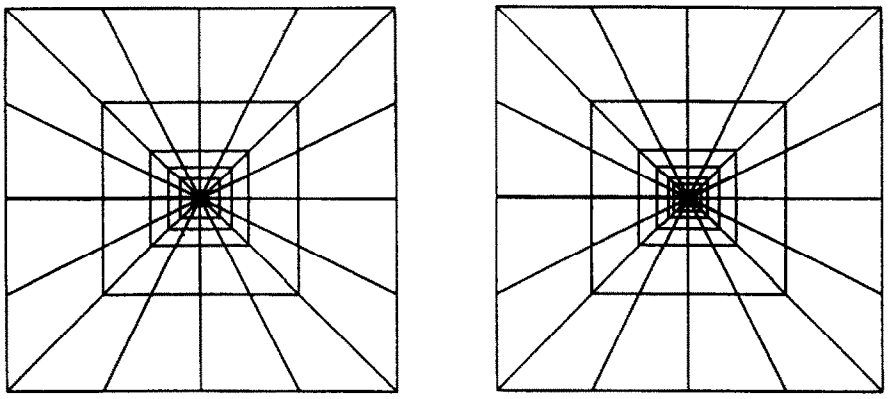

Fig. 5. Meshes with different number of elements in focused region.

Table 3. Results for different number of elements in the focused mesh

Number of elements	Case 1			Case 2			Case 3		
	K_I	K_{II}	ψ	K_I	K_{II}	ψ	K_I	K_{II}	ψ
64	1.8110	−0.2193	−6.9057	1.8107	−0.2265	−7.1304	1.8165	−0.2652	−8.3066
96	1.8108	−0.2190	−6.8965	1.8109	−0.2261	−7.1155	1.8169	−0.2648	−8.2914
128	1.8107	−0.2189	−6.8940	1.8109	−0.2260	−7.1148	1.8169	−0.2647	−8.2904

that the same results are obtained with different values of ΔK_I and ΔK_{II}. It is a natural outcome. Several calculations were done using different meshes focused at the crack tip to test the convergence and mesh dependence. Figure 5 shows the two focused meshes with 96 and 128 elements. The number of elements for the rest of the mesh was kept constant at 96 as shown in Fig. 4. Numerical results for different numbers of elements in the focused mesh are shown in Table 3. As Table 3 shows, K_I, K_{II} and ψ are insensitive to the number of focused elements.

CONCLUSIONS

A new method has been presented for calculating K_I and K_{II} in bimaterial fracture. The method produces very accurate results, even for relatively coarse meshes, and has a low sensitivity to the degree of mesh refinement near the crack tip.

The cost of computing K_I and K_{II} is generally less than a few per cent of the cost of initial elastic analysis. Furthermore, the modularity of the new algorithm permits it to be easily incorporated in any existing finite element code without changing the main body of the program.

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