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VIBRATION ANALYSIS OF LAMINATED PLATES USING A REFINED SHEAR DEFORMATION THEORY

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A previously published discrete-layer shear deformation theory is used to analyze free vibration of laminated plates. The theory includes the assumption that the transverse shear strains across any two layers are linearly dependent on each other. The theory has the same dependent variables as first order shear deformation theory, but the set of governing differential equations is of twelfth order. No shear correction factors are required. Free vibration of simply supported symmetric and antisymmetric cross-ply plates is calculated. The numerical results are in good agreement with those from three-dimensional elasticity theory.

1. INTRODUCTION

Shear deformation effects are of great importance in the vibration analysis of laminated plates. The first order shear deformation theory for laminated plates [1, 2] is the well known Reissner–Mindlin type theory. It is much more accurate than classical laminated plate theory for the prediction of natural frequencies. However, there is no improvement in the accuracy of the modal in-plane displacements and modal stresses. Furthermore, shear correction factors have to be used in order to adjust the transverse shear stiffnesses. To overcome these drawbacks, several refined shear deformation theories (displacement based) have been presented. These include the higher order laminate models based on non-linear through thickness in-plane displacement assumptions [3, 4] and the discrete layer models based on piecewise linear in-plane displacement assumptions [5–7]. As the number of layers is large in most cases, the discrete layer models are more capable of modelling the warpage of the cross-section during bending and vibration and of predicting in-plane responses. However, since the number of field equations and edge boundary conditions depends upon the number of layers, the discrete layer models are computationally expensive. In an attempt to overcome this difficulty, another model based upon a piecewise linear in-plane displacement field has been proposed [5, 8] which allows the contact conditions for the displacements and the transverse shearing stresses at the interfaces to be satisfied simultaneously. However, since in that theory the transverse shear stresses are assumed implicitly to be constant, this is not entirely satisfactory. Recently, the senior author presented a refined shear deformation theory [9]. The in-plane

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displacements are also assumed to be piecewise linear and the transverse displacement constant across the thickness. In addition, the transverse shear strains across any two different layers are assumed to be linearly dependent on each other. The theory has been used to analyze bending of laminated plates [10, 11]. The solutions of the theory were found to be in good agreement with the exact solutions of the three-dimensional theory of elasticity. The present paper deals with the free vibration analysis of laminated plates. Analytical solutions for simply supported symmetric and antisymmetric cross-ply plates are obtained. The results are also compared with those from three-dimensional elasticity theory.

2. FORMULATION

Consider a plate of constant thickness h composed of N_1+N_2+1 thin layers of anisotropic material bonded together. The origin of a Cartesian co-ordinate system is located within the central plane (x–y) with the z-axis being normal to this plane. The thickness of the ith layer is t_i ($i=-N_2,\ldots,0,\ldots,N_1$). The co-ordinate in the z-direction of the mid-plane of the ith layer is z_i. The layer corresponding to $i=0$ is determined from the condition $-t_0/2<z_0\leqslant t_0/2$; i.e., it includes the central plane of the plate. We begin with the displacement field of the ith layer,

$$u^{(i)}(x,y,z)=u_m^{(i)}(x,y)+(z-z_i)\psi_x^{(i)}(x,y),$$
$$v^{(i)}(x,y,z)=v_m^{(i)}(x,y)+(z-z_i)\psi_y^{(i)}(x,y),\qquad w^{(i)}(x,y,z)=W(x,y), \tag{1}$$

where $u_m^{(i)}$, $v_m^{(i)}$ and W denote the displacements of a point (x,y) in the mid-plane of the ith layer, and $\psi_x^{(i)}$ and $\psi_y^{(i)}$ are the rotations of the normals to the mid-plane about the y- and x-axes respectively. In fact, we assume that the deflection W is constant through the thickness of the plate. The transverse shear strain components $\gamma_{zx}^{(i)}$ and $\gamma_{yz}^{(i)}$ within the ith layer are

$$\gamma_{zx}^{(i)}=\psi_x^{(i)}+\partial W/\partial x,\qquad \gamma_{yz}^{(i)}=\psi_y^{(i)}+\partial W/\partial y. \tag{2}$$

As real transverse shear stresses are continuous between layers, we assume that the transverse shear strain components across any two different layers are linearly dependent on each other. Therefore we have

$$\gamma_{zx}^{(i)}=\lambda_{11}^{(i)}\gamma_{zx}^{(0)}+\lambda_{12}^{(i)}\gamma_{yz}^{(0)},\qquad \gamma_{yz}^{(i)}=\lambda_{21}^{(i)}\gamma_{zx}^{(0)}+\lambda_{22}^{(i)}\gamma_{yz}^{(0)}, \tag{3}$$

where $\gamma_{zx}^{(0)}$ and $\gamma_{yz}^{(0)}$ represent the transverse shear strain components within layer zero. $\lambda_{rs}^{(i)}$ ($rs=11,22,12,21$) are undetermined constants. Furthermore, the continuity of interlaminar in-plane displacements has to be preserved. Hence, the in-plane displacements of points in every layer can be expressed in terms of five unknown functions: the displacements of points on the central plane of the plate, $U(x,y)$, $V(x,y)$ and $W(x,y)$, and the rotations of the normals to the mid-plane of layer zero, $\psi_x^{(0)}(x,y)$ and $\psi_y^{(0)}(x,y)$. The expressions are

$$\begin{aligned}u^{(i)}(x,y,z)=\;&U(x,y)+[z_0+s(i)t_{11}^{(i)}]\psi_x^{(0)}(x,y)+s(i)t_{12}^{(i)}\psi_y^{(0)}(x,y)\\&-[z_i-z_0-s(i)t_{11}^{(i)}]\,\partial W/\partial x+s(i)t_{12}^{(i)}\,\partial W/\partial y+(z-z_i)[\lambda_{11}^{(i)}\psi_x^{(0)}(x,y)\\&+\lambda_{12}^{(i)}\psi_y^{(0)}(x,y)-(1-\lambda_{11}^{(i)})\,\partial W/\partial x+\lambda_{12}^{(i)}\,\partial W/\partial y],\\v^{(i)}(x,y,z)=\;&V(x,y)+[z_0+s(i)t_{22}^{(i)}]\psi_y^{(0)}(x,y)+s(i)t_{21}^{(i)}\psi_x^{(0)}(x,y)\\&-[z_i-z_0-s(i)t_{22}^{(i)}]\,\partial W/\partial y+s(i)t_{21}^{(i)}\,\partial W/\partial x+(z-z_i)[\lambda_{22}^{(i)}\psi_y^{(0)}(x,y)\\&+\lambda_{21}^{(i)}\psi_x^{(0)}(x,y)-(1-\lambda_{22}^{(i)})\,\partial W/\partial y+\lambda_{21}^{(i)}\,\partial W/\partial x],\end{aligned} \tag{4}$$

where

$$t_{rs}^{(i)} = \left\{ \begin{array}{ll} \frac{1}{2}\delta_{rs}t_0 + \sum\limits_{k=1}^{i-1} \lambda_{rs}^{(k)} t_k + \frac{1}{2}\lambda_{rs}^{(i)} t_i, & i > 0 \\ \frac{1}{2}\delta_{rs}t_0 + \sum\limits_{k=i+1}^{-1} \lambda_{rs}^{(k)} t_k + \frac{1}{2}\lambda_{rs}^{(i)} t_i, & i < 0 \end{array} \right\}, \tag{5}$$

$$\delta_{rs} = \begin{Bmatrix} 1, & r = s \\ 0, & r \neq s \end{Bmatrix}, \quad s(i) = \begin{Bmatrix} 1, & i > 0 \\ 0, & i = 0 \\ -1, & i < 0 \end{Bmatrix}, \qquad rs = 11, 22, 12, 21. \tag{6, 7}$$

The sum in equation (5) vanishes in the case of $i = 1$ or -1 respectively. Define

$$\begin{gathered} \epsilon_x = \partial U/\partial x, \qquad \epsilon_y = \partial V/\partial y, \qquad \gamma_{xy} = \partial U/\partial y + \partial V/\partial x, \\ \kappa'_x = \partial\psi_x^{(0)}/\partial x, \qquad \kappa'_y = \partial\psi_y^{(0)}/\partial y, \qquad \kappa'_{xy} = \partial\psi_y^{(0)}/\partial x, \qquad \kappa'_{yx} = \partial\psi_x^{(0)}/\partial y, \\ \kappa''_x = -\partial^2 W/\partial x^2, \qquad \kappa''_y = -\partial^2 W/\partial y^2, \qquad \kappa''_{xy} = \kappa''_{yx} = -\partial^2 W/\partial x\,\partial y. \end{gathered} \tag{8}$$

Then the expression for the strain energy density per unit area of the central plane of the laminated plate is

$$\begin{aligned} E = \tfrac{1}{2}[& N_x\epsilon_x + N_y\epsilon_y + N_{xy}\gamma_{xy} + M'_x\kappa'_x + M'_y\kappa'_y + M'_{xy}\kappa'_{xy} + M'_{yx}\kappa'_{yx} + M''_x\kappa''_x \\ & + M''_y\kappa''_y + M''_{xy}(2\kappa''_{xy}) + Q'_x\gamma_{zx}^{(0)} + Q'_y\gamma_{yz}^{(0)}]. \end{aligned} \tag{9}$$

Here $N_x, N_y, \ldots, Q'_y$ are all generalized internal forces. Upon defining

$$\begin{gathered} \{N\} = [N_x \quad N_y \quad N_{xy}]^{\mathrm{T}}, \qquad \{M'\} = [M'_x \quad M'_y \quad M'_{xy} \quad M'_{yx}]^{\mathrm{T}}, \\ \{M''\} = [M''_x \quad M''_y \quad M''_{xy}]^{\mathrm{T}}, \qquad \{Q'\} = [Q'_x \quad Q'_y]^{\mathrm{T}}, \qquad \{\epsilon\} = [\epsilon_x \quad \epsilon_y \quad \gamma_{xy}]^{\mathrm{T}}, \\ \{\kappa'\} = [\kappa'_x \quad \kappa'_y \quad \kappa'_{xy} \quad \kappa'_{yx}]^{\mathrm{T}}, \qquad \{\kappa''\} = [\kappa''_x \quad \kappa''_y \quad 2\kappa''_{xy}]^{\mathrm{T}}, \qquad \{\gamma\} = [\gamma_{zx}^{(0)} \quad \gamma_{yz}^{(0)}]^{\mathrm{T}}, \end{gathered} \tag{10}$$

TABLE 1

Non-dimensional fundamental frequencies $\lambda = 10\,\omega(\rho h^2/E_T)^{1/2}$ of cross-ply square laminated plates

Lamination	N†	Analysis	E_L/E_T 3	10	20	30	40
Symmetric	3	Exact [14]	2·6474	3·2841	3·8241	4·1089	4·3006
		Present	2·6357	3·3342	3·8457	4·1464	4·3510
		FSDT [15]	2·6278	3·3192	3·8268	4·1303	4·3415
	9	Exact [14]	2·6640	3·4432	4·0547	4·4210	4·6679
		Present	2·6390	3·4169	4·0310	4·4008	4·6510
		FSDT [15]	2·6384	3·4169	4·0334	4·4058	4·6580
Antisymmetric	2	Exact [14]	2·5031	2·7938	3·0698	3·2705	3·4250
		Present	2·5174	2·8129	3·1011	3·3166	3·4860
		FSDT [15]	2·4834	2·7757	3·0824	3·3284	3·5333
	10	Exact [14]	2·6583	3·4250	4·0337	4·4011	4·6498
		Present	2·6329	3·3974	4·0075	4·3774	4·6285
		FSDT [15]	2·6335	3·4053	4·0255	4·4023	4·6577

† N = number of layers.

TABLE 2

Corresponding values of $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ for symmetric cross-ply square laminated plates

N		i	E_L/E_T 3	10	20	30	40
3	$\lambda_{11}^{(i)}$	± 1	0·421	0·457	0·477	0·491	0·502
	$\lambda_{22}^{(i)}$	± 1	0·477	0·310	0·198	0·144	0·113
9	$\lambda_{11}^{(i)}$	± 1	1·167	1·182	1·187	1·190	1·182
		± 2	0·825	0·844	0·856	0·866	0·868
		± 3	0·717	0·782	0·819	0·844	0·858
		± 4	0·248	0·312	0·344	0·363	0·375
	$\lambda_{22}^{(i)}$	± 1	0·778	0·767	0·763	0·756	0·767
		± 2	0·789	0·781	0·782	0·781	0·798
		± 3	0·424	0·386	0·374	0·371	0·378
		± 4	0·138	0·069	0·040	0·028	0·022

the expression for the overall generalized force–strain relations becomes

$$\begin{bmatrix} \{N\} \\ \{M'\} \\ \{M''\} \\ \{Q'\} \end{bmatrix} = \begin{bmatrix} [A] & [B'] & [B''] & [0] \\ [B']^{\mathrm{T}} & [D'] & [D_c] & [0] \\ [B'']^{\mathrm{T}} & [D_c]^{\mathrm{T}} & [D''] & [0] \\ [0] & [0] & [0] & [G] \end{bmatrix} \begin{bmatrix} \{\epsilon\} \\ \{\kappa'\} \\ \{\kappa''\} \\ \{\gamma\} \end{bmatrix}. \tag{11}$$

Except for the zero submatrices, the expressions for the other submatrices are given in Appendix A.

We use the principle of stationary potential energy [12] to derive the differential equations of free vibration and the equations that $\lambda_{rs}^{(i)}$ must satisfy. The former can be given as follows:

$$\begin{aligned}
&\partial N_x/\partial x + \partial N_{xy}/\partial y + \omega^2(M_{11}U + M_{13}W + M_{14}\psi_x^{(0)} + M_{15}\psi_y^{(0)}) = 0,\\
&\partial N_{xy}/\partial x + \partial N_y/\partial y + \omega^2(M_{22}V + M_{23}W + M_{24}\psi_x^{(0)} + M_{25}\psi_y^{(0)}) = 0,\\
&-\partial(Q'_x + Q''_x)/\partial x - \partial(Q'_y + Q''_y)/\partial y\\
&\qquad + \omega^2(M_{13}U + M_{23}V + M_{33}W + M_{34}\psi_x^{(0)} + M_{35}\psi_y^{(0)}) = 0,\\
&\partial M'_x/\partial x + \partial M'_{yx}/\partial y - Q'_x + \omega^2(M_{14}U + M_{24}V + M_{34}W + M_{44}\psi_x^{(0)} + M_{45}\psi_y^{(0)}) = 0,\\
&\partial M'_{xy}/\partial x + \partial M'_y/\partial y - Q'_y + \omega^2(M_{15}U + M_{25}V + M_{35}W + M_{45}\psi_x^{(0)} + M_{55}\psi_y^{(0)}) = 0,
\end{aligned} \tag{12}$$

where

$$Q''_x = \partial M''_x/\partial x + \partial M''_{xy}/\partial y, \qquad Q''_y = \partial M''_{xy}/\partial x + \partial M''_y/\partial y, \tag{13}$$

In equation (12) ω is the natural circular frequency, and M_{11}, etc., are coefficients and differential operators with respect to x and y. The expressions for them are also given in Appendix A. The set of equations (12) can be expressed in terms of the amplitudes of the displacements U, V, W and the rotations $\psi_x^{(0)}$ and $\psi_y^{(0)}$. It is of twelfth order and no shear correction factors are introduced. The consistent homogeneous boundary conditions are of the form

$$N_n = 0 \quad \text{or} \quad U_n = 0, \qquad N_{ns} = 0 \quad \text{or} \quad U_s = 0,$$

$$Q_n' + Q_n'' + \partial M_{ns}''/\partial s - \omega^2[(R_{11} - R)U + R_{21}V + (I + I_{1111} + I_{2121} - 2I_{11})\,\partial W/\partial x$$
$$+ (I_{1112} + I_{2221} - I_{12} - I_{21})\,\partial W/\partial y + (I_{1111} + I_{2121} - I_{11})\psi_x^{(0)}$$
$$+ (I_{1112} + I_{2221} - I_{12})\psi_y^{(0)}]\cos(n, x) - \omega^2[R_{12}U + (R_{22} - R)V$$
$$+ (I_{1112} + I_{2221} - I_{12} - I_{21})\,\partial W/\partial x + (I + I_{2222} + I_{1212} - 2I_{22})\,\partial W/\partial y$$
$$+ (I_{1112} + I_{2221} - I_{21})\psi_x^{(0)} + (I_{2222} + I_{1212} - I_{22})\psi_y^{(0)}]\cos(n, y) = 0,$$
$$\text{or} \quad W = 0, \qquad M_n' = 0 \quad \text{or} \quad \psi_n^{(0)} = 0, \qquad M_{ns}' = 0 \quad \text{or} \quad \psi_s^{(0)} = 0,$$
$$M_n'' = 0 \quad \text{or} \quad \partial W/\partial n = 0. \tag{14}$$

The expressions for the coefficients R_{11}, etc., are also given in Appendix A. At each corner of the plate there is the additional requirement that

$$M_{ns}''(s + 0) - M_{ns}''(s - 0) = 0 \qquad \text{or} \qquad W = 0. \tag{15}$$

Furthermore, two independent sets of simultaneous linear algebraic equations which $\lambda_{rs}^{(i)}$ ($i > 0$ or $i < 0$) must satisfy, respectively, can also be obtained. The coefficients of the

TABLE 3

Corresponding values of $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ for antisymmetric cross-ply square laminated plates

			E_L/E_T				
N		i	3	10	20	30	40
2	$\lambda_{11}^{(i)}$	−1	1·003	0·661	0·454	0·352	0·292
	$\lambda_{22}^{(i)}$	−1	0·997	1·514	2·204	2·841	3·429
10	$\lambda_{11}^{(i)}$	4	0·245	0·309	0·341	0·360	0·373
		3	0·685	0·762	0·804	0·829	0·849
		2	0·790	0·825	0·844	0·855	0·864
		1	1·131	1·162	1·174	1·178	1·180
		−1	1·196	1·189	1·185	1·184	1·183
		−2	0·897	0·871	0·866	0·868	0·872
		−3	0·882	0·849	0·848	0·856	0·868
		−4	0·473	0·416	0·400	0·399	0·403
		−5	0·166	0·082	0·047	0·034	0·026
	$\lambda_{22}^{(i)}$	4	0·139	0·069	0·040	0·028	0·022
		3	0·396	0·350	0·337	0·337	0·340
		2	0·739	0·715	0·715	0·723	0·733
		1	0·751	0·733	0·731	0·733	0·737
		−1	0·836	0·841	0·844	0·844	0·845
		−2	0·947	0·978	0·990	0·995	0·997
		−3	0·661	0·694	0·712	0·722	0·730
		−4	0·574	0·641	0·678	0·700	0·717
		−5	0·206	0·260	0·288	0·304	0·315

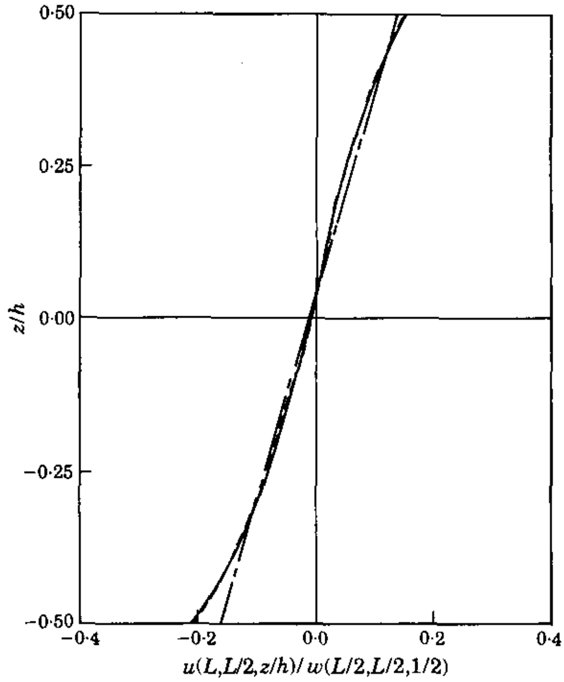

Figure 1. Mode shapes associated with the fundamental frequency for a ten-layer cross-ply laminate with $E_L/E_T = 30$ and $h/L = 0{\cdot}2$. ——, Exact [14]; – – –, present; — · —, FSDT [14].

algebraic equations consist of ω^2 and area integrals defined with respect to the region of the plate. The integrated functions are expressed in terms of the amplitudes of the displacements and rotations and the strain components in equation (8). The details are given in Appendix B. Coupling between the set of differential equations (12) and the two sets of algebraic equations arises through the coefficients in those equations. We have to solve them together using iteration methods. A procedure is suggested as follows.

At first an approximate distribution (e.g., a parabola) of each transverse shear stress component across the plate thickness is assumed. The transverse shear stress–strain relations for each layer are

$$\tau_{zx}^{(i)} = Q_{55}^{(i)}\gamma_{zx}^{(i)} + Q_{45}^{(i)}\gamma_{yz}^{(i)}, \qquad \tau_{yz}^{(i)} = Q_{45}^{(i)}\gamma_{zx}^{(i)} + Q_{44}^{(i)}\gamma_{yz}^{(i)}. \tag{16}$$

If one regards the values of the transverse shear stresses at the mid-plane of each layer as the representative ones, then for a parabolic distribution one can obtain the expressions

$$\lambda_{11}^{(i)} = a^{(i)}(Q_{44}^{(i)}Q_{55}^{(0)} - Q_{45}^{(i)}Q_{45}^{(0)})/b^{(i)}, \qquad \lambda_{22}^{(i)} = a^{(i)}(Q_{55}^{(i)}Q_{44}^{(0)} - Q_{45}^{(i)}Q_{45}^{(0)})/b^{(i)},$$
$$\lambda_{12}^{(i)} = a^{(i)}(Q_{44}^{(i)}Q_{45}^{(0)} - Q_{45}^{(i)}Q_{44}^{(0)})/b^{(i)}, \qquad \lambda_{21}^{(i)} = a^{(i)}(Q_{55}^{(i)}Q_{45}^{(0)} - Q_{45}^{(i)}Q_{55}^{(0)})/b^{(i)},$$
$$a^{(i)} = [1 - (2z_i/h)^2]/[1 - (2z_0/h)^2], \qquad b^{(i)} = Q_{44}^{(i)}Q_{55}^{(i)} - Q_{45}^{(i)2}, \tag{17}$$

as the initial values for $\lambda_{rs}^{(i)}$. Utilizing them in the calculations, we can solve equation (12) with determined coefficients. Thus one can obtain an initial solution (analytical or numerical) for every natural frequency ω and corresponding mode U, V, W, $\psi_x^{(0)}$ and $\psi_y^{(0)}$ of equation (12) and the related boundary conditions. The mode can be so normalized that

$$2T^* = \int_\Omega \sum_{i=-N_2}^{N_1} \left\{ \int_{z_i - t_i/2}^{z_i + t_i/2} \rho_i (u^{(i)2} + v^{(i)2} + w^{(i)2})\, dz \right\} d\Omega = 1,$$

where ρ_i is the mass density of the ith layer. According to each natural frequency ω and corresponding normal mode, one calculates the coefficients in the algebraic equations. Then one solves the equations to obtain new values of $\lambda_{rs}^{(i)}$ which are different for different natural frequencies. Thereafter, equation (12) is re-solved by using each new value of $\lambda_{rs}^{(i)}$ to obtain a new value of each natural frequency and corresponding mode shape, and so on. In general, the iteration process converges quickly. In some cases other more approximate distributions of the transverse shear stresses across the thickness can be assumed, and the initial values of $\lambda_{rs}^{(i)}$ are better. Thus the iteration process can converge more quickly. If less accuracy is acceptable, the iteration process can be stopped after a few cycles. Sometimes even the prescribed values of $\lambda_{rs}^{(i)}$ obtained from equation (17) can be used to give acceptable results [13] and the iteration process can be omitted.

3. NUMERICAL RESULTS FOR SIMPLY SUPPORTED SYMMETRIC AND ANTISYMMETRIC CROSS-PLY PLATES

The exact analytical solution of equation (12) for a general laminated plate under arbitrary boundary conditions is a difficult task. Here the free vibrations of simply supported, symmetric and antisymmetric cross-ply square plates are to be considered. Upon introducing a co-ordinate system and defining the edges of the plate to be $x = 0, L$ and $y = 0, L$, the following simply supported boundary conditions are adopted:

$x = 0, L$:

$$N_x = V = W = M'_x = \psi_y^{(0)} = M''_x = 0,$$

$y = 0, L$:

$$N_y = U = W = M'_y = \psi_x^{(0)} = M''_y = 0. \tag{18}$$

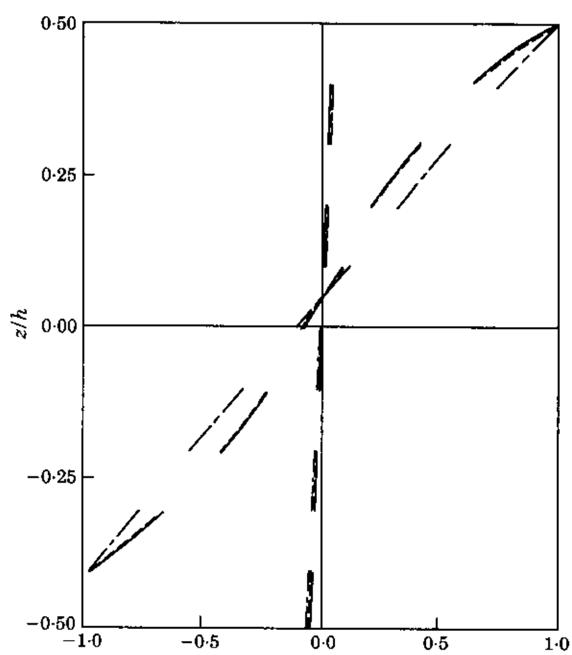

Figure 2. Modal normal stresses corresponding to the mode shapes of Figure 1.

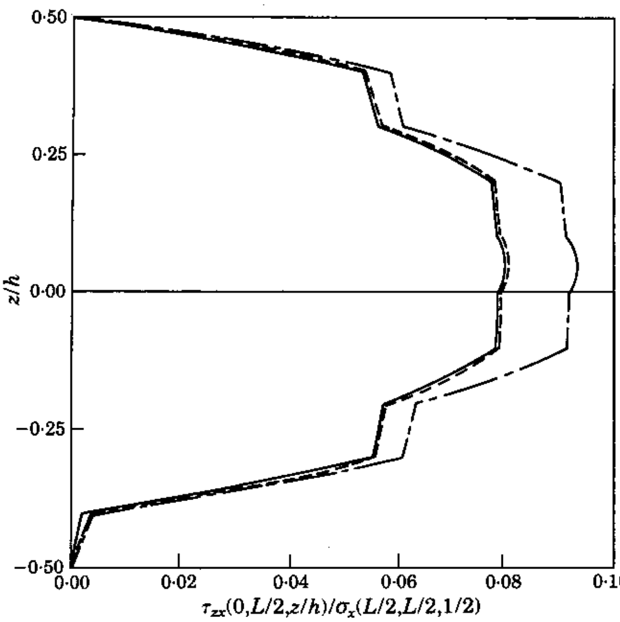

Figure 3. Modal shear stresses corresponding to the mode shapes of Figure 1.

To obtain the fundamental frequency and the associated mode shape, one can assume the following form of spatial variation of $(U, V, W, \psi_x^{(0)}, \psi_y^{(0)})$ that satisfies boundary conditions in equation (18):

$$U(x,y) = U\cos(\pi x/L)\sin(\pi y/L), \qquad V(x,y) = V\sin(\pi x/L)\cos(\pi y/L),$$
$$W(x,y) = W\sin(\pi x/L)\sin(\pi y/L), \qquad \psi_x^{(0)}(x,y) = \psi_x^{(0)}\cos(\pi x/L)\sin(\pi y/L),$$
$$\psi_y^{(0)}(x,y) = \psi_y^{(0)}\sin(\pi x/L)\cos(\pi y/L). \tag{19}$$

It can be verified that in this case $\lambda_{12}^{(i)} = \lambda_{21}^{(i)} = 0$. Substituting equation (19) into equation (12) for cross-ply plates, one can obtain the solution for ω^2, U, V, W, $\psi_x^{(0)}$ and $\psi_y^{(0)}$ and $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ by the iteration procedure. Cross-ply laminated plates having both symmetric and antisymmetric laminations with respect to the middle plane were considered. The fiber orientations of the different laminas alternate between 0° and 90° with respect to the x-axis, and in the symmetrical laminates the 0° layers are at the outer surfaces of the laminate. In the antisymmetrical laminates the $+N_1$th layer is a 0° layer. The total thickness of the 0° and 90° layers in each laminate are the same. The material characteristics of the individual layers taken in reference [14] are considered here and they are $(G_{LT}/E_T) = 0{\cdot}6$, $(G_{TT}/E_T) = 0{\cdot}5$ and $\nu_{LT} = \nu_{TT} = 0{\cdot}25$. Subscript L refers to the direction of fibers and subscript T refers to the transverse direction. The ratio E_L/E_T is varied between 3 and 40, and h/L is fixed to be 0·2. Two layer, three-layer, nine-layer and ten-layer plates were considered. The non-dimensionalized fundamental frequencies are presented in Table 1. They are compared with the results of an exact solution [14] (a finite difference solution of the three-dimensional elasticity equations) and the results obtained by the first order shear deformation theory (FSDT) [15]. The shear correction factors for FSDT are taken to be 5/6 in reference [15]. The corresponding values of $\lambda_{11}^{(i)}$ and $\lambda_{22}^{(i)}$ for symmetric and antisymmetric cross-ply square laminated plates are given in Tables 2 and 3, respectively. For these sample examples the iteration processes were operated through dozens of cycles until the values of $\lambda_{rs}^{(i)}$ converged to three decimal places. The natural frequencies converged much faster than the values of $\lambda_{rs}^{(i)}$. Plots of the mode shapes and modal normal and shear stresses associated with the fundamental frequency for a ten-layered plate with $E_L/E_T = 30$ are shown in Figures 1–3. The modal in-plane displacement u was normalized by dividing it by the transverse displacement w at the surface $z = h/2$. The modal normal and shear stresses, σ_x and τ_{zx}, were normalized by dividing them by $\sigma_{x\,max}$. The results show that the present study is in good agreement with the elasticity solution [14]. However, the mode shapes and modal stresses obtained by the FSDT are not as accurate as the frequencies.

4. CONCLUSIONS

A refined shear deformation laminated plate theory first developed in reference [9] has been used here to analyze the free vibration of laminated plates. The theory contains the same number of dependent variables as first order shear deformation theory, but the set of governing differential equations is of twelfth order. No shear correction factors are required. The theory can be used to analyze the free vibration of arbitrary laminated plates without limitation on the materials and the number of layers and the direction of the ply angle. The numerical results for simply supported, symmetric and antisymmetric cross-ply laminates have been compared with those given by elasticity theory. From the results it can be concluded that the present theory gives accurate predictions of both the natural frequencies and the modal shapes and modal stresses even for fairly thick laminates with a span-to-depth ratio equal to 5. Although the analytical solution of the equations can be obtained only in a few cases, one can use approximate methods, e.g., finite element methods, to obtain numerical solutions in other cases.

If one prescribes $\lambda_{11}^{(i)} = \lambda_{22}^{(i)} = 1$ and $\lambda_{12}^{(i)} = \lambda_{21}^{(i)} = 0$, then the displacement model in the present theory becomes identical with that of the first order shear deformation theory. Therefore the present theory can be regarded as a direct generalization and improvement of the first order shear deformation theory.

In the present theory the undetermined constants $\lambda_{rs}^{(i)}$ may be regarded as average values for the whole laminate. The values of $\lambda_{rs}^{(i)}$ are different for each normal mode. As every normal mode of a laminated plate can be approximately obtained with the present theory, then the orthogonality relations between any two different normal modes also exist approximately. If the responses of the laminated plate to dynamic loads have to be calculated, they can be described by the sum of the product of the response of each normal co-ordinate of the plate with the corresponding generalized force and the corresponding normal mode. Using the present theory, one can predict accurately not only the transverse displacements but also the in-plane stresses, which are also important for dynamic analysis of laminated plates.

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APPENDIX A: ELEMENTS OF THE STIFFNESS MATRIX AND EXPRESSIONS OF COEFFICIENTS AND DIFFERENTIAL OPERATORS M_{ij}

The expressions for the non-zero submatrices in equation (11) are as follows:

$$[A]=\begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ \text{symm.} & & A_{66} \end{bmatrix}, \qquad [B']=\begin{bmatrix} B_{11'} & B_{12'} & B_{16'} & B_{16''} \\ B_{21'} & B_{22'} & B_{26'} & B_{26''} \\ B_{61'} & B_{62'} & B_{66'} & B_{66''} \end{bmatrix},$$

$$[B'']=\begin{bmatrix} B_{11}-B_{11'} & B_{12}-B_{12'} & B_{16}-\frac{1}{2}B_{16'}-\frac{1}{2}B_{16''} \\ B_{21}-B_{21'} & B_{22}-B_{22'} & B_{26}-\frac{1}{2}B_{26'}-\frac{1}{2}B_{26''} \\ B_{61}-B_{61'} & B_{62}-B_{62'} & B_{66}-\frac{1}{2}B_{66'}-\frac{1}{2}B_{66''} \end{bmatrix},$$

$$[D']=\begin{bmatrix} D_{1'1'} & D_{1'2'} & D_{1'6'} & D_{1'6''} \\ & D_{2'2'} & D_{2'6'} & D_{2'6''} \\ & & D_{6'6'} & D_{6'6''} \\ \text{symm.} & & & D_{6''6''} \end{bmatrix},$$

$$[D_c]=\begin{bmatrix} D_{11'}-D_{1'1'} & D_{21'}-D_{1'2'} & D_{61'}-\frac{1}{2}D_{1'6'}-\frac{1}{2}D_{1'6''} \\ D_{12'}-D_{1'2'} & D_{22'}-D_{2'2'} & D_{62'}-\frac{1}{2}D_{2'6'}-\frac{1}{2}D_{2'6''} \\ D_{16'}-D_{1'6'} & D_{26'}-D_{2'6'} & D_{66'}-\frac{1}{2}D_{6'6'}-\frac{1}{2}D_{6'6''} \\ D_{16''}-D_{1'6''} & D_{26''}-D_{2'6''} & D_{66''}-\frac{1}{2}D_{6'6''}-\frac{1}{2}D_{6''6''} \end{bmatrix},$$

$$[D''] = \begin{bmatrix} D_{11} - 2D_{11'} + D_{1'1'} & D_{12} - D_{12'} - D_{21'} + D_{1'2'} & D_{16} - \frac{1}{2} D_{16'} - \frac{1}{2} D_{16''} - D_{61'} + \frac{1}{2} D_{1'6'} + \frac{1}{2} D_{1'6''} \\ & D_{22} - 2D_{22'} + D_{2'2'} & D_{26} - \frac{1}{2} D_{26'} - \frac{1}{2} D_{26''} - D_{62'} + \frac{1}{2} D_{2'6'} + \frac{1}{2} D_{2'6''} \\ \text{symm.} & & D_{66} - D_{66'} - D_{66''} + \frac{1}{4} D_{6'6'} + \frac{1}{2} D_{6'6''} + \frac{1}{4} D_{6''6''} \end{bmatrix},$$

$$[G] = \begin{bmatrix} G_{55} & G_{45} \\ G_{45} & G_{44} \end{bmatrix}. \tag{A1}$$

The elements in the submatrices are expressed as follows. Define a notation $\Sigma'(\cdots)$ to designate a sum which is given by summing all the quantities associated with their superscripts or subscripts of all the non-zero numbers i. Denoting by $Q_{pq}^{(i)}$ the usual transformed, plane-stress-reduced elastic constants of the ith layer, then we have

$$A_{pq} = Q_{pq}^{(0)} t_0 + \sum{}' (Q_{pq}^{(i)} t_i), \qquad D_{pq} = Q_{pq}^{(0)} J_0 + \sum{}' (Q_{pq}^{(i)} J_i), \qquad pq = 11, 22, 66, 12, 16, 26, \tag{A2}$$

where

$$J_i = t_i z_i^2 + \tfrac{1}{12} t_i^3, \qquad i \geqslant 0, \quad i \leqslant 0. \tag{A3}$$

Defining

$$\tilde{t}_{pq}^{(i)} = \begin{Bmatrix} t_{pq}^{(i)} + s(i) z_0, & p = q \\ t_{pq}^{(i)}, & p \neq q \end{Bmatrix}, \qquad J_{pq}^{(i)} = s(i) z_i t_i \tilde{t}_{pq}^{(i)} + \tfrac{1}{12} \lambda_{pq}^{(i)} t_i^3,$$

$$J_{pqrs}^{(i)} = t_i \tilde{t}_{pq}^{(i)} \tilde{t}_{rs}^{(i)} + \tfrac{1}{12} \lambda_{pq}^{(i)} \lambda_{rs}^{(i)} t_i^3, \qquad pq, rs = 11, 22, 12, 21, \tag{A4}$$

we have

$$D_{11'} = Q_{11}^{(0)} J_0 + \sum{}' (Q_{11}^{(i)} J_{11}^{(i)} + Q_{16}^{(i)} J_{21}^{(i)}), \qquad D_{12'} = Q_{12}^{(0)} J_0 + \sum{}' (Q_{12}^{(i)} J_{22}^{(i)} + Q_{16}^{(i)} J_{12}^{(i)}),$$
$$D_{16'} = Q_{16}^{(0)} J_0 + \sum{}' (Q_{11}^{(i)} J_{12}^{(i)} + Q_{16}^{(i)} J_{22}^{(i)}), \qquad D_{16''} = Q_{16}^{(0)} J_0 + \sum{}' (Q_{12}^{(i)} J_{21}^{(i)} + Q_{16}^{(i)} J_{11}^{(i)}),$$
$$D_{21'} = Q_{12}^{(0)} J_0 + \sum{}' (Q_{12}^{(i)} J_{11}^{(i)} + Q_{26}^{(i)} J_{21}^{(i)}), \qquad D_{22'} = Q_{22}^{(0)} J_0 + \sum{}' (Q_{22}^{(i)} J_{22}^{(i)} + Q_{26}^{(i)} J_{12}^{(i)}),$$
$$D_{26'} = Q_{26}^{(0)} J_0 + \sum{}' (Q_{12}^{(i)} J_{12}^{(i)} + Q_{26}^{(i)} J_{22}^{(i)}), \qquad D_{26''} = Q_{26}^{(0)} J_0 + \sum{}' (Q_{22}^{(i)} J_{21}^{(i)} + Q_{26}^{(i)} J_{11}^{(i)}),$$
$$D_{61'} = Q_{16}^{(0)} J_0 + \sum{}' (Q_{66}^{(i)} J_{21}^{(i)} + Q_{16}^{(i)} J_{11}^{(i)}), \qquad D_{62'} = Q_{26}^{(0)} J_0 + \sum{}' (Q_{66}^{(i)} J_{12}^{(i)} + Q_{26}^{(i)} J_{22}^{(i)}),$$
$$D_{66'} = Q_{66}^{(0)} J_0 + \sum{}' (Q_{66}^{(i)} J_{22}^{(i)} + Q_{16}^{(i)} J_{12}^{(i)}), \qquad D_{66''} = Q_{66}^{(0)} J_0 + \sum{}' (Q_{66}^{(i)} J_{11}^{(i)} + Q_{26}^{(i)} J_{21}^{(i)}), \tag{A5}$$
$$D_{1'1'} = Q_{11}^{(0)} J_0 + \sum{}' [Q_{11}^{(i)} J_{1111}^{(i)} + Q_{66}^{(i)} J_{2121}^{(i)} + 2Q_{16}^{(i)} J_{1121}^{(i)}],$$
$$D_{1'2'} = Q_{12}^{(0)} J_0 + \sum{}' [Q_{12}^{(i)} J_{1122}^{(i)} + Q_{66}^{(i)} J_{1221}^{(i)} + Q_{16}^{(i)} J_{1112}^{(i)} + Q_{26}^{(i)} J_{2221}^{(i)}],$$
$$D_{1'6'} = Q_{16}^{(0)} J_0 + \sum{}' [Q_{11}^{(i)} J_{1112}^{(i)} + Q_{66}^{(i)} J_{2221}^{(i)} + Q_{16}^{(i)} (J_{1122}^{(i)} + J_{1221}^{(i)})],$$
$$D_{1'6''} = Q_{16}^{(0)} J_0 + \sum{}' [(Q_{12}^{(i)} + Q_{66}^{(i)}) J_{1121}^{(i)} + Q_{16}^{(i)} J_{1111} + Q_{26}^{(i)} J_{2121}^{(i)}],$$
$$D_{2'2'} = Q_{22}^{(0)} J_0 + \sum{}' [Q_{22}^{(i)} J_{2222}^{(i)} + Q_{66}^{(i)} J_{1212}^{(i)} + 2Q_{26}^{(i)} J_{2212}^{(i)}],$$
$$D_{2'6'} = Q_{26}^{(0)} J_0 + \sum{}' [(Q_{12}^{(i)} + Q_{66}^{(i)}) J_{2212}^{(i)} + Q_{16}^{(i)} J_{1212}^{(i)} + Q_{26}^{(i)} J_{2222}^{(i)}],$$
$$D_{2'6''} = Q_{26}^{(0)} J_0 + \sum{}' [Q_{22}^{(i)} J_{2221}^{(i)} + Q_{66}^{(i)} J_{1112}^{(i)} + Q_{26}^{(i)} (J_{1122}^{(i)} + J_{1221}^{(i)})],$$
$$D_{6'6'} = Q_{66}^{(0)} J_0 + \sum{}' [Q_{11}^{(i)} J_{1212}^{(i)} + Q_{66}^{(i)} J_{2222}^{(i)} + 2Q_{16}^{(i)} J_{2212}^{(i)}],$$
$$D_{6'6''} = Q_{66}^{(0)} J_0 + \sum{}' [Q_{12}^{(i)} J_{1221}^{(i)} + Q_{66}^{(i)} J_{1122}^{(i)} + Q_{16}^{(i)} J_{1112}^{(i)} + Q_{26}^{(i)} J_{2221}^{(i)}],$$

$$D_{6''6''} = Q_{66}^{(0)} J_0 + \sum{}' [Q_{22}^{(i)} J_{2121}^{(i)} + Q_{66}^{(i)} J_{1111}^{(i)} + 2Q_{26}^{(i)} J_{1121}^{(i)}], \tag{A6}$$

Defining

$$S_i = t_i z_i, \qquad i \geqslant 0, \quad i \leqslant 0, \tag{A7}$$

$$S_{pq}^{(i)} = s(i) t_i \bar{t}_{pq}^{(i)}, \qquad pq = 11, 22, 12, 21, \tag{A8}$$

we have

$$B_{pq} = Q_{pq}^{(0)} S_0 + \sum{}' (Q_{pq}^{(i)} S_i), \tag{A9}$$

The expressions for B_{11}, etc., are similar to those for D_{11}, etc., in equation (A5) but with J_0 and $J_{pq}^{(i)}$ replaced by S_0 and $S_{pq}^{(i)}$. Then

$$G_{55} = Q_{55}^{(0)} t_0 + \sum{}' [t_i (Q_{55}^{(i)} \lambda_{11}^{(i)2} + Q_{44}^{(i)} \lambda_{21}^{(i)2} + 2Q_{45}^{(i)} \lambda_{11}^{(i)} \lambda_{21}^{(i)})],$$

$$G_{44} = Q_{44}^{(0)} t_0 + \sum{}' [t_i (Q_{44}^{(i)} \lambda_{22}^{(i)2} + Q_{55}^{(i)} \lambda_{12}^{(i)2} + 2Q_{45}^{(i)} \lambda_{22}^{(i)} \lambda_{12}^{(i)})],$$

$$G_{45} = Q_{45}^{(0)} t_0 + \sum{}' \{t_i [Q_{44}^{(i)} \lambda_{22}^{(i)} \lambda_{21}^{(i)} + Q_{55}^{(i)} \lambda_{11}^{(i)} \lambda_{12}^{(i)} + Q_{45}^{(i)} (\lambda_{11}^{(i)} \lambda_{22}^{(i)} + \lambda_{12}^{(i)} \lambda_{21}^{(i)})]\}. \tag{A10}$$

The coefficients and differential operators M_{11}, etc., are

$$M_{11} = \rho_A, \qquad M_{13} = (R_{11} - R)\,\partial/\partial x + R_{12}\,\partial/\partial y, \qquad M_{14} = R_{11}, \qquad M_{15} = R_{12},$$

$$M_{22} = \rho_A, \qquad M_{23} = R_{21}\,\partial/\partial x + (R_{22} - R)\,\partial/\partial y, \qquad M_{24} = R_{21}, \qquad M_{25} = R_{22},$$

$$M_{33} = (I + I_{1111} + I_{2121} - 2I_{11})\,\partial^2/\partial x^2 + 2(I_{1112} + I_{2221} - I_{12} - I_{21})\,\partial^2/\partial x\,\partial y$$

$$+ (I + I_{2222} + I_{1212} - 2I_{22})\,\partial^2/\partial y^2 - \rho_A,$$

$$M_{34} = (I_{1111} + I_{2121} - I_{11})\,\partial/\partial x + (I_{1112} + I_{2221} - I_{21})\,\partial/\partial y,$$

$$M_{35} = (I_{1112} + I_{2221} - I_{12})\,\partial/\partial x + (I_{2222} + I_{1212} - I_{22})\,\partial/\partial y,$$

$$M_{44} = I_{1111} + I_{2121}, \qquad M_{45} = I_{1112} + I_{2221}, \qquad M_{55} = I_{2222} + I_{1212}, \tag{A11}$$

where

$$\rho_A = \rho_0 t_0 + \sum{}' (\rho_i t_i), \qquad R = \rho_0 S_0 + \sum{}' (\rho_i S_i), \qquad R_{11} = \rho_0 S_0 + \sum{}' (\rho_i S_{11}^{(i)}),$$

$$R_{22} = \rho_0 S_0 + \sum{}' (\rho_i S_{22}^{(i)}), \qquad R_{12} = \sum{}' (\rho_i S_{12}^{(i)}), \qquad R_{21} = \sum{}' (\rho_i S_{21}^{(i)}),$$

$$I = \rho_0 J_0 + \sum{}' (\rho_i J_i), \qquad I_{11} = \rho_0 J_0 + \sum{}' (\rho_i J_{11}^{(i)}), \qquad I_{22} = \rho_0 J_0 + \sum{}' (\rho_i J_{22}^{(i)}),$$

$$I_{12} = \sum{}' (\rho_i J_{12}^{(i)}), \qquad I_{21} = \sum{}' (\rho_i J_{21}^{(i)}), \qquad I_{1111} = \rho_0 J_0 + \sum{}' (\rho_i J_{1111}^{(i)}),$$

$$I_{2222} = \rho_0 J_0 + \sum{}' (\rho_i J_{2222}^{(i)}), \qquad I_{1112} = \sum{}' (\rho_i J_{1112}^{(i)}), \qquad I_{2221} = \sum{}' (\rho_i J_{2221}^{(i)}),$$

$$I_{1212} = \sum{}' (\rho_i J_{1212}^{(i)}), \qquad I_{2121} = \sum{}' (\rho_i J_{2121}^{(i)}). \tag{A12}$$

APPENDIX B: SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS WHICH $\lambda_{rs}^{(i)}$ MUST SATISFY

Two sets of simultaneous linear algebraic equations that the unknown constants $\lambda_{rs}^{(i)}$ ($i > 0$ or $i < 0$) must satisfy, respectively, can also be obtained in accordance with the principle of stationary potential energy. They are as follows:

$$\sum_{j=1}^{N_1}\begin{bmatrix} C_{1111}^{(kj)} & C_{1122}^{(kj)} & C_{1112}^{(kj)} & C_{1121}^{(kj)} \\ C_{2211}^{(kj)} & C_{2222}^{(kj)} & C_{2212}^{(kj)} & C_{2221}^{(kj)} \\ C_{1211}^{(kj)} & C_{1222}^{(kj)} & C_{1212}^{(kj)} & C_{1221}^{(kj)} \\ C_{2111}^{(kj)} & C_{2122}^{(kj)} & C_{2112}^{(kj)} & C_{2121}^{(kj)} \end{bmatrix}\begin{bmatrix} \lambda_{11}^{(j)} \\ \lambda_{22}^{(j)} \\ \lambda_{12}^{(j)} \\ \lambda_{21}^{(j)} \end{bmatrix} + \begin{bmatrix} d_{11}^{(k)} \\ d_{22}^{(k)} \\ d_{12}^{(k)} \\ d_{21}^{(k)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad k = 1, 2, \ldots N_1, \tag{B1}$$

$$\sum_{j=-1}^{-N_2}\begin{bmatrix} C_{1111}^{(kj)} & C_{1122}^{(kj)} & C_{1112}^{(kj)} & C_{1121}^{(kj)} \\ C_{2211}^{(kj)} & C_{2222}^{(kj)} & C_{2212}^{(kj)} & C_{2221}^{(kj)} \\ C_{1211}^{(kj)} & C_{1222}^{(kj)} & C_{1212}^{(kj)} & C_{1221}^{(kj)} \\ C_{2111}^{(kj)} & C_{2122}^{(kj)} & C_{2112}^{(kj)} & C_{2121}^{(kj)} \end{bmatrix}\begin{bmatrix} \lambda_{11}^{(j)} \\ \lambda_{22}^{(j)} \\ \lambda_{12}^{(j)} \\ \lambda_{21}^{(j)} \end{bmatrix} + \begin{bmatrix} d_{11}^{(k)} \\ d_{22}^{(k)} \\ d_{12}^{(k)} \\ d_{21}^{(k)} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad k = -1, -2, \ldots -N_2, \tag{B2}$$

$$C_{rspq}^{(kj)} = C_{Erspq}^{(kj)} - \omega^2 C_{T^*rspq}^{(kj)}, \qquad kj > 0, \quad rs, pq = 11, 22, 12, 21, \tag{B3}$$

$$d_{rs}^{(k)} = d_{Ers}^{(k)} - \omega^2 d_{T^*rs}^{(k)}, \qquad k > 0 \quad \text{or} \quad k < 0, \quad rs = 11, 22, 12, 21. \tag{B4}$$

Defining

$$r_x = \kappa_x' - \kappa_x'', \qquad r_y = \kappa_y' - \kappa_y'', \qquad r_{xy} = \kappa_{xy}' - \kappa_{xy}'', \qquad r_{yx} = \kappa_{yx}' - \kappa_{yx}'', \tag{B5}$$

we have

$$C_{Erspq}^{(kj)} = \int_\Omega \frac{\partial}{\partial \lambda_{pq}^{(j)}}\left(\frac{\partial E}{\partial \lambda_{rs}^{(k)}}\right) \mathrm{d}\Omega, \qquad kj > 0, \quad rs, pq = 11, 22, 12, 21, \tag{B6}$$

$$\frac{\partial}{\partial \lambda_{pq}^{(j)}}\left(\frac{\partial E}{\partial \lambda_{rs}^{(k)}}\right) = \left\{\begin{array}{ll} t_j\left\{\sum\limits_m \sum\limits_n \left[\left(\frac{1}{2} Q_{mn}^{(k)} t_k^2 + t_k \sum_k'' (t_i Q_{mn}^{(i)})\right) d_{m,rs} d_{n,pq}\right]\right\}, & kj > 0, |j| < |k| \\ t_k\left\{\sum\limits_m \sum\limits_n \left[\left(\frac{1}{3} Q_{mn}^{(k)} t_k^2 + t_k \sum_k'' (t_i Q_{mn}^{(i)})\right) d_{m,rs} d_{n,pq}\right] + a_{rspq}^{(k)}\right\}, & j = k \\ t_k\left\{\sum\limits_m \sum\limits_n \left[\left(\frac{1}{2} Q_{mn}^{(j)} t_j^2 + t_j \sum_j'' (t_i Q_{mn}^{(i)})\right) d_{m,rs} d_{n,pq}\right]\right\}, & kj > 0, \quad |j| > |k| \end{array}\right\}$$

$$m, n = 1, 2, 6, \tag{B7}$$

$$\begin{aligned} &d_{1,11} = r_x, && d_{2,11} = 0, && d_{6,11} = r_{yx}, && d_{1,22} = 0, && d_{2,22} = r_y, && d_{6,22} = r_{xy}, \\ &d_{1,12} = r_{xy}, && d_{2,12} = 0, && d_{6,12} = r_y, && d_{1,21} = 0, && d_{2,21} = r_{yx}, && d_{6,21} = r_x. \end{aligned} \tag{B8}$$

We perform area integrations with respect to the region Ω of the plate in equation (B6). The notation $\sum_k''(\cdots)$ designates a sum which is given by summing all the quantities associated with their superscripts or subscripts of the numbers i the values of which satisfy both the conditions of $ik > 0$ and $|i| > |k|$. $a^{(k)}_{rspq}$ in equation (B7) are corresponding elements of the matrix

$$\begin{bmatrix} Q^{(k)}_{55}\gamma^{(0)2}_{xz} & Q^{(k)}_{45}\gamma^{(0)}_{xz}\gamma^{(0)}_{yz} & Q^{(k)}_{55}\gamma^{(0)}_{xz}\gamma^{(0)}_{yz} & Q^{(k)}_{45}\gamma^{(0)2}_{xz} \\ & Q^{(k)}_{44}\gamma^{(0)2}_{yz} & Q^{(k)}_{45}\gamma^{(0)2}_{yz} & Q^{(k)}_{44}\gamma^{(0)}_{xz}\gamma^{(0)}_{yz} \\ & & Q^{(k)}_{55}\gamma^{(0)2}_{yz} & Q^{(k)}_{45}\gamma^{(0)}_{xz}\gamma^{(0)}_{yz} \\ & \text{symmetric} & & Q^{(k)}_{44}\gamma^{(0)2}_{xz} \end{bmatrix}, \tag{B9}$$

$$C^{(kj)}_{T^*rspq} = \int_\Omega \frac{\partial}{\partial\lambda^{(j)}_{pq}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{rs}}\right)\mathrm{d}\Omega, \tag{B10}$$

$$\frac{\partial}{\partial\lambda^{(j)}_{11}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{11}}\right) = \frac{\partial}{\partial\lambda^{(j)}_{21}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{21}}\right) = I^{(kj)}\gamma^{(0)2}_{zx}, \qquad \frac{\partial}{\partial\lambda^{(j)}_{22}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{22}}\right) = \frac{\partial}{\partial\lambda^{(j)}_{12}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{12}}\right) = I^{(kj)}\gamma^{(0)2}_{yz},$$

$$\frac{\partial}{\partial\lambda^{(j)}_{12}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{11}}\right) = \frac{\partial}{\partial\lambda^{(j)}_{11}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{12}}\right) = \frac{\partial}{\partial\lambda^{(j)}_{21}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{22}}\right) = \frac{\partial}{\partial\lambda^{(j)}_{22}}\left(\frac{\partial T^*}{\partial\lambda^{(k)}_{21}}\right) = I^{(kj)}\gamma^{(0)}_{zx}\gamma^{(0)}_{yz}, \tag{B11}$$

$$I^{(kj)} = \left\{\begin{array}{ll} t_j\left[\frac{1}{2}\rho_k t_k^2 + t_k\sum_k''(\rho_i t_i)\right], & kj > 0,\quad |j| < |k| \\ t_k\left[\frac{1}{3}\rho_k t_k^2 + t_k\sum_k''(\rho_i t_i)\right], & j = k, \\ t_k\left[\frac{1}{2}\rho_j t_j^2 + t_j\sum_j''(\rho_i t_i)\right], & kj > 0,\quad |j| > |k| \end{array}\right\}. \tag{B12}$$

All other $(\partial/\partial\lambda^{(j)}_{pq})(\partial T^*/\partial\lambda^{(k)}_{rs})$ are zeros. Also,

$$\begin{aligned} d^{(k)}_{Ers} = \int_\Omega \sum_m d_{m,rs}\Big\{ & s(k)\left[\tfrac{1}{2}Q^{(k)}_{m1}t_k^2 + t_k\textstyle\sum_k''(t_iQ^{(i)}_{m1})\right][\epsilon_x + (z_0 + \tfrac{1}{2}s(k)t_0)r_x] \\ & + s(k)\left[\tfrac{1}{2}Q^{(k)}_{m2}t_k^2 + t_k\textstyle\sum_k''(t_iQ^{(i)}_{m2})\right][\epsilon_y + (z_0 + \tfrac{1}{2}s(k)t_0)r_y] \\ & + s(k)\left[\tfrac{1}{2}Q^{(k)}_{m6}t_k^2 + t_k\textstyle\sum_k''(t_iQ^{(i)}_{m6})\right][\gamma_{xy} + (z_0 + \tfrac{1}{2}s(k)t_0)(r_{xy} + r_{yx})] \\ & + \left[Q^{(k)}_{m1}(\tfrac{1}{2}s(k)z_kt_k^2 + \tfrac{1}{12}t_k^3) + s(k)t_k\textstyle\sum_k''(z_it_iQ^{(i)}_{m1})\right]\kappa_x'' \\ & + \left[Q^{(k)}_{m2}(\tfrac{1}{2}s(k)z_kt_k^2 + \tfrac{1}{12}t_k^3) + s(k)t_k\textstyle\sum_k''(z_it_iQ^{(i)}_{m2})\right]\kappa_y'' \\ & + \left[Q^{(k)}_{m6}(\tfrac{1}{2}s(k)z_kt_k^2 + \tfrac{1}{12}t_k^3) + s(k)t_k\textstyle\sum_k''(z_it_iQ^{(i)}_{m6})\right]2\kappa_{xy}''\Big\}\,\mathrm{d}\Omega, \qquad m = 1, 2, 6, \end{aligned} \tag{B13}$$

$$d^{(k)}_{T^*11} = \int_\Omega \gamma^{(0)}_{zx}\left(\eta^{(k)}_1 U - \eta^{(k)}_2\frac{\partial W}{\partial x} + \eta^{(k)}_3\gamma^{(0)}_{zx}\right)\mathrm{d}\Omega,$$

$$d^{(k)}_{T^*22} = \int_\Omega \gamma^{(0)}_{yz}\left(\eta^{(k)}_1 V - \eta^{(k)}_2\frac{\partial W}{\partial y} + \eta^{(k)}_3\gamma^{(0)}_{yz}\right)\mathrm{d}\Omega,$$

$$d^{(k)}_{T^*12} = \int_\Omega \gamma^{(0)}_{yz}\left(\eta^{(k)}_1 U - \eta^{(k)}_2 \frac{\partial W}{\partial x} + \eta^{(k)}_3 \gamma^0_{zx}\right) \mathrm{d}\Omega,$$

$$d^{(k)}_{T^*21} = \int_\Omega \gamma^{(0)}_{zx}\left(\eta^{(k)}_1 V - \eta^{(k)}_2 \frac{\partial W}{\partial y} + \eta^{(k)}_3 \gamma^{(0)}_{yz}\right) \mathrm{d}\Omega, \tag{B14}$$

$$\eta^{(k)}_1 = s(k)\left[\tfrac{1}{2}\rho_k t_k^2 + t_k \textstyle\sum''_k (\rho_i t_i)\right], \qquad \eta^{(k)}_2 = s(k)\left[\tfrac{1}{2}\rho_k t_k^2 z_k + t_k \textstyle\sum''_k (\rho_i t_i z_i)\right] + \tfrac{1}{12}\rho_k t_k^3,$$

$$\eta^{(k)}_3 = [\tfrac{1}{2} t_0 + s(k) z_0]\left[\tfrac{1}{2}\rho_k t_k^2 + t_k \textstyle\sum''_k (\rho_i t_i)\right]. \tag{B15}$$