Higher-Order Analysis of Crack-Tip Fields in Power Law Hardening Materials*

XIA Lin (夏 霖) and WANG Zi-Qiang (王自强)

(*LNM, Institute of Mechanics, Academia Sinica, Beijing* 100080, *PRC*)

Received March 8, 1993.

Abstract In this paper, we present an exact higher-order asymptotic analysis on the near-crack-tip fields in elastic-plastic materials under plane strain, Mode I. A four- or five-term asymptotic series of the solutions is derived. It is found that when $1.6 < n \leqslant 2.8$ (here, n is the hardening exponent), the elastic effect enters the third-order stress field; but when $2.8 < n \leqslant 3.7$ this effect turns to enter the fourth-order field, with the fifth-order field independent. Moreover, if $n > 3.7$, the elasticity only affects the fields whose order is higher than 4. In this case, the fourth-order field remains independent. Our investigation also shows that as long as n is larger than 1.6, the third-order field is always not independent, whose amplitude coefficient K_3 depends either on K_1 or on both K_1 and K_2 (K_1 and K_2 are the amplitude coefficients of the first- and second-order fields, respectively). Finally, good agreement is found between our results and O'Dowd and Shih's numerical ones[8] by comparison.

Keywords: higher-order asymptotic analysis, the near-crack-tip fields.

1 Introduction

It is well known that the critical value of the J-integral can be used to predict crack initiation and a small amount of growth. This idea has an essential prerequisite that the region of dominance of the HRR-singular fields must be greater than the fracture process zone[1-6].

Li and Wang[7] showed that the near-tip fields developed under different geometries and different scale yielding conditions can be characterized together by J and k_2. Here k_2 is the amplitude of the second term of the asymptotic series for the near-tip fields Their work has been confirmed by Sharma and Aravas[12].

Recently, O'Dowd and Shih[8-10] proposed a concept of the J-Q annulus. They found that the full range of near-tip stress fields associated with different scale yielding and crack geometries are identified with the members of the Q-family of solutions.

For Mode II plane strain crack in hardening materials, Xia and Wang[13] made an accurate high-order asymptotic analysis, and found that the second term of the asymptotic stress field have slight influence on the near-crack-tip field compared with Mode I.

* Project supported by the National Natural Science Foundation of China.

For Mode I plane stress crack, the analogous analysis has also been made independently by Li[14], Li and Wang[15] and Sharma and Aravas[12].

Some other developments in this field can be found in Ref. [11].

2 Basic Equations

Consider a power law hardening material, whose deformation obeys the Ramberg-Osgood formula under uniaxial tension, i.e.

$$\varepsilon/\varepsilon_0=\sigma/\sigma_0+\alpha(\sigma/\sigma_0)^n, \tag{2.1}$$

where σ_0 is the yield stress, and the corresponding yield strain is given by $\varepsilon_0=\sigma_0/E$ with E being the Young's elastic modulus. n and α are hardening exponent and hardening coefficient, respectively.

By means of the plastic deformation theory, the general stress-strain relations can be expressed as

$$\varepsilon_{ij}=\frac{1+\nu}{E}\,s_{ij}+\frac{1-2\nu}{3E}\,\sigma_{kk}\delta_{ij}+\frac{3}{2}\,\alpha\varepsilon_0\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\frac{s_{ij}}{\sigma_0}, \tag{2.2}$$

where s_{ij} is the stress deviator, $\sigma_e=\sqrt{3s_{ij}s_{ij}/2}$ is the effective stress, and ν is Poisson's ratio.

Under the plane strain condition, we have

$$\varepsilon_{\beta\gamma}=\frac{1+\nu}{E}\,\sigma_{\beta\gamma}+\delta_{\beta\gamma}\frac{\Gamma}{E}\sigma_{\rho\rho}+\frac{3}{2}\,\alpha\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\frac{P_{\beta\gamma}}{E}, \tag{2.3}$$

where $\sigma_{\rho\rho}=\sigma_r+\sigma_\theta$, $P_{\beta\gamma}=\sigma_{\beta\gamma}-\frac{1}{2}\,\sigma_{\rho\rho}\delta_{\beta\gamma}$, and

$$\Gamma\doteq-(1+\nu)\nu+\left(\frac{1}{2}-\nu\right)^2.$$

As for the effective stress, we have

$$\sigma_e^2\doteq\frac{3}{4}\,(\sigma_r-\sigma_\theta)^2+3\tau_{r\theta}^2. \tag{2.4}$$

Equilibrium can be guaranteed provided that the stress function φ is introduced:

$$\begin{cases}\sigma_r=\dfrac{1}{r}\left(\dfrac{\partial\varphi}{\partial r}+\dfrac{1}{r}\,\dfrac{\partial^2\varphi}{\partial\theta^2}\right),\\[2mm] \sigma_\theta=\dfrac{\partial^2\varphi}{\partial r^2},\\[2mm] \tau_{r\theta}=-\dfrac{\partial}{\partial r}\left(\dfrac{1}{r}\,\dfrac{\partial\varphi}{\partial\theta}\right).\end{cases} \tag{2.5}$$

The strain compatibility equation is

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\varepsilon_\theta)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\varepsilon_r-\frac{1}{r}\frac{\partial}{\partial r}\varepsilon_r-\frac{2}{r^2}\frac{\partial^2}{\partial r\partial\theta}(r\varepsilon_{r\theta})=0. \tag{2.6}$$

Let

$$\varphi=\sigma_0\sum_{i=1}^{5}K_i r^{s_i+2}\tilde{\varphi}_i(\theta). \tag{2.7}$$

Substituting (2.7) into (2.5) yields

$$\frac{\sigma_{\beta\gamma}}{\sigma_0}=\sum_{i=1}^{5}K_i r^{s_i}\tilde{\sigma}_{\beta\gamma_i}(\theta)=K_1 r^{s_1}[\tilde{\sigma}_{\beta\gamma_1}+\eta_1 r^{\Delta s_2}\tilde{\sigma}_{\beta\gamma_2}+\eta_2 r^{\Delta s_3}\tilde{\sigma}_{\beta\gamma_3}+\eta_3 r^{\Delta s_4}\tilde{\sigma}_{\beta\gamma_4}+\eta_4 r^{\Delta s_5}\tilde{\sigma}_{\beta\gamma_5}], \tag{2.8}$$

where

$$\begin{cases}\tilde{\sigma}_{r_i}=\ddot{\tilde{\varphi}}_i+(s_i+2)\tilde{\varphi}_i, \\ \tilde{\sigma}_{\theta_i}=(s_i+2)(s_i+1)\tilde{\varphi}_i, \qquad i=1,2,3,4,5 \\ \tilde{\tau}_{r\theta_i}=-(s_i+1)\dot{\tilde{\varphi}}_i\end{cases} \tag{2.9}$$

and $\Delta s_i=s_i-s_1\,(i=2,3,4,5)$. $\eta_j=K_{j+1}/K_1\ (j=1,2,3,4)$.

For the effective stress, we have

$$\begin{aligned}\left(\frac{\sigma_e}{\sigma_0}\right)^2=K_1^2 r^{2s_1}\tilde{\sigma}_{e_{11}}\Bigg[1+\Bigg(&2\eta_1 r^{\Delta s_2}\frac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e11}}+\eta_1^2 r^{2\Delta s_2}\frac{\tilde{\sigma}_{e22}}{\tilde{\sigma}_{e11}}+2\eta_2 r^{\Delta s_3}\frac{\tilde{\sigma}_{e13}}{\tilde{\sigma}_{e11}}+\eta_2^2 r^{2\Delta s_3}\frac{\tilde{\sigma}_{e33}}{\tilde{\sigma}_{e11}}\\&+2\eta_3 r^{\Delta s_4}\frac{\tilde{\sigma}_{e14}}{\tilde{\sigma}_{e11}}+\eta_3^2 r^{2\Delta s_4}\frac{\tilde{\sigma}_{e44}}{\tilde{\sigma}_{e11}}+2\eta_4 r^{\Delta s_5}\frac{\tilde{\sigma}_{e15}}{\tilde{\sigma}_{e11}}+\eta_4^2 r^{2\Delta s_5}\frac{\tilde{\sigma}_{e55}}{\tilde{\sigma}_{e11}}\\&+2\eta_1\eta_2 r^{\Delta s_2+\Delta s_3}\frac{\tilde{\sigma}_{e23}}{\tilde{\sigma}_{e11}}+2\eta_1\eta_3 r^{\Delta s_2+\Delta s_4}\frac{\tilde{\sigma}_{e24}}{\tilde{\sigma}_{e11}}+2\eta_1\eta_4 r^{\Delta s_2+\Delta s_5}\frac{\tilde{\sigma}_{e25}}{\tilde{\sigma}_{e11}}\\&+2\eta_2\eta_3 r^{\Delta s_3+\Delta s_4}\frac{\tilde{\sigma}_{e34}}{\tilde{\sigma}_{e11}}+2\eta_2\eta_4 r^{\Delta s_3+\Delta s_5}\frac{\tilde{\sigma}_{e35}}{\tilde{\sigma}_{e11}}+2\eta_3\eta_4 r^{\Delta s_4+\Delta s_5}\frac{\tilde{\sigma}_{e45}}{\tilde{\sigma}_{e11}}\Bigg)\Bigg],\end{aligned} \tag{2.10}$$

where

$$\tilde{\sigma}_{e_{ij}}=\frac{3}{4}(\tilde{\sigma}_{r_i}-\tilde{\sigma}_{\theta_i})(\tilde{\sigma}_{r_j}-\tilde{\sigma}_{\theta_j})+3\tilde{\tau}_{r\theta_i}\tilde{\tau}_{r\theta_j}, \qquad i,j=1,2,3,4,5. \tag{2.11}$$

As follows, $\tilde{\sigma}_{e_{11}}$ is replaced by $\tilde{\sigma}_{e_1}^2$ for the sake of writing simplicity.

As $r\to 0$, the quantity in parentheses is much smaller than the first term in (2.10). Therefore,

$$\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}=K_1^{n-1}r^{(n-1)s_1}\tilde{\sigma}_{e_1}^{n-1}\Bigg\{1+\eta_1 r^{\Delta s_2}(n-1)\frac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}+\eta_2 r^{\Delta s_3}(n-1)\frac{\tilde{\sigma}_{e13}}{\tilde{\sigma}_{e_1}^2}$$

$$+\eta_3 r^{\Delta s_4}(n-1)\frac{\tilde{\sigma}_{e14}}{\tilde{\sigma}_{e_1}^2}+\eta_4 r^{\Delta s_5}(n-1)\frac{\tilde{\sigma}_{e15}}{\tilde{\sigma}_{e_1}^2}$$

$$\eta_1^2 r^{2\Delta s_2}\frac{(n-1)}{2}\left[\frac{\tilde{\sigma}_{e22}}{\tilde{\sigma}_{e_1}^2}+(n-3)\frac{\tilde{\sigma}_{e12}^2}{\tilde{\sigma}_{e_1}^4}\right]$$

$$+\eta_1\eta_2 r^{\Delta s_2+\Delta s_3}(n-1)\left[\frac{\tilde{\sigma}_{e23}}{\tilde{\sigma}_{e_1}^2}+(n-3)\frac{\tilde{\sigma}_{e12}\sigma_{e13}}{\tilde{\sigma}_{e_1}^4}\right]\Bigg\}+\cdots. \tag{2.12}$$

Substituting (2.8) and (2.12) into (2.3), we obtain

$$\varepsilon_{\beta\gamma}=\varepsilon_{\beta\gamma}^e+\varepsilon_{\beta\gamma}^p\,, \tag{2.13}$$

$$\varepsilon_{\beta\gamma}^e=\varepsilon_0 K_1 r^{s_1}(\tilde{\varepsilon}_{\beta\gamma_1}^e+\eta_1 r^{\Delta s_2}\tilde{\varepsilon}_{\beta\gamma_2}^e+\eta_2 r^{\Delta s_3}\tilde{\varepsilon}_{\beta\gamma_3}^e+\eta_3 r^{\Delta s_4}\tilde{\varepsilon}_{\beta\gamma_4}^e+\eta_4 r^{\Delta s_5}\tilde{\varepsilon}_{\beta\gamma_5}^e), \tag{2.14}$$

where

$$\tilde{\varepsilon}_{\beta\gamma_i}^e=(1+\nu)\tilde{\sigma}_{\beta\gamma_i}+\delta_{\beta\gamma}\Gamma\tilde{\sigma}_{\beta\gamma_i}, \qquad i=1,2,3,4,5 \tag{2.15}$$

$$\varepsilon_{\beta\gamma}^p=\frac{3}{2}\alpha\varepsilon_0\left(\frac{\sigma_e}{\sigma_0}\right)^{n-1}\frac{P_{\beta\gamma}}{\sigma_0}=\alpha\varepsilon_0 K_1^n r^{ns_1}[\tilde{\varepsilon}_{\beta\gamma_1}^p+\eta_1 r^{\Delta s_2}\tilde{\varepsilon}_{\beta\gamma_2}^p+\eta_1^2 r^{2\Delta s_2}\tilde{\varepsilon}_{\beta\gamma_{22}}^p$$

$$+\eta_2 r^{\Delta s_3}\tilde{\varepsilon}_{\beta\gamma_3}^p+\eta_1\eta_2 r^{\Delta s_2+\Delta s_3}\tilde{\varepsilon}_{\beta\gamma_{23}}^p+\eta_3 r^{\Delta s_4}\tilde{\varepsilon}_{\beta\gamma_4}^p+\eta_4 r^{\Delta s_5}\tilde{\varepsilon}_{\beta\gamma_5}^p], \tag{2.16}$$

with

$$\begin{cases}
\tilde{\varepsilon}_{\beta\gamma_1}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\tilde{s}_{\beta\gamma_1}, \\[2ex]
\tilde{\varepsilon}_{\beta\gamma_2}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\left[(n-1)\dfrac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}\,\tilde{s}\,\tilde{\varepsilon}_{\beta\gamma_1}+\tilde{s}_{\beta\gamma_2}\right], \\[2ex]
\tilde{\varepsilon}_{\beta\gamma_{22}}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\left\{\dfrac{(n-1)}{2}\left[\dfrac{\tilde{\sigma}_{e22}}{\tilde{\sigma}_{e_1}^2}+(n-3)\dfrac{\tilde{\sigma}_{e12}^2}{\tilde{\sigma}_{e_1}^4}\right]\tilde{s}_{\beta\gamma_1}+(n-1)\dfrac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}\,\tilde{s}_{\beta\gamma_2}\right\}, \\[2ex]
\tilde{\varepsilon}_{\beta\gamma_3}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\left[(n-1)\dfrac{\tilde{\sigma}_{e13}}{\tilde{\sigma}_{e_1}^2}\tilde{s}_{\beta\gamma_1}+\tilde{s}_{\beta\gamma_3}\right], \\[2ex]
\tilde{\varepsilon}_{\beta\gamma_4}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\left[(n-1)\dfrac{\tilde{\sigma}_{e14}}{\tilde{\sigma}_{e_1}^2}\tilde{s}_{\beta\gamma_1}+\tilde{s}_{\beta\gamma_4}\right], \\[2ex]
\tilde{\varepsilon}_{\beta\gamma_{23}}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}(n-1)\left\{\left[\dfrac{\tilde{\sigma}_{e23}}{\tilde{\sigma}_{e_1}^2}+(n-3)\dfrac{\tilde{\sigma}_{e12}\tilde{\sigma}_{e13}}{\tilde{\sigma}_{e_1}^4}\right]\tilde{s}_{\beta\gamma_1}+\dfrac{\tilde{\sigma}_{e13}}{\tilde{\sigma}_{e_1}^2}\tilde{s}_{\beta\gamma_2}+\dfrac{\tilde{\sigma}_{e12}}{\tilde{\sigma}_{e_1}^2}\,\tilde{s}_{\beta\gamma_3}\right\}, \\[2ex]
\tilde{\varepsilon}_{\beta\gamma_5}^p=\dfrac{3}{2}\tilde{\sigma}_{e_1}^{n-1}\left[(n-1)\dfrac{\tilde{\sigma}_{e15}}{\tilde{\sigma}_{e_1}^2}\tilde{s}_{\beta\gamma_1}+\tilde{s}_{\beta\gamma_5}\right].
\end{cases} \tag{2.17}$$

Here

$$\tilde{s}_{\beta\gamma_i} = \tilde{\sigma}_{\beta\gamma_i} - \frac{1}{2}\tilde{\sigma}_{\rho\rho_i}\delta_{\beta\gamma}, \qquad i=1, 2, 3, 4, 5 .$$

Substituting (2.13) into (2.6), we can obtain

$$\begin{aligned}&\alpha K_1^n r^{ns_1-2}\prod\nolimits_1^p + \alpha K_1^n \eta_1 r^{ns_1+\Delta s_2-2}\prod\nolimits_2^p + \alpha K_1^n \eta_1^2 r^{ns_1+2\Delta s_2-2}\prod\nolimits_3^p \\ &+\alpha K_1^n \eta_2 r^{ns_1+\Delta s_3-2}\prod\nolimits_4^p + \alpha K_1^n \eta_1\eta_2 r^{ns_1+\Delta s_2+\Delta s_3-2}\prod\nolimits_5^p + \alpha K_1^n \eta_3 r^{ns_1+\Delta s_4-2}\prod\nolimits_6^p \\ &+\alpha K_1^n \eta_4 r^{ns_1+\Delta s_5-2}\prod\nolimits_7^p + K_1 r^{s_1-2}\prod\nolimits_1^e + K_1\eta_1 r^{s_1+\Delta s_2-2}\prod\nolimits_2^e = 0,\end{aligned} \tag{2.18}$$

where

$$\left\{\begin{aligned}&\prod\nolimits_1^p = \ddot{\tilde{\varepsilon}}_{r_1}^p - ns_1(ns_1+2)\tilde{\varepsilon}_{r_1}^p - 2(ns_1+1)\dot{\tilde{\varepsilon}}_{r\theta_1}^p, \\ &\prod\nolimits_2^p = \ddot{\tilde{\varepsilon}}_{r_2}^p - (ns_1+\Delta s_2)(ns_1+\Delta s_2+2)\tilde{\varepsilon}_{r_2}^p - 2(ns_1+\Delta s_2+1)\dot{\tilde{\varepsilon}}_{r\theta_2}^p, \\ &\prod\nolimits_3^p = \ddot{\tilde{\varepsilon}}_{r_{22}}^p - (ns_1+2\Delta s_2)(ns_1+2\Delta s_2+2)\tilde{\varepsilon}_{r_{22}}^p - 2(ns_1+2\Delta s_2+1)\dot{\tilde{\varepsilon}}_{r\theta_{22}}^p, \\ &\prod\nolimits_4^p = \ddot{\tilde{\varepsilon}}_{r_3}^p - (ns_1+\Delta s_3)(ns_1+\Delta s_3+2)\tilde{\varepsilon}_{r_3}^p - 2(ns_1+\Delta s_3+1)\dot{\tilde{\varepsilon}}_{r\theta_3}^p, \\ &\prod\nolimits_5^p = \ddot{\tilde{\varepsilon}}_{r_{23}}^p - (ns_1+\Delta s_2+\Delta s_3)(ns_1+\Delta s_2+\Delta s_3+2)\tilde{\varepsilon}_{r_{23}}^p - 2(ns_1+\Delta s_2+\Delta s_3+1)\dot{\tilde{\varepsilon}}_{r\theta_{23}}^p, \\ &\prod\nolimits_6^p = \ddot{\tilde{\varepsilon}}_{r_4}^p - (ns_1+\Delta s_4)(ns_1+\Delta s_4+2)\tilde{\varepsilon}_{r_4}^p - 2(ns_1+\Delta s_4+1)\dot{\tilde{\varepsilon}}_{r\theta_4}^p, \\ &\prod\nolimits_7^p = \ddot{\tilde{\varepsilon}}_{r_5}^p - (ns_1+\Delta s_5)(ns_1+\Delta s_5+2)\tilde{\varepsilon}_{r_5}^p - 2(ns_1+\Delta s_5+1)\dot{\tilde{\varepsilon}}_{r\theta_5}^p, \\ &\prod\nolimits_1^e = \ddot{\tilde{\varepsilon}}_{r_1}^e - s_1\tilde{\varepsilon}_{r_1}^e + s_1(s_1+1)\tilde{\varepsilon}_{\theta_1}^e - 2(s_1+1)\dot{\tilde{\varepsilon}}_{r\theta_1}^e, \\ &\prod\nolimits_2^e = \ddot{\tilde{\varepsilon}}_{r_2}^e - (s_1+\Delta s_2)\tilde{\varepsilon}_{r_2}^e + (s_1+\Delta s_2)(s_1+\Delta s_2+1)\tilde{\varepsilon}_{\theta_2}^e - 2(s_1+\Delta s_2+1)\dot{\tilde{\varepsilon}}_{r\theta_2}^e.\end{aligned}\right. \tag{2.19}$$

The traction free conditions on crack face require

$$\tilde{\varphi}_i(\pi) = \dot{\tilde{\varphi}}_i(\pi) = 0, \tag{2.20}$$

and at $\theta=0$, we have the symmetrical conditions

$$\dot{\tilde{\varphi}}_i(0) = \dddot{\tilde{\varphi}}_i(0) = 0. \tag{2.21}$$

The above formulae (2.18) — (2.21) comprise the governing equations for the asymptotic field.

3 Solution of Governing Equation

The first-order field has been solved by Rice and Rosengren[1] and Hutchinson[2]. Its stress singular exponent is equal to $s_1 = -\dfrac{1}{n+1}$.

For the second-order stress field, Li and Wang[7] obtained its solutions, which was further confirmed by Sharma and Aravas[12]. The eigenvalue of second-order field, Δs_2, is given in Fig. 1.

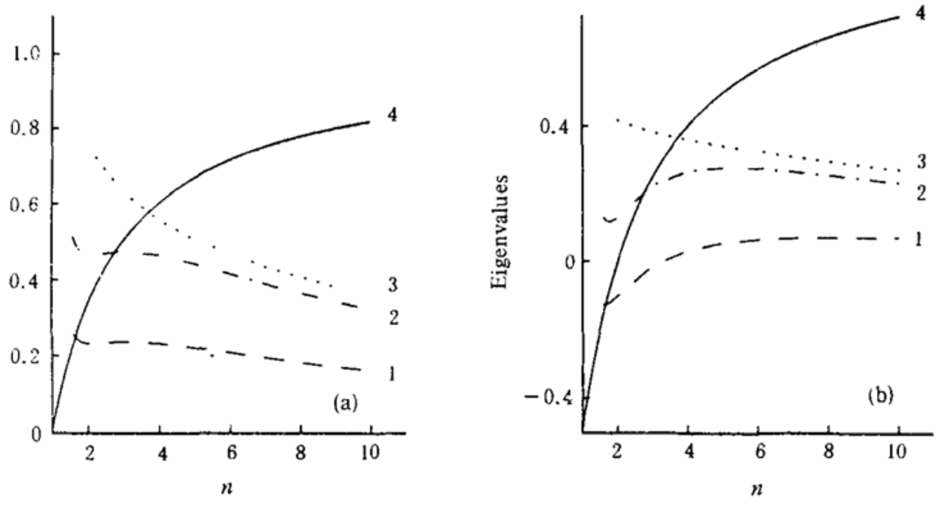

Fig. 1. n-variation of eigenvalues.

In what follows, the third-order field will be analyzed in detail.

From (2.18) we can find that one (or more) of three terms can constitute the third-order stress field. They are the third term, the fourth term and the eighth term of (2.18), i.e.

$$\begin{cases} \alpha K_1^n \eta_1^2 r^{ns_1+2\Delta s_2-2} \prod_3^p, \\ \alpha K_1^n \eta_2 r^{ns_1+\Delta s_3-2} \prod_4^p, \\ K_1 r^{s_1-2} \prod_1^e. \end{cases}$$

What we need to do first is to find out the smallest one among the three exponents:

$$\begin{cases} ns_1+2\Delta s_2-2\,, \\ ns_1+\Delta s_3-2\,, \\ s_1-2\,. \end{cases}$$

It is equivalent to searching for the smallest one among the following three numbers:

$$2\Delta s_2, \qquad \Delta s_3, \qquad (n-1)/(n+1)\,.$$

In the above formulae, the values of $2\Delta s_2$ have been known after the second-order field was determined, which are shown as curve 2 of Fig. 1(a). Curve 4 corresponds to $(n-1)/(n+1)$.

In order to find the value of Δs_3, let the fourth term of (2.18) equal zero, which leads to

$$\prod_4^p = 0 . \tag{3.1}$$

Using (2.19), (3.1) can be transformed into a linear equation:

$$\prod_4^p = D_1 \ddddot{\tilde{\varphi}}_3 + D_2 \dddot{\tilde{\varphi}}_3 + D_3 \ddot{\tilde{\varphi}}_3 + D_4 \dot{\tilde{\varphi}}_3 + D_5 \tilde{\varphi}_3 = 0 , \tag{3.2}$$

where $D_1 \sim D_5$ are functions of $\tilde{\varphi}_1 \sim \ddddot{\tilde{\varphi}}_1$, s_3 and Δs_3, which are given in Appendix.

Equation (3.2) is a homogeneous ordinary differential equation. Therefore, the Eigenvalue Δs_3(or s_3) can be decided by solving this equation.

Figure 1(a) gives n-variations curves of Δs_2, $2\Delta s_2$, Δs_3^* and $(n-1)/(n+1)$ simultaneously. The stress exponent of the third-order field is determined by curve 4 when $1.6 < n \leqslant 2.8$ (i.e. $(n-1)/(n+1)$), but when $n > 2.8$ it is decided by curve 2 (i.e. $2\Delta s_2$).

Analogous analyses can be performed for the fourth- and fifth-order fields.

4 Results

Figures 2 and 3 show the plots of θ-variation for different order stress fields. Here

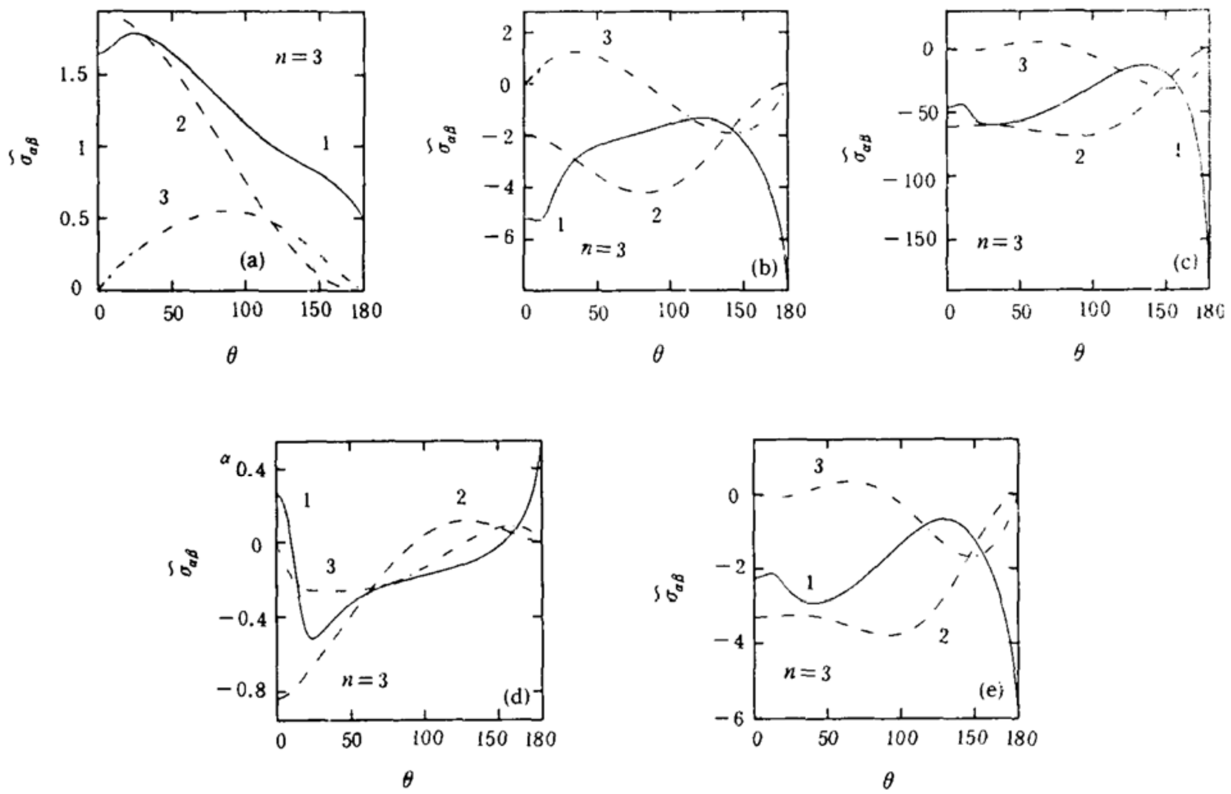

Fig. 2. Angular distribution of the first- to fifth-order stress fields for $n=3$. 1, $\tilde{\sigma}_{ri}$; 2, $\tilde{\sigma}_{\theta i}$, 3, $\tilde{\tau}_{r\theta i}$ $(i=1, 2, 3, 4, 5)$.

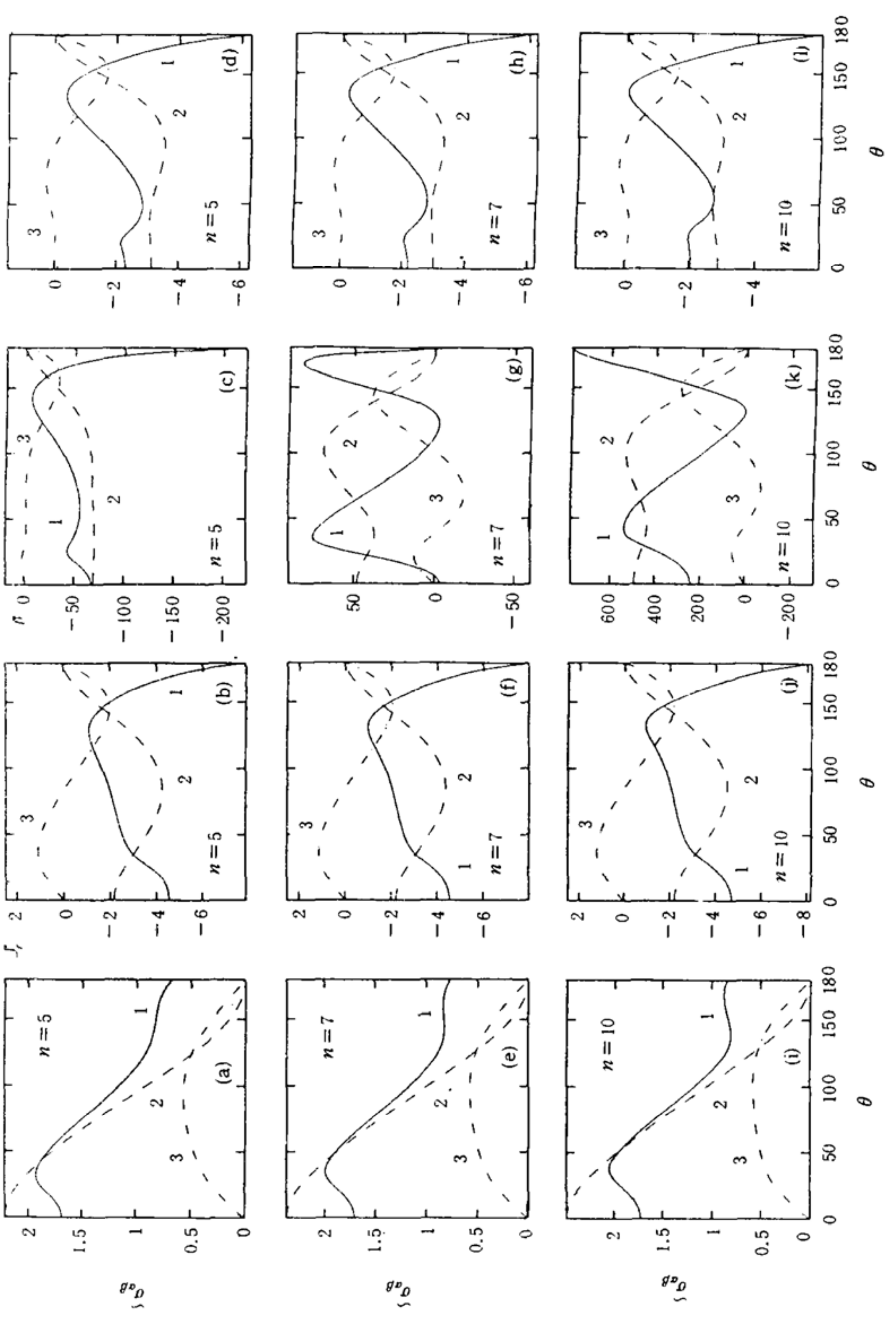

Fig. 3. Angular distribution of the first- to fourth-order stress fields for $n=5$, $n=7$ and $n=10$. 1, $\tilde{\sigma}_{r_i}$; 2. $\tilde{\sigma}_{\theta_i}$; 3, $\tilde{\tau}_{r\theta_i}$ $(i=1, 2, 3, 4)$.

four cases for $n=3$, $n=5$, $n=7$ and $n=10$ are investigated.

For $n=3$, the first- to the fifth-order stress fields are obtained, as Fig. 2 shows. It is noted that the elastic effects enter the fourth-order stress field. Here Poisson's ratio is taken as $\nu=0.5$. The stress solution can be expressed as

$$\begin{aligned}\frac{\sigma_{\beta\gamma}}{\sigma_0} &= \sum_{i=1}^{5} K_i r^{s_i}\tilde{\sigma}_{\beta\gamma_i}(\theta)\\ &= K_1 r^{-1/(n+1)}\tilde{\sigma}_{\beta\gamma_1}(\theta) + K_2 r^{s_2}\tilde{\sigma}_{\beta\gamma_2}(\theta) + \frac{K_2^2}{K_1} r^{2s_2+1/(n+1)}\tilde{\sigma}_{\beta\gamma_3}(\theta)\\ &\quad + \frac{1}{\alpha K_1^{n-2}} r^{(n-2)/(n+1)}\tilde{\sigma}_{\beta\gamma_4}(\theta) + K_5 r^{s_5}\tilde{\sigma}_{\beta\gamma_5}(\theta)\\ &= \left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n r}\right)^{0.25}\tilde{\sigma}_{\beta\gamma_1}(\theta) + K_2 r^{-0.01284028}\tilde{\sigma}_{\beta\gamma_2}(\theta)\\ &\quad + \left(\frac{\alpha\varepsilon_0\sigma_0 I_n}{J}\right)^{0.25} K_2^2 r^{0.22431944}\tilde{\sigma}_{\beta\gamma_3}(\theta) + \frac{1}{\alpha}\left(\frac{\alpha\varepsilon_0\sigma_0 I_n}{J}\right)^{0.25} r^{0.25}\tilde{\sigma}_{\beta\gamma_4}(\theta)\\ &\quad + K_5 r^{0.38221612}\tilde{\sigma}_{\beta\gamma_5}(\theta).\end{aligned} \tag{4.1}$$

Here we can find that $K_3=K_2^2/K_1$ and $K_4=1/(\alpha K_1^{n-2})$. That is to say, among five amplitude coefficients K_i ($i=1, 2, 3, 4, 5$) there are only three independent ones; they are K_1, K_2, K_5. For the fifth-order stress field, its normalized condition is taken as $\tilde{\varphi}_5(0)=-1$.

Figure 3 shows the angular distributions of the first- to fourth-order stress fields for $n=5$, $n=7$, $n=10$. Comparisons indicate that the first-, second- and fourth-order stress fields all have slight changes with different hardening exponents n, but there are relatively large difference among their third-order stress fields. The total stress fields can be represented as

$$\begin{aligned}\frac{\sigma_{\beta\gamma}}{\sigma_0} &= \sum_{i=1}^{4} K_i r^{s_i}\tilde{\sigma}_{\beta\gamma_i}(\theta)\\ &= K_1 r^{-1/(n+1)}\tilde{\sigma}_{\beta\gamma_1}(\theta) + K_2 r^{s_2}\tilde{\sigma}_{\beta\gamma_2}(\theta) + \frac{K_2^2}{K_1} r^{2s_2+1/(n+1)}\tilde{\sigma}_{\beta\gamma_3}(\theta) + K_4 r^{s_4}\tilde{\sigma}_{\beta\gamma_4}(\theta)\\ &= \left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n r}\right)^{1/(n+1)}\tilde{\sigma}_{\beta\gamma_1}(\theta) + K_2 r^{s_2}\tilde{\sigma}_{\beta\gamma_2}(\theta)\\ &\quad + \left(\frac{\alpha\varepsilon_0\sigma_0 I_n}{J}\right)^{1/(n+1)} K_2^2 r^{s_3}\tilde{\sigma}_{\beta\gamma_3}(\theta) + K_4 r^{s_4}\tilde{\sigma}_{\beta\gamma_4}(\theta),\end{aligned} \tag{4.2}$$

from which it can also be seen that K_3 is equal to K_2^2/K_1, and K_4 is another independent amplitude coefficient besides K_1 and K_2. The correspondent normalized condition of the fourth-order field is $\tilde{\varphi}_4(0)=-1$. Table 1 gives detailed values of stress exponents for the first- to fourth-order stress fields at three different hardening exponents $n=5$, $n=7$ and $n=10$. The stress exponents for $n=3$ are also given in Table 1.

Table 1

	s_1	s_2	s_3	s_4
$n=3$	-0.25000000	-0.01284028	0.22431944	0.25000000
$n=5$	-0.16666667	0.05455957	0.27578581	0.34072164
$n=7$	-0.12500000	0.06937479	0.26374958	0.30845004
$n=10$	-0.09090909	0.06976616	0.23044141	0.26959457

Lastly, comparisons will be made between our results and O'Dowd and Shih's[8]. Figure 4 is the difference stress fields for $n=10$ given by their numerical calculations.

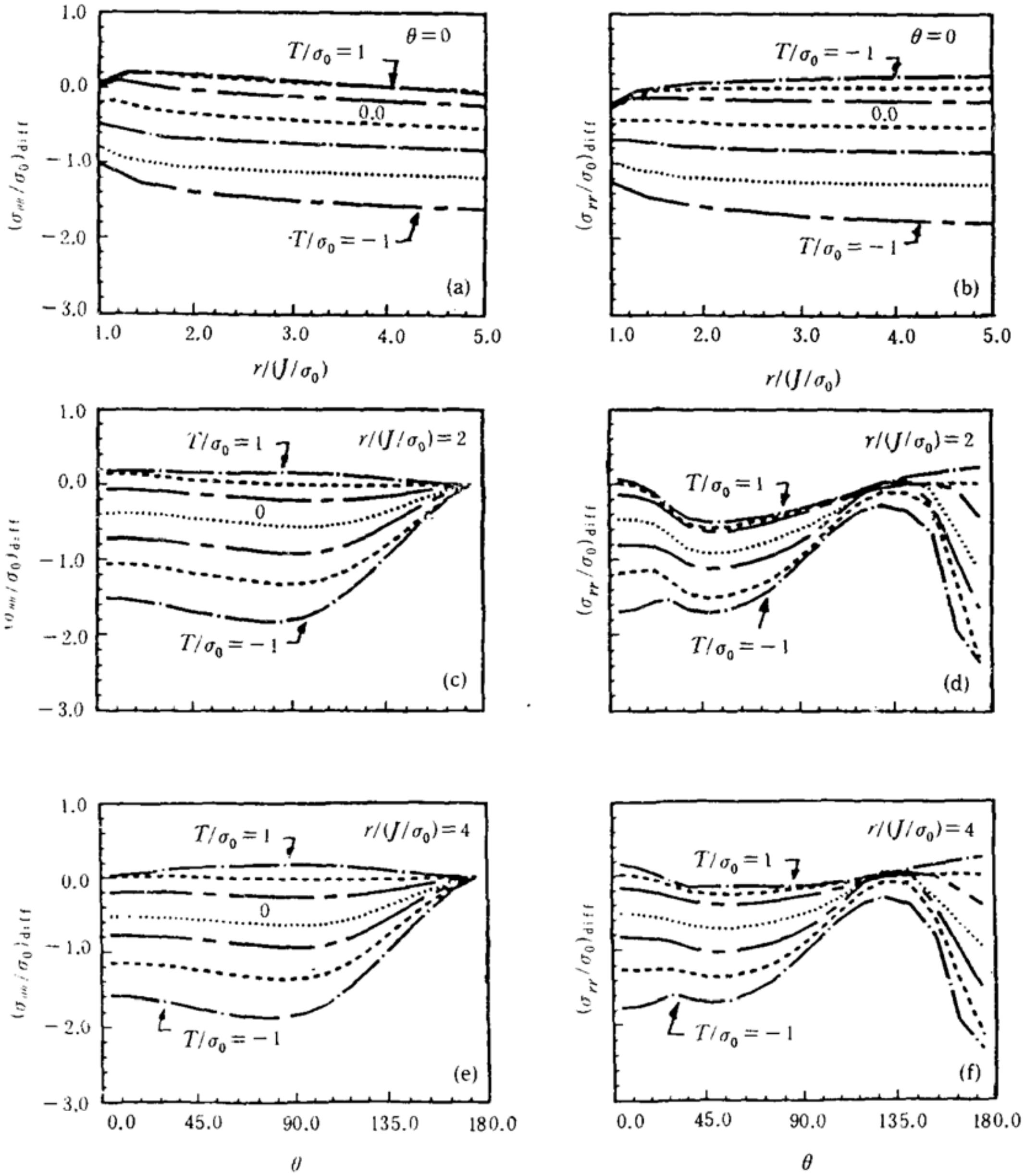

Fig. 4. Figure 5 of O'Dowd and Shih (1991).

According to O'Dowd and Shih[8], the difference stress field is defined as the difference between the numerical solutions generated by the modified layer formulation (in which the remote tractions are given by $\sigma_{\beta\gamma}=\frac{K_I}{\sqrt{2\pi r}}f_{\beta\gamma}(\theta)+T\delta_{1\beta}\delta_{1\gamma}$) and the HRR field. Correspondingly, the sum of our second-, third- and fourth-order stress fields for $n=10$ can be expressed as

$$\left(\frac{\sigma_{\beta\gamma}}{\sigma_0}\right)_{\rm diff}=\sum_{i=1}^{4}K_i r^{s_i}\tilde{\sigma}_{\beta\gamma_i}(\theta)$$

$$=K_1\eta_1 r^{s_2}\tilde{\sigma}_{\beta\gamma_2}(\theta)+K_1\eta_1^2 r^{2s_2+1/(n+1)}\tilde{\sigma}_{\beta\gamma_3}(\theta)+K_1\eta_3 r^{s_4}\tilde{\sigma}_{\beta\gamma_4}(\theta)$$

$$=k_2\left(\frac{r}{J/\sigma_0}\right)^{0.06976616}\tilde{\sigma}_{\beta\gamma_2}(\theta)+k_3\left(\frac{r}{J/\sigma_0}\right)^{0.23044141}\tilde{\sigma}_{\beta\gamma_3}(\theta)$$

$$+k_4\left(\frac{r}{J/\sigma_0}\right)^{0.26959457}\tilde{\sigma}_{\beta\gamma_4}(\theta), \tag{4.3}$$

where

$$\begin{cases} k_2=K_1\eta_1\left(\dfrac{J}{\sigma_0}\right)^{s_2}, \\ k_3=K_1\eta_1^2\left(\dfrac{J}{\sigma_0}\right)^{2s_2-s_1}, \\ k_4=K_1\eta_3\left(\dfrac{J}{\sigma_0}\right)^{s_4}, \end{cases} \tag{4.4}$$

$$K_1=\left(\frac{J}{\alpha\varepsilon_0\sigma_0 I_n}\right)^{1/(n+1)}. \tag{4.5}$$

It is easily seen that

$$k_3=(\alpha\varepsilon_0 I_n)^{1/(n+1)}k_2^2, \tag{4.6}$$

where $\alpha=1.0$, $\varepsilon_0=1/300$, and $I_n=4.54$, the same as those of O'Dowd and Shih's (1991).

Table 2

$n=10$

	$Q\hat{\tau}_{\theta\theta}/\sigma_0$	$Q\hat{\tau}_{rr}/\sigma_0$	k_2	k_3	k_4
1	−1.50	−1.70	0.06844	0.00320	0.91558
2	−1.06	−1.20	0.05513	0.00208	0.61191
3	−0.74	−0.80	0.04027	0.00111	0.36945
4	−0.40	−0.48	0.03212	0.00071	0.21034
5	−0.08	−0.13	0.01571	0.00017	0.04041
6	0.20	0.10	0.01035	0.00007	−0.05275

Therefore, there are only two independent coefficients k_2 and k_4 in (4.3). We can adjust them to match (4.3) with $Q\hat{\tau}_{\theta\theta}/\sigma_0$ of Fig. 4(c) and $Q\hat{\tau}_{rr}/\sigma_0$ of Fig. 4(d) at $\theta=0.0$. It is noted that $r/(J/\sigma_0)=2$ in Fig. 4(c) and Fig. 4(d). The calculated results of k_2 and k_4 together with k_3 are shown in Table 2.

Figure 5 is plotted according to (4.3) and Table 2. Comparison of Fig. 5 with Fig. 4 shows that there exists a good agreement between both results. Moreover, Fig. 6 gives our results ((a), (b)) and O'Dowd and Shih's ((c), (d)) for $\tau_{r\theta}/\sigma_0$ simultaneously. It can be seen that two results are very similar to each other. Fig. 6(a)—(b) reveals that the first-order shear stress field, i.e. HRR solution for $\tau_{r\theta}$, is dominant within the forward sector $-\pi/2\leqslant\theta\leqslant\pi/2$. This point indeed agrees with Fig. 6(c), (d).

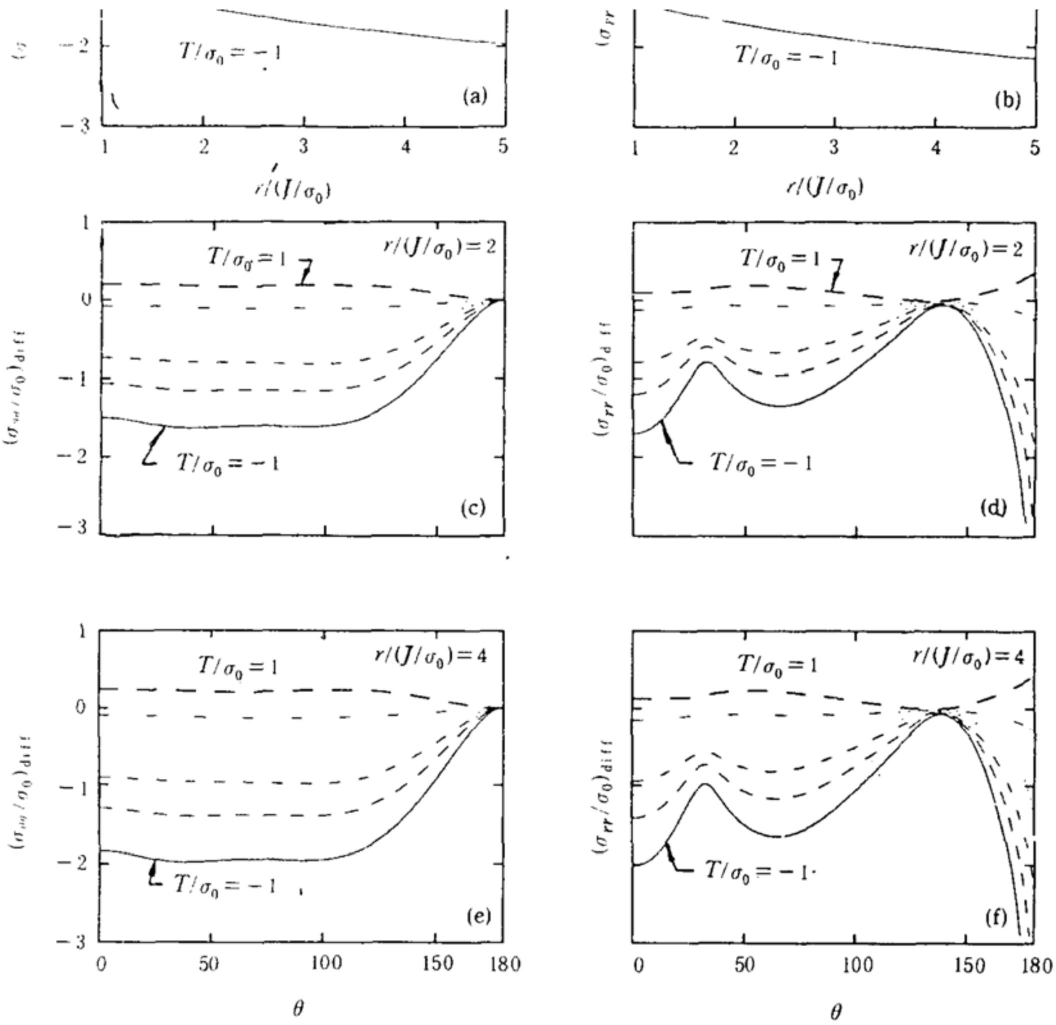

Fig. 5. The sum of the second- to the fourth-order stress fields of our results.

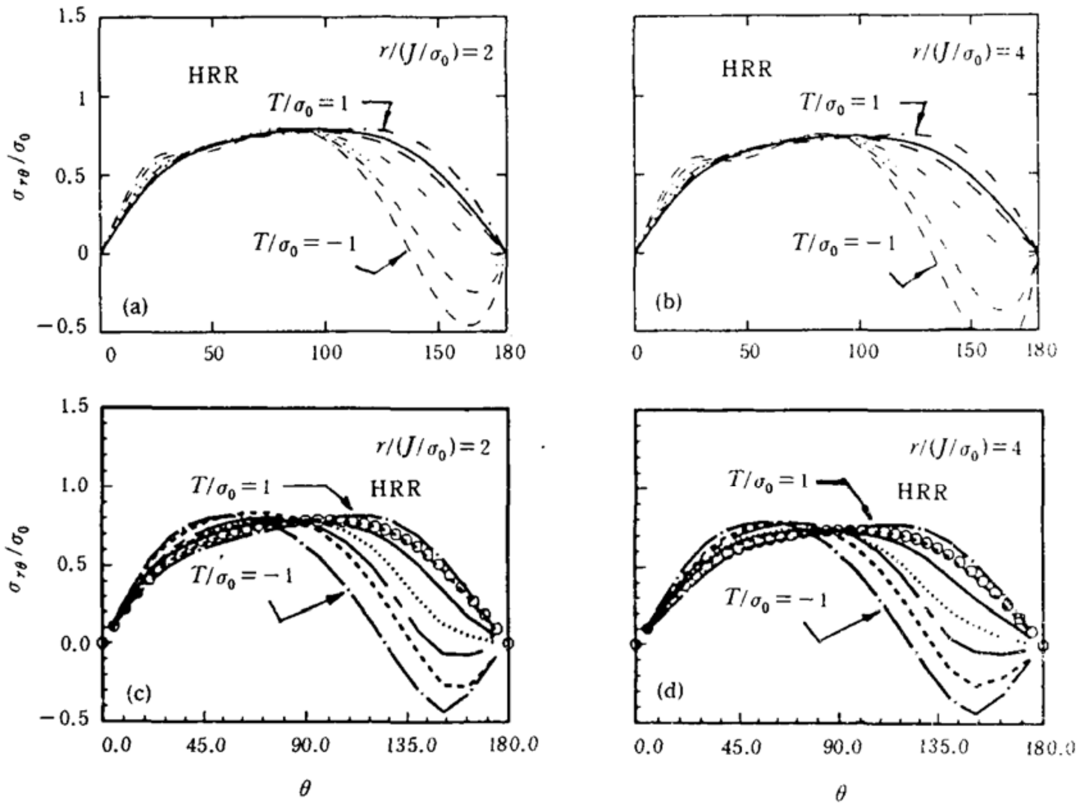

Fig. 6. Angular distribution of shear stress. (a) and (b) are our results; (c) and (d) are O'Dowd and Shih's results (1991) (their Fig. 3).

5 Conclusions

For Mode I plane strain crack, a higher-order asymptotic analysis has been made for the near-crack-tip stress fields in power law hardening materials. A five-term expansion of asymptotic series of stress fields has been obtained for $n=3$. And for $n=5$, $n=7$, and $n=10$, the similar four-term expansion of solutions has also been derived. Our investigations show that when $1.6<n\leqslant 2.8$ the elastic effect enters the third-order stress field. But as $2.8<n\leqslant 3.5$ this effect turns to enter the fourth-order stress field, with the amplitude coefficient of the fifth-order field independent. Furthermore, we have observed that if $n>3.7$, the elastic effect will enter the fields whose order is higher than 4, while the fourth-order field is independent. We also found that $s_3=2s_2-s_1$ and the amplitude K_3 depends on both K_1 and K_2 so long as $n>2.8$. Moreover, K_4 will become independent when $n\geqslant 3.7$. Therefore, it can be concluded that whether or not K_i ($i=2, 3, 4, 5$) is independent depends on what value the hardening exponent n is.

For $n=10$, comparisons have been made between our results and O'Dowd and Shih's[8], which show a good agreement. This fact reflects that the higher terms of asymptotic expansion, more or less, have some contributions to the near-crack-tip stress fields.

Appendix

Using Eqs. (2.20), the plastic strains can be written as

$$\begin{cases} \tilde{\varepsilon}^p_{r_3} = A(\tilde{\sigma}_{r_3} - \tilde{\sigma}_{\theta_3}) + B\tilde{\tau}_{r\theta_3}, \\ \tilde{\varepsilon}^p_{r\theta_3} = B(\tilde{\sigma}_{r_3} - \tilde{\sigma}_{\theta_3})/2 + C\tilde{\tau}_{r\theta_3}, \end{cases} \tag{A1}$$

where

$$\begin{cases} A = \Omega\{[(n-1)\tilde{S}^2_{r_1} + \tilde{g}]\tilde{g}^2\}, \\ B = \Omega\{2(n-1)\tilde{S}_{r_1}\tilde{\tau}_{r\theta_1}\tilde{g}^2\}, \\ C = \Omega\{2[(n-1)\tilde{\tau}^2_{r\theta_1} + \tilde{g}]\tilde{g}^2\}, \end{cases} \tag{A2}$$

$$\tilde{S}_{r_1} = \frac{1}{2}(\tilde{\sigma}_{r_1} - \tilde{\sigma}_{\theta_1}), \qquad \tilde{g} = \tilde{S}^2_{r_1} + \tilde{\tau}^2_{r\theta_1}, \tag{A3}$$

with $\Omega = (3^{\frac{n+1}{2}}/4)g^{\frac{n-7}{2}}$.

In (3.7), the coefficients $D_1 \sim D_5$ can be expressed as

$$\begin{cases} D_1 = A, \\ D_2 = 2A' - (ns_1 + \Delta s_3 + s_3 + 2)B, \\ D_3 = A'' - (ns_1 + \Delta s_3 + 2s_3 + 3)B' - [(ns_1 + \Delta s_3)(ns_1 + \Delta s_3 + 2) \\ \qquad + (s_3 + 2)s_3]A + 2(ns_1 + \Delta s_3 + 1)(s_3 + 1)C, \\ D_4 = -(s_3 + 1)B'' - 2(s_3 + 2)s_3A' + [(ns_1 + \Delta s_3)(ns_1 + \Delta s_3 + 2)(s_3 + 1) \\ \qquad + (ns_1 + \Delta s_3 + 1)(s_3 + 2)s_3]B + 2(ns_1 + \Delta s_3 + 1)(s_3 + 1)C', \\ D_5 = -(s_3 + 2)s_3A'' + (ns_1 + \Delta s_3)(ns_1 + \Delta s_3 + 2)(s_3 + 2)s_3A \\ \qquad + (ns_1 + \Delta s_3 + 1)(s_3 + 2)s_3B'. \end{cases} \tag{A4}$$

References

1 Rice, J. R. & Rosengren, G. F., *J. Mech. Phys. Solids*, 1986, **16**:1968.
2 Hutchinson, J. W., *J. Mech. Phys. Solids*, 1968, **16**:13.
3 Shih, C F. & German, M. D., *Int. J. Fract. Mech.*, 1981, **17**:27.
4 McMeeking, R. M. & Parks, D. M., *ASME STP* 668, 1979, pp.175–194.
5 Needleman, A. & Tvergaard, V., *ASME STP* 803, 1983, pp.80–115.
6 Wang T C , *Proceedings of ICF International Symposium on Fracture Mechanics*, Beijing, 1983, pp.243–248.
7 Li, Y C. & Wang, T. C., *Scientia Sinica* (Series A), 1986, **29**:941.
8 O'Dowd, N. P. & Shih, C. F., *J. Mech. Phys. Solids*, 1991, **39A**:989.
9 O'Dowd, N. P. & Shih, C. F., *J, Mech. Phys. Solids*, 1991, b, in press.
10 Shih, C. F., O'Dowd, N. P. & Kirk, M T., in *ASTM STP*. December, 1991.
11 Betegón, C. & Hancock, J W., *J. Appl. Mech.*, 1991. **58**:104.
12 Sharma, S. M. & Aravas, N., *J. Mech. Phys. Solids*, 1991, **39**:1043.
13 Xia, L. & Wang, T. C., *Acta Mechanica Sinica*, 1992, (8).
14 Li, S. T., 1988, *Master Thesis* (in Chinese).
15 Li, S. T & Wang, T. C., *Science in China* (Series A), 1992, **35**(4):395.