Statistical Description of Pattern Evolution in Damage-Fracture*

XIA Meng-Fen** (夏蒙棼), BAI Yi-Long (白以龙) and KE Fu-Jiu (柯孚久)***

(Laboratory for Nonlinear Mechanics of Continuous Media, Institute of Mechanics, Academia Sinica, Beijing 100080, PRC)

Received August 8, 1993

picted by probability distribution function.

Keywords: pattern evolution in damage-fracture, evolution-induced catastrophe, stochastic jump, pattern dynamics.

1 Introduction

So far there has been no universal theoretical framework for the damage-fracture processes in a solid under external load, which, as a kind of complex pattern evolution, generally involve the nucleation and extension of microdamages, coalescence between microdamages and the formation of a main crack that leads to the eventual fracture.

Pattern evolution is a very common problem in nonlinear systems; it has aroused a good deal of attention in recent years. An important characteristic of pattern evolution in damage-fracture is that the systems are controlled by both deterministic dynamics and disorder effects. Experimental results and numerical simulation show that the nucleated microdamages are distributed randomly over a material element, and the coalescence can occur only for those microdamages whose ligaments are shorter than or of the same order of the scale as the microdamages, suggesting that the nucleation is closely related to the disordered distribution of meso-structure in materials, and the coalescence is mainly determined by the local enhancement of stress acting on the

* Work supported by the National Basic Research Project Nonlinear Science.

** Department of Physics, Center of Nonlinear Science, Peking University, Beijing 100871, PRC.

*** Department of Applied Mathematics and Physics, Beijing University of Aeronautics and Astronautics, Beijing 100080, PRC.

ligament, following a deterministic rule. In this paper, we discuss the general characteristics of pattern evolution in a system controlled by both deterministic dynamics and stochastic jumps. We assume that the nucleation is a stochastic jump and the coalescence follows a deterministic dynamical rule.

For simplicity, we consider a finite-one-dimensional lattice, whose mechanical background may be a sheet coupon or a bundle of fibres.

Let us examine a chain consisting of N sites. There are two possible states in each site. One is $x_i=1$, called occupied state, and the other is $x_i=0$, called empty state. A state, or a pattern, of the system is expressed by $X=\{x_i=0 \text{ or } 1 \mid i=1,2,\cdots, N\}$. The number of occupied sites in a state is

$$n=\sum_{i=1}^{N} x_i, \tag{1}$$

and $p=n/N$ is the fraction of occupation, p can be regarded as damage fraction: $p=0$ describes a solid state; whereas for a damage state, the value of p is within the range of $0<p<1$. Moreover, a cluster of occupied sites can be regarded as a microcrack; $p=1$ represents the fracture state $X_F=\{x_i=1 \mid i=1, 2,\cdots, N\}$.

The phase space of the system consists of

$$\Omega=2^N \tag{2}$$

state points. All the states in the phase space can be devided into $(N+1)$ groups according to the value of n (or p). The number of states in group n is

$$\Omega_n=N!/n!(N-n)!. \tag{3}$$

The fracture state X_F is the unique one in group $n=N$.

2 Pattern Dynamics

In the light of observation, it is assumed that the coalescence of microdamages follows a deterministic, irreversible dynamical rule. The deterministicity implies that a dynamical trajectory in phase space is uniquely determined by its initial state, while the irreversibility refers to the fact that the change in a site from $x_i=0$ to $x_i=1$ is permissible but the reverse is forbidden, i.e. the annihilation of microdamages is negligible.

For a system controlled by deterministic and irreversible dynamics, the states in phase space can be classified into two classes: transient states and fixed points. A pattern evolution is completed after a system goes through a sequence of transient states and approaches one of the fixed points. The set of final states is identical to the set of fixed points in phase space.

The pattern dynamics of the system can be described by the flow and the structure

in phase space. The flow in phase space refers to the set of trajectories starting from every state. The structure of phase space is specified by a series of fixed point attractors and their basins of attraction. In order to investigate the global behaviour of the system, we shall introduce a statistical description for the dynamical evolution of the system.

In general, the final states of the evolution starting from Ω_n states in group n may belong to several different groups. Letting $y_{nn'}$ be the number of evolution processes with starting state in group n and final state in group n', we get an $(N+1)\cdot(N+1)$ matrix Y with elements $y_{nn'}$. Hence Y is named evolution matrix.

Due to the irreversibility of the dynamics, the occupation fraction of final state should not be less than that of the initial state, so we have $y_{nn'}=0$ for $n>n'$. The diagonal element y_{nn} gives the number of fixed points in group n. The total number of fixed points $Z=\sum y_{nn}$ is the total number of possible final states of the system. The total number of states in the basins of attractors belonging to group n can be calculated by

$$M_n=\sum_{n_1=0}^{N} y_{n_1 n}=\sum_{n_1=0}^{n} y_{n_1 n}\,. \tag{4}$$

The fracture state X_F is the unique state in group $n=N$ and it is a fixed point. The element y_{nN} gives the number of states belonging to the basin of X_F in group n, and the total number of the states in the basin of X_F is

$$M_1=\sum_{n=0}^{N} y_{nN}. \tag{5}$$

Define two probability distribution functions concerning the states as follows:

$$\Phi(p)=\Phi_n=y_{nN}/\Omega_n \tag{6}$$

and

$$\xi(p)=\xi_n=y_{nN}/M_F, \tag{7}$$

where $p=n/N$. Φ and ξ describe the distribution of the basin of X_F in the phase space. $\Phi(p)$ gives the fraction of states belonging to the basin of X_F in total states of group n, and $\xi(p)$ gives the ratio of the number of states in the basin of X_F that belongs to group n to that of the total states of the basin of X_F.

According to the final states of evolution, the flows in phase space belong to two kinds of evolution modes: globally stable modes (GS modes) and evolution induced

to fracture. Dynamically, an evolution mode is exclusively determined by its initial state. The states in phase space may also be classified as GS states and EIC states according to their evolution modes.

Now we introduce a dynamical evolution rule to model the coalescence between microcracks. We assume that the stress released by nucleated damage will be equally borne by its two sides, and the coalescence of adjacent microcracks will occur if the average stress on their ligament exceeds the strength. Based on these assumption, the coalescence rule can be derived as follows[2]: Let r- and s-occupied clusters be separated by an l-empty cluster. Then the l-empty cluster will be occupied if

$$l \leqslant (r+s) * G/2, \tag{8}$$

where G is a parameter. If the empty cluster is at the end of the lattice, then r or s is zero in Eq. (8). The rule can be illustrated in a geometric way. There are two influence regions with size $G \cdot r/2$ ahead of both ends of an r-occupied cluster. Coalescence will occur if the influence regions of two adjacent cracks contact or overlap each other. The pattern evolution dynamics governed by rule (8) is deterministic, irreversible, nonlinear and nonlocal. The nonlocality indicates that the change in a site is relevant not only to its local conditions but also to distant sites. Eq. (8) represents

may be a certain relationship between EIC modes and percolation of effective occupation region because a system will belong to EIC mode if it is fully covered by effective occupation regions. In one-dimensional lattice, the threshold of effective percolation is $p_c' = 1$, which gives the threshold of real occupation fraction

$$p_c = 1/(1+G). \tag{9}$$

However, there are some inherent differences between EIC mode and percolation. The percolation is an equilibrium phase transition controlled by parameter p, and EIC mode describes an evolution far from equilibrium, where p is a variable. The evolution is a pattern-specific phenomenon, which cannot be determined only by the value of p. One will see that only in the sense of averaging or most probability can the threshold condition (9) be applicable to EIC mode.

Now, we present several examples of a system governed by the dynamical rule (8). For a lattice with $N=10$, we have $\Omega = 1024$ and all the states in phase space are divided into 11 groups. In Table 1, we list Ω_n and the elements of the evolution matrix Y for the case of $N=10$ and $G=1$. The distribution functions Φ_n and ξ_n are shown in Table 2.

Table 1 shows that the total number of fixed points $Z=179$. Let it be noted that

Table 1 Evolution Matrix Y and Ω_n, $N=10$, $G=1$

n \ n'	0	1	2	3	4	5	6	7	8	9	10	Ω_n
0	1	0	0	0	0	0	0	0	0	0	0	1
1	0	10	0	0	0	0	0	0	0	0	0	10
2	0	0	35	8	2	0	0	0	0	0	0	45
3	0	0	0	58	30	12	12	0	0	0	8	120

$$\begin{cases} 0<\Phi(p)<1 & \text{for } p_L<p<p_U, \\ \Phi(p)=1 & \text{for } p_U\leqslant p\leqslant 1. \end{cases} \tag{10}$$

The upper and lower bounds of the transitional region are $p_U=0.7$ and $p_L=0.2$ for the case of $N=10$ and $G=1$, respectively. The total number of states in the basin of λ_1 is $M_F=682$; among them there are $M_F'=506$ states in the transitional region. We have $M_F/\Omega=0.66602$ and $M_F'/M_F=0.74194$; the latter value means that the transition region plays an important role in fracture phenomena.

Figure 1 is a schematic diagram of the phase space for $N=10$ and $G=1$, where the boundary between GS states and EIC states is represented by a solid line and the

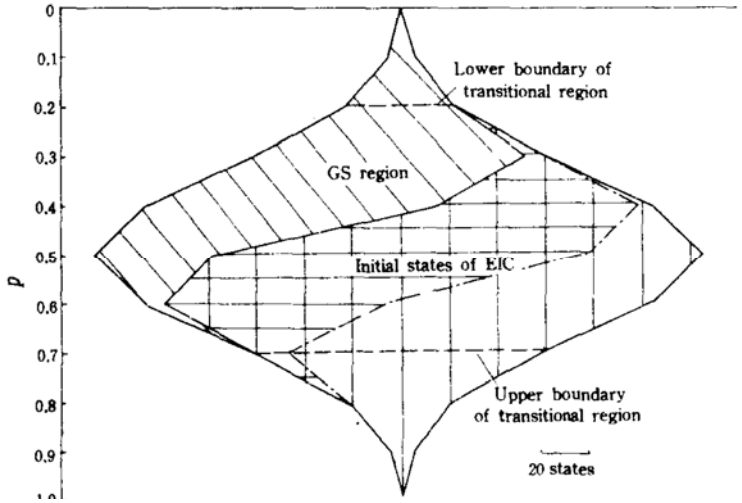

Table 3 Transition Region in Different Cases

N	5	6	7	8	9	10	10	10
G	1	1	1	1	1	1	0.5	2
p_L	0.2	0.1667	0.2857	0.25	0.2222	0.2	0.4	0.1

$$\lim_{N\to\infty} p_U = 1/(1+G/2). \tag{12}$$

The two boundaries of the transitional region are related to the evolution rule.

The existence of transitional region indicates that there is no explicit distinction between GS modes and EIC modes, and the distribution of these modes in phase space should be described by probability distribution function $\Phi(p)$ or $\xi(p)$.

3 Stochastic Pattern Jump

In our damage-fracture model, on the one hand the pattern evolution follows a deterministic dynamical rule, and on the other hand the pattern evolution may also be caused by stochastic jumps. The latter models the stochastic nucleation of microdamages. The dynamical evolution of a pattern can be affected by the stochastic jump in various ways. It can jump from a trajectory to another trajectory, from a basin

to another basin, and especially can convert from GS mode to EIC mode, and then to fracture.

Here, the selection rule of stochastic pattern jumps and the jump probability are assumed as follows:

(i) irreversibility, i.e. a pattern jump from occupied site to empty is not permissible;

(ii) change in occupation number is limited by $n=1$ in each pattern jump;

(iii) all possible jumps have the same probability.

Now, let us discuss the number of a jump from one state to another. There are $(N-n+1)$ empty sites in a state with occupation number $(n-1)$, so the number of possible jumps $\Delta n=1$ is $(N-n+1)$, giving rise to $(N-n+1)$ distinct states with occupation number n.

Since there are Ω_{n-1} states in group $(n-1)$, the total number of possible jumps from group $(n-1)$ to group n is

$$v=\sum_{n=1} v_n . \tag{14}$$

v is the total number of all possible jumps in the system.

It is easy to see that a jump-driven conversion from EIC mode to GS mode is impossible. The jumps may be classified into three groups: GS → GS, EIC → EIC and GS → EIC. There are $\Omega_{n-1}\Phi_{n-1}$ EIC states in group $(n-1)$. The number of all the possible jumps is $(N-n+1)\Omega_{n-1}\Phi_{n-1}$ and all of them lead to EIC states in group n. In group n, the number of EIC states is $\Omega_n\Phi_n$, and the number of all the possible jumps to these states from the states in group $(n-1)$ is $n\Omega_n\Phi_n$. Clearly among them the number of jumps EIC → EIC is $(N-n+1)\Omega_{n-1}\Phi_{n-1}$, and the rest should belong to the jumps from GS to EIC. Then, we get the number of the jumps from GS states in group $(n-1)$ to EIC states in group n as follows:

$$\mu_n \equiv \mu_{n-1,n}=n\Omega_n\Phi_n-(N-n+1)\Omega_{n-1}\Phi_{n-1}=(\Phi_n-\Phi_{n-1})v_n. \tag{15}$$

Let

$$\mu=\sum_{n=1}^{N}\mu_n . \tag{16}$$

μ is the total number of all possible jumps GS → EIC.

Define two probability distribution functions concerning jump

$$\psi(p) \equiv \psi_n \equiv \psi_{n-1}, n = \mu_n / \nu_n = \Phi_n - \Phi_{n-1} \tag{17}$$

and

$$\eta(p) \equiv \eta_n \equiv \eta_{n-1,n} = \mu_n/\mu = \frac{\nu_n}{\mu}\ (\Phi_n - \Phi_{n-1}). \tag{18}$$

ψ_n is the ratio of the number of jumps GS→ EIC to that of all jumps from group $(n-1)$ to group n; and η_n is the ratio of number jumps GS → EIC from group $(n-1)$ to group n to that of all the possible jumps GS → EIC. ψ_n and η_n satisfy the normalizing condition.

The functions $\psi(p)$ and $\eta(p)$ describe the probability distribution of mode conversion from GS to EIC modes through jump; they have a non-zero region:

$$\psi(p)>0,\ \eta(p)>0 \text{ for } p_L < p \leqslant p_U. \tag{19}$$

This non-zero region nearly coincides with the transitional region defined by function $\Phi(p)$. In general, $\psi(p)$ is a single peak function. We let the position of the peak $p=p_m$. Define

$$\bar{p} = \sum_p p\psi(p) \tag{20}$$

and

$$\sigma = [\sum_p (p-\bar{p})^2 \psi(p)]^{1/2}, \tag{21}$$

where $\bar{p}$ and $\sigma/\bar{p}$ give the global shape of the function $\psi(p)$.

In the case $N=10$ and $G=1$, we get $\nu=5120$, $\mu=1032$, and $\gamma \equiv \mu/\nu=0.20156$. ν_n, μ_n, ψ_n and η_n are shown in Table 4. In Table 5, we show p_m, $\bar{p}$, $\sigma/\bar{p}$, γ and p_c for several cases.

Table 4 ν_n, μ_n, ψ_n and η_n for $N=10$ and $G=1$

n	0	1	2	3	4	5	6	7	8	9	10
ν_n	/	10	90	360	840	1260	1260	840	360	90	10
μ_n	/	0	0	24	308	474	198	28	0	0	0
ψ_n	/	0	0	0.0667	0.3667	0.3762	0.1571	0.0333	0	0	0
η_n	/	0	0	0.0233	0.2983	0.4593	0.1919	0.0271	0	0	0

Table 5 p_m, $\bar{p}$, $\sigma/\bar{p}$, γ and p_c

N	5	6	7	8	9	10	10	10
G	1	1	1	1	1	1	0.5	2
p_c	0.5	0.5	0.5	0.5	0.5	0.5	0.6667	0.3333
p_m	0.6	0.5	0.5714	0.5	0.4444	0.5	0.7	0.3
$\bar{p}$	0.66	0.5278	0.5388	0.5366	0.4714	0.4724	0.6789	0.3010
$\sigma/\bar{p}$	0.1940	0.2011	0.1802	0.2148	0.2016	0.1949	0.1001	0.2175
γ	0.3125	0.2917	0.2723	0.2383	0.2131	0.2016	0.1762	0.0793

The meaning of $\psi(p)$ is that if the occupied sites are cumulated stochastically, the probability of mode conversion from GS to EIC at occupation fraction p is given by $\psi(p)$. This is merely a probability of fracture in the case of stochastic cumulation of microdamages. From Table 5 we can see that p_m and $\bar{p}$ are close to p_c, indicating a relationship between fracture and percolation, that is to say, the percolation, of effective region of occupation, describes the mode conversion from GS to EIC in the sense of averaging and most probability. However, Tables 4 and 5 show that there is a broad non-zero region in function $\psi(p)$ resulting from the pattern-specific behaviour of the dynamical evolution, and the boundary between GS modes and EIC modes cannot be specified by a threshold value of p. Therefore, as a non-equilibrium evolution phenomenon, the fracture is inherently different from percolation phenomenon.

The conversion from GS modes to EIC modes has a crucial meaning in the processes from stochastic cumulation of microdamages to fracture. However, only the EIC states, which can be attained by the jump GS → EIC and are not the intermediate states in a dynamical flow line, are the true initial states of EIC modes. These states govern the lifetime of a system decisively. Let t_n be the number of the initial states of EIC modes in group n, and $f_n = t_n/\Omega_n$. In Table 6, we show t_n and f_n for $N=10$ and $G=1$. We can see that the initial states of EIC modes are distributed in the area of the transitional region (i.e. $f(p)=f_n>0$ for $p_L < p \leqslant p_U$). It means once more that the transitional region plays an important role in fracture phenomena. The distribution of initial states of EIC modes in phase space is also shown in Fig.1 (the area between the solid line and dash-dotted line).

Table 6 The Distribution of Initial States of EIC Modes for the Case of $N=10$ and $G=1$

n	0	1	2	3	4	5	6	7	8	9	10
t_n		0	0	8	85	156	92	12	0	0	0
f_n		0	0	0.0667	0.4048	0.6191	0.4381	0.1	0	0	0

4 Conclusion

In this paper, the fracture is treated as a problem of pattern evolution. A model governed by both deterministic dynamics and stochastic jump is adopted. In the model, the coalescence of microdamages is taken into account by a deterministic, irreversible, nonlocal and nonlinear dynamical rule, and the nucleation of microdamages is described by irreversible stochastic jump. Such a kind of pattern evolution involving both dynamics and disorder is a very common phenomenon.

We lay emphasis on the fracture condition, which should be attributed to conversion from GS mode to EIC and the formation of a specific pattern (the fracture state). As was pointed out, the statistical description is a powerful technique. In this paper, we have analyzed the flow and structure of phase space, defined an evolution matrix,

and introduced the distribution function $\Phi(p)$ and $\xi(p)$ to depict the distribution of states in the basin of fracture in phase space. We also introduced distribution functions $\psi(p)$ and $\eta(p)$ to describe the probability distribution of mode conversion from GS to EIC. Finally we have given a distribution function $f(p)$ to describe the initial states of EIC modes, which are related to the lifetime of the materials.

In our evolution model, an important characteristic is the existence of a transitional region in the space of occupation fraction p. In the transitional region, both GS and EIC modes exist. The mode conversion from GS to EIC has non-zero probability, where the initial states of EIC modes are also distributed there. The transitional region plays an essential role in fracture phenomena.

The theoretical framework and the results obtained in this paper may serve as a starting point for a deep-going study of the damage-fracture processes.

References

1 Curran, D. R., Seaman, L. & Shockey, D. A., *Phys. Rep.*, 1987, **147**: 253.

2 Bai Yilong, Xia Mengfen, Ke Fujiu *et al*, *Chin. Phys. Lett*, 1993, **10**: 155.

3 Stauffer, D. & Aharony, A., *Introduction to Percolation Theory*, 2nd ed., Taylor & Francis Ltd. 1991